Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Distributed Control for a Class of Spatially Interconnected Discrete-time Systems

Saulat S. Chughtai, Herbert Werner

Institute of Control Systems, Hamburg University of Technology, Germany

Abstract: This paper considers the analysis and synthesis of a spatially distributed controller for discrete-time spatially interconnected parameter-varying system. The system under consideration has both discrete time and space dynamics. The concept of quadratic separators (Iwasaki and Shibata, 2001), (Chughtai and Werner, 2007) has been extended to compute a measure of worst-case performance for such systems by solving an LMI problem. The use of quadratic separator allows a systematic search for a parameter dependent Lyapunov function, thus resulting in less conservative controllers. The problem of synthesizing controllers leads to a nonlinear matrix inequality, and a hybrid evolutionary-LMI approach to solving this problem, based on LMI solvers and genetic algorithms, is proposed in this paper. A design example illustrates the efficiency of the proposed method.

Keywords: Spatially interconnected Systems, Genetic Algorithms, Gain Scheduled Control Systems.

1. INTRODUCTION

The problem of finding suitable controllers for large scale interconnected systems has attracted researchers for more than three decades. The main problem associated with such systems is that modern MIMO controller design approaches may fail if the number of subsystems becomes very large.

In some cases interacting systems are distributed spatially and their interactions depend on the spatial location of one subsystem with respect to another. In (D'Andrea and Dullerud, 2003), a detailed analysis and synthesis framework is presented to deal with systems which consists of identical interconnected subsystems located at the nodes of a fixed lattice as shown in Figure 1.

The framework developed in (D'Andrea and Dullerud, 2003) requires that the spatially distributed system $\mathbf{G}$ be represented in its state space form as

$$\begin{bmatrix} \mathbf{T}x^t(t,s) \\ \mathbf{S}x^s(t,s) \\ z \end{bmatrix} = \begin{bmatrix} A^{tt} & A^{ts} & B_1^t \\ A^{st} & A^{ss} & B_1^s \\ C_1^t & C_1^s & D_{11} \end{bmatrix} \begin{bmatrix} x^t \\ x^s \\ w \end{bmatrix} \tag{1}$$

where $\mathbf{T}$ and $\mathbf{S}$ are the temporal differential and spatial shift operators defined as:

$$\mathbf{T}x^t(t,s) = \frac{dx(t,s_1,\ldots,s_L)}{dt}$$
$$\mathbf{S_i}x^s(t,s) = x(t,s_1,\ldots,s_i+1,\ldots,s_L)$$

and $s = [s_1, ..., s_L]$ represents the spatial variable. Physically, the spatial states x^s represent the interactions among the subsystems.

The assumption of having identical subsystems can be relaxed to near identical subsystems if these can be represented in Linear Fractional Transformation (LFT) form,

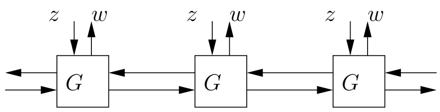

Fig. 1. Spatially Distributed System

see (Wu and Yildizoglu, 2005). The approach proposed in that work searches for a fixed Lyapunov function over complete variations and any conservatism which may arise is reduced by scaling or multipliers. The controller designed using the approach has structure as shown in Figure 2, which requires considerable communication among the local controllers. Which may limit its application.

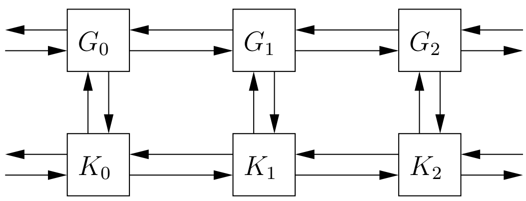

Fig. 2. Spatially Gain Scheduled Control Structure

This paper presents new analysis conditions to find the worst-case $\mathcal{L}_2$-norm of spatially interconnected parameter varying (SIPV) discrete-time systems. The approach presented here is based on the concept of quadratic separation (Iwasaki and Shibata, 2001). Thus, it systematically searches for a parameter dependent Lyapunov function along with an extra degree of freedom which arises in the form of multipliers. The analysis condition is used to synthesize fixed structured controllers using LMI solvers and evolutionary search. Experience suggests that this approach is quite suitable for real-world applications, as it is independent of controller order and structure. It allows a systematic search for less conservative distributed controllers which are easily implementable. Specifically we have considered the following types of controller.

1. As shown in Figure 3, where the controller is scheduled according to the spatial variations of parameters but different control modules do not exchange information, thus reducing the communication burden.
2. As shown in Figure. 4, where the same controller is used at each node point.

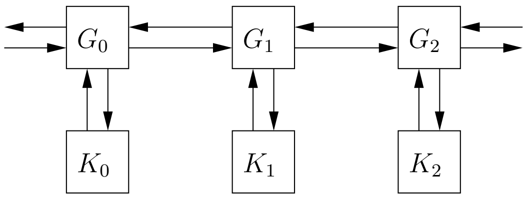

Fig. 3. Spatially Decentralized Gain Scheduled Control Structure

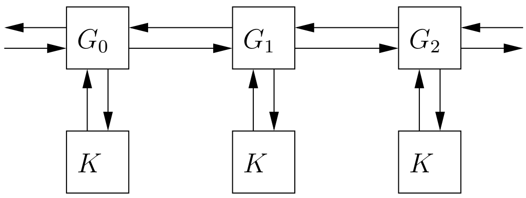

Fig. 4. Decentralized Robust Control Structure

The paper is organized as follows: Section II summarizes some preliminary concepts which will be used in Section III to obtain an $\mathcal{L}_2$-norm analysis result. Section IV presents an approach to controller synthesis using the analysis LMIs developed in section III. In section IV the infinite-dimensional problem of Theorem 3.1 is converted to a finite dimensional problem using the D-G scaling approach of (Iwasaki and Shibata, 2001). The approach is applied to a simple example to demonstrate its usage. Finally in section V some conclusions are drawn.

2. NOTATION AND PRELIMINARIES

Let the state space representation of an interconnected system be given as

$$\begin{bmatrix} \mathbf{T}x^t(t,s) \\ \mathbf{S}x^s(t,s) \\ q \\ z \end{bmatrix} = \begin{bmatrix} A^{tt} & A^{ts} & B_0^t & B_1^t \\ A^{st} & A^{ss} & B_0^s & B_1^s \\ C_0^t & C_0^s & D_{00} & 0 \\ C_1^t & C_1^s & 0 & D_{11} \end{bmatrix} \begin{bmatrix} x^t \\ x^s \\ p \\ w \end{bmatrix} \qquad (2)$$

with feedback

$$p = \Delta q$$

where $p, q \in R^m$, $\Delta \in R^{m\times m}$, $\Delta \in \mathbf{\Delta}$. We are dealing with signals which are vector valued functions of $L+1$ independent variables: $d = d(t, s_1, \ldots, s_L)$, where t denotes the temporal variable and s_i the spatial variable. For the systems considered here both t and s_i are integer valued.

For brevity let us define $x^T = \begin{bmatrix} x^{tT} & x^{sT} \end{bmatrix}$, where $x^t \in R^{n_t}$, $n_s = \sum_{i=1}^{L} n_{s_i} + \sum_{i=1}^{L} n_{s_{-i}}$, $x^{s_1} \in R^{n_{s_i}}$, $w \in R^l$, $z \in R^k$ and,

$$A = \begin{bmatrix} A^{tt} & A^{ts} \\ A^{st} & A^{ss} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_1^t \\ B_1^s \end{bmatrix} \quad C_1 = \begin{bmatrix} C_1^t & C_1^s \end{bmatrix} \qquad (3)$$

Using the discrete-time stability condition presented in (Curtain and Zwart, 1995), Theorem 2 of (D'Andrea and Dullerud, 2003) can be modified for discrete-time discrete-space systems as follows.

Theorem 2.1. A system $\mathbf{G}$ with state space representation (1), is well-posed, stable and has $\mathcal{L}_2$-norm $< \gamma$ if there exist $X_t \in \mathcal{X}_t$ and $X_s \in \mathcal{X}_s$ such that the following inequality is satisfied:

$$\begin{bmatrix} A^T X A - X & A^T X B_1 & C_1^T \\ B_1^T X A & -\gamma^2 I & D_{11}^T \\ C_1 & D_{11} & -I \end{bmatrix} < 0 \qquad (4)$$

where

$$X := diag(X^t, X^s)$$
$$\mathcal{X}^t := \{X^t \in R^{n_t \times n_t} : X^t = X^{tT} > 0\}$$

$$\mathcal{X}^s := \{X^s = diag(X^{s_1}, ..., X^{s_L}) : X^{s_i} = X^{s_i T} \in R^{n_{s_i} \times n_{s_i}}\}$$

3. WORST CASE $\mathcal{L}_2$-NORM OF SIPV SYSTEMS

Consider an SIPV system with state space representation as given in (2). We can define

$$A^{\Delta} = A + \begin{bmatrix} B_0^t & B_0^t \\ B_0^s & B_0^s \end{bmatrix} \begin{bmatrix} N^t & 0 \\ 0 & N^s \end{bmatrix} \qquad (5)$$

where

$$\begin{aligned} N^t &:= (I - \Delta D_{00})^{-1} \Delta C_0^t \\ N^s &:= diag(N_0^{s_1}, \ldots, N_0^{s_L}) \\ N^{s_i} &= (I - \Delta D_{00})^{-1} \Delta C_0^{s_i} \\ C_0^s &= [C_0^{s_1} \ \ldots \ C_0^{s_L}] \end{aligned} \qquad (6)$$

Furthermore, let the parameter-dependent Lyapunov matrix X^{Δ} be represented as shown in Figure 5, where $\Delta(t,s)$

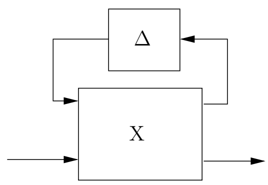

Fig. 5. Lyapunov matrix X^{Δ}

is the same as that associated with the plant. Then

$$X^{\Delta} = diag(X^t_{\Delta}, X^s_{\Delta}) \qquad (7)$$

where

$$\xi^T \left((*)^T X \begin{bmatrix} A^{tt} & A^{ts} & B_0^t & \tilde{B}_0^t & B_1^t & 0 & 0 \\ A^{st} & A^{ss} & B_0^s & \tilde{B}_0^s & B_1^s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix} - \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} X (*)^T \begin{bmatrix} C_1^{tT} C_1^t & C_1^{tT} C_1^s & 0 & 0 & C^{tT} D_{11} & 0 \\ C_1^{sT} C_1^t & C_1^{sT} C_1^s & 0 & 0 & C_1^{sT} D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ D_{11}^T C_1^t & D_{11}^T C_1^s & 0 & 0 & D_{11}^T D_{11} - \gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \xi < 0 \tag{14}$$

$$\begin{bmatrix} \Delta C_0^t & \Delta C_0^s & (\Delta D_{00} - I) & (\Delta \tilde{D} - \tilde{I}) & 0 & 0 & 0 & 0 & 0 \\ \Delta C_0^t A^{tt} & \Delta C_0^t A^{ts} & \Delta C_0^t B_0^t & \Delta C_0^t \tilde{B}_0^t & \Delta C_0^t B_1^t & (\Delta D_{00} - I) & 0 & I & 0 \\ \tilde{\Delta} \tilde{C}_0^s A^{st} & \tilde{\Delta} \tilde{C}_0^s A^{ss} & \tilde{\Delta} \tilde{C}_0^s B_0^s & \tilde{\Delta} \tilde{C}_0^s \tilde{B}_0^s & \tilde{\Delta} \tilde{C}_0^s B_1^s & 0 & (\tilde{\Delta} \tilde{D}_{00} - I) & 0 & I \\ \Delta^\delta C_0^t A^{tt} & \Delta^\delta C_0^t A^{ts} & \Delta^\delta C_0^t B_0^t & \Delta^\delta C_0^t \tilde{B}_0^t & \Delta^\delta C_0^t B_1^t & \tilde{\Delta}^\delta D_{00} & 0 & -I & 0 \\ \tilde{\Delta}^\delta \tilde{C}_0^s A^{st} & \tilde{\Delta}^\delta \tilde{C}_0^s A^{ss} & \tilde{\Delta}^\delta \tilde{C}_0^s B_0^s & \tilde{\Delta}^\delta \tilde{C}_0^s \tilde{B}_0^s & \tilde{\Delta}^\delta \tilde{C}_0^s B_1^s & 0 & \tilde{\Delta}^\delta \tilde{D}_{00} & 0 & -I \end{bmatrix} \xi = 0 \tag{15}$$

$$\begin{aligned} X_\Delta^t &= \begin{bmatrix} I \\ N^t \end{bmatrix}^T X^t \begin{bmatrix} I \\ N^t \end{bmatrix}, \\ X_\Delta^s &= diag(X_\Delta^{s_1}, ..., X_\Delta^{s_L}) \\ X_\Delta^{s_i} &= \begin{bmatrix} I \\ N^{s_i} \end{bmatrix}^T X^{s_i} \begin{bmatrix} I \\ N^{s_i} \end{bmatrix}, X^{s_i} \in R^{n_{s_i} \times n_{s_i}} \end{aligned} \tag{8}$$

Applying Theorem 2.1 to the system whose dynamics are governed by A^Δ will leads to an LMI condition where (4) is replaced by the following inequality

$$\begin{bmatrix} A^{\Delta T} X_{k+1}^\Delta A^\Delta - X_k^\Delta & A^{\Delta T} X_{k+1}^\Delta B_1 & C_1^T \\ B_1^T X_{k+1}^\Delta A^\Delta & -\gamma^2 I & D_{11}^T \\ C_1 & D_{11} & -I \end{bmatrix} < 0 \tag{9}$$

where the index k is introduced to distinguish between X^Δ at different times, since only temporally causal systems are considered here. Applying the Schur complement to (9) results in

$$J < 0 \tag{10}$$

where

$$J = \begin{bmatrix} A^{\Delta T} X_{k+1}^\Delta A^\Delta - X_k^\Delta + C_1^T C_1 & A^{\Delta T} X_k^\Delta B_1 + C_1^T D_{11} \\ B_1^T X_k^\Delta A^\Delta + D_{11}^T C_1 & D_{11}^T D_{11} - \gamma^2 I \end{bmatrix}$$

Let us further define $\Delta_k^\delta = \Delta_{k+1} - \Delta_k$ as the change in uncertainty during one interval and

$$\begin{aligned} \eta_k^t &= N_{k+1}^t \mathbf{T} x_k^t \\ \phi_k^t &= \Delta_k^\delta C_0^t \begin{bmatrix} A^{tt} & A^{ts} \end{bmatrix} x_k + \Delta_k^\delta C_0^t B_0^t p_k^t + \Delta_k^\delta C_0^t \tilde{B}_1^t w_k \\ &\quad + \Delta_k^\delta D_{00} \eta_k \\ \eta_k^s &= N_{k+1}^s \mathbf{S} x^s \\ \phi_k^s &= \tilde{\Delta}_k^\delta \tilde{C}_0^s \begin{bmatrix} A^{ss} & A^{st} \end{bmatrix} + B_0^s p_k^t + \tilde{B}_0^s p_k^s + B_1^s w \\ &\quad + \Delta_k^\delta \tilde{D}_{00} \eta_k^s \end{aligned} \tag{11}$$

where

$$\begin{aligned} p_k^t &= N_k^t x_k^t \\ p^s &= [p^{s_1}, \ldots, p^{s_L}]^T \\ p^{s_i} &= N^{s_i} x^{s_i}, \forall i = \{1, \ldots, L\} \end{aligned} \tag{12}$$

and $\tilde{D_{11}}$, $\tilde{D}$, $\tilde{I}$, $\tilde{B}_0^t$ and $\tilde{B}_0^s$ are matrices with D_{11}, D_{00}, I, B_0^t, B_0^s repeated L times, and

$$\begin{aligned} \tilde{\Delta} &= diag(\Delta, \ldots, \Delta), \tilde{\Delta^\delta} = diag(\Delta^\delta, \ldots, \Delta^\delta) \\ \tilde{C}_0^s &= diag(C_0^{s_1}, \ldots, C_0^{s_L}), \tilde{D}_{00} = diag(D_{00}, \ldots, D_{00}) \end{aligned} \tag{13}$$

Remark: Here for generality it is assumed that $\Delta^\delta \neq 0$. Indeed, if only uncertain systems are considered, then $\Delta^\delta = 0$ can be considered as a special case.

Now, for a system defined by (2) and using (5) and (7), (10) can be written as (14). From (6), (11) and (12) the vectors x, η^t, η^s, ϕ^t and ϕ^s must satisfy the constraint given in (15), where $\xi^T = [x^t, x^s, p^t, p^s, w, \eta^t, \eta^s, \phi^t, \phi^s]$.

Let

$$\mathcal{A} = \begin{bmatrix} A^{tt} & A^{ts} & B_0^t & \tilde{B}_0^t \\ A^{st} & A^{ss} & B_0^s & \tilde{B}_0^s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} B_1^t & 0 & 0 & 0 & 0 \\ B_1^s & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} C_0^t & C_0^s & D_{00} & \tilde{D} \\ C_0^t A^{tt} & C_0^t A^{ts} & C_0^t B_0^t & C_1^t \tilde{B}_0^t \\ \tilde{C}_0^s A^{st} & \tilde{C}_0^s A^{ss} & \tilde{C}_0^s B_0^s & \tilde{C}_1^s \tilde{B}_0^s \\ C_0^t A^{tt} & C_0^t A^{ts} & C_0^t B_0^t & C_1^t \tilde{B}_0^t \\ \tilde{C}_0^s A^{st} & \tilde{C}_0^s A^{ss} & \tilde{C}_0^s B_0^s & \tilde{C}_1^s \tilde{B}_0^s \\ 0 & 0 & I & \tilde{I} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ C_0^t B_1^t & D_{00} & 0 & 0 & 0 \\ \tilde{C}_0^s B_1^s & 0 & \tilde{D}_{00} & 0 & 0 \\ C_0^t B_1^t & D_{00} & 0 & 0 & 0 \\ \tilde{C}_0^s B_1^s & 0 & \tilde{D}_{00} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & -I & 0 \\ 0 & 0 & I & 0 & -I \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

$$\mathcal{E} = \begin{bmatrix} C_1^t & C_1^s & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{F} = \begin{bmatrix} D_{11} & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \quad (16)$$

then, (14) and (15) can be written as

$$\xi^T \left(\begin{bmatrix} \mathcal{A}^T \\ \mathcal{B}^T \end{bmatrix} X \left[\mathcal{A}\ \mathcal{B}\right] - \begin{bmatrix} I \\ 0 \end{bmatrix} X \left[I\ 0\right] + \begin{bmatrix} \mathcal{E}^T \\ \mathcal{F}^T \end{bmatrix} \Gamma \left[\mathcal{E}\ \mathcal{F}\right] \right) \xi < 0 \quad (17)$$

$$\left[\nabla\ -I\right]\left[\mathcal{C}\ \mathcal{D}\right] \xi = 0 \quad (18)$$

where $\nabla := diag(\Delta, \Delta, \tilde{\Delta}, \Delta^\delta, \tilde{\Delta}^\delta)$ and

$$\Gamma_\gamma = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$$

Applying the Generalized Finsler's Lemma (Iwasaki and Shibata, 2001) , (17) and (18) are equivalent to

$$\begin{bmatrix} \mathcal{A}^T \\ \mathcal{B}^T \end{bmatrix} X \left[\mathcal{A}\ \mathcal{B}\right] - \begin{bmatrix} I \\ 0 \end{bmatrix} X \left[I\ 0\right] + \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}^T \end{bmatrix} \Theta \left[\mathcal{C}\ \mathcal{D}\right] + \begin{bmatrix} \mathcal{E}^T \\ \mathcal{F}^T \end{bmatrix} \Gamma \left[\mathcal{E}\ \mathcal{F}\right] < 0 \quad (19)$$

where $\Theta \in \boldsymbol{\Theta}$

$$\boldsymbol{\Theta} = \left\{ \boldsymbol{\Theta} : \begin{bmatrix} I \\ \nabla \end{bmatrix}^{\mathbf{T}} \boldsymbol{\Theta} \begin{bmatrix} I \\ \nabla \end{bmatrix} > \mathbf{0} \right\} \quad (20)$$

The above result can formally be presented as the following theorem.

Theorem 3.1. Consider a SIPV system with state space realization (2). The system is well-posed and internally stable and the worst-case $\mathcal{L}_2$-norm from w to z is less than γ if there exists $X := diag(X^t, X^s)$, where

$$\mathcal{X}^t := \{X^t \in R^{(n_t+m)\times(n_t+m)} : X^t = X^{tT} > 0\}$$
$$\mathcal{X}^s := \{X^s = diag(X^{s_1}, ..., X^{s_L}) : X^{s_i} = X^{s_i T} \in R^{(n_{s_i}+m)\times(n_{s_i}+m)}\} \quad (21)$$

and the following holds for each admissible $\Delta \in \boldsymbol{\Delta}$.

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \\ \mathcal{C} & \mathcal{D} \\ \mathcal{E} & \mathcal{F} \end{bmatrix}^T \begin{bmatrix} X & 0 & 0 & 0 \\ 0 & -X & 0 & 0 \\ 0 & 0 & \Theta & 0 \\ 0 & 0 & 0 & \Gamma_\gamma \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \\ \mathcal{C} & \mathcal{D} \\ \mathcal{E} & \mathcal{F} \end{bmatrix} < 0$$

$$\begin{bmatrix} I \\ \nabla \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \nabla \end{bmatrix} > 0 \quad (22)$$

4. CONTROLLER SYNTHESIS

In this section it will be shown how the analysis condition presented in the previous section can be utilized for controller synthesis purposes. Let an SIPV system (2) be given in generalized plant form as

$$\begin{bmatrix} \mathbf{T}x^t \\ \mathbf{S}x^s \\ q \\ z \\ y \end{bmatrix} = \begin{bmatrix} A^{tt} & A^{ts} & B_0^t & B_1^t & B_2^t \\ A^{st} & A^{ss} & B_0^s & B_1^s & B_2^s \\ C_0^t & C_0^s & D_{00} & 0 & D_{02} \\ C_1^t & C_1^s & 0 & D_{11} & D_{12} \\ C_y^t & C_y^s & D_{yp} & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x^t \\ x^s \\ p \\ w \\ u \end{bmatrix} \quad (23)$$
$$e = r - y$$

with feedback

$$p = \Delta q$$

Checking the condition of Theorem 3.1 is an infinite-dimensional problem, which can be reduced by using the D-G scaling approach as proposed in (Iwasaki and Shibata, 2001). This approach is however conservative as the set $\boldsymbol{\Theta}$ will be replaced by a subset to gain tractability.

4.1 D-G scaling

Let Δ and Δ^δ of a SIPV system have the form

$$\Delta(t,s) = diag\left\{q_1(t)I_{k_1}, \ldots, q_v(t)I_{k_v}, q_{v+1}(s)I_{k_{v+1}}, \ldots, q_m(s)I_{k_m}\right\}$$
$$\Delta^\delta(t,s) = diag\left\{q_1^\delta(t)I_{k_1}, \ldots, q_v^\delta(t)I_{k_v}, 0, \ldots, 0\right\} \quad (24)$$

and all the varying parameters q_i are bounded such that $|q_i| < \delta_i\ \forall i = 1, \ldots, m$. Moreover, changes in the varying parameter in single time interval q_i^δ are bounded such that $\left|q_i^\delta\right| < \rho_i$, $\forall i = 1, \ldots, v$, while the spatially varying parameters are considered fixed over time.

Let us also define sets of scaling matrices that commute with the structure of ∇ as follows:

$$D := \{D : D\nabla = \nabla D, D = D' > 0\}$$
$$G = \{G : G\nabla = \nabla G, G + G' = 0\} \quad (25)$$

Using D-G scaling leads to the following result.

Theorem 4.1. The LPV system described by (2) and (24) is stable and has worst-case performance less than γ if there exists $X = diag(X^t, X^s)$, where X^t and X^s are defined in (21) and $\Theta \in \Theta_{DG}$ such that (22) holds, where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ are defined in (16) and

$$\Theta_{DG} := \left\{ \begin{bmatrix} \Upsilon R \Upsilon & S \\ S' & -R \end{bmatrix} : R \in D, S \in G \right\} \quad (26)$$

where

$$\Upsilon := diag\left(\Upsilon_\delta, \Upsilon_\delta, \tilde{\Upsilon}_\delta, \Upsilon_\rho, \tilde{\Upsilon}_\rho\right)$$
$$\Upsilon_\delta := diag\left(\delta_1 I_{k_1}, \ldots, \delta_m I_{k_m}\right)$$
$$\Upsilon_\rho := diag\left(\rho_1 I_{k_1}, \ldots, \rho_v I_{k_v}, 0, \ldots, 0\right)$$
$$\tilde{\Upsilon}_\delta = diag(\Upsilon_\delta, \ldots, \Upsilon_\delta)$$
$$\tilde{\Upsilon}_\rho = diag(\Upsilon_\rho, \ldots, \Upsilon_\rho) \quad (27)$$

The synthesis problem - the LMI (19) - is turned into a non-linear matrix inequality in P, Θ, Γ_γ and K. However, for fixed K, (19) can be solved using standard LMI solvers. This motivates a hybrid evolutionary-LMI approach, where K is searched for using evolutionary search methods.

4.2 Hybrid Evolutionary-Algebraic Algorithm for Controller Synthesis

A hybrid evolutionary approach was proposed in (Farag and Werner, 2004) for solving non-convex synthesis problems involving lumped LTI systems. Here, this approach is used to construct K, and the LMI solver is applied to calculate P, Θ and γ. A complete algorithm to find a controller that minimizes the worst-case performance γ over the complete operational envelope of an SIPV system, can be given as follows:

- **Generate** an initial random population of controllers $\{K_1, \ldots, K_\mu\}$,
  where $K_i = \{{A_k}^i, {B_k}^i, {C_k}^i, {D_k}^i\}$
- **Evaluate** the objective function:
  $$f(K_i) = \begin{cases} \gamma & \text{if } A_{cl} \text{ is stable} \\ \kappa(A_{cl}) + \beta & \text{if } A_{cl} \text{ is unstable} \end{cases} \quad (28)$$
  where A_{cl} is the A-matrix of the closed loop system, $\kappa(A_{cl})$ is the maximum real part of the eigenvalues of A_{cl}, β is a penalty on destabilizing controllers, and γ is the worst-case performance obtained by solving (22) with (26) using standard LMI solvers
- **Evolve** the current generation using evolutionary operators to produce the next generation
- **Repeat** evaluate and evolve steps until a stopping criterion is met.

Note that the above algorithm is independent of the controller structure, thus controllers with arbitrary order and structure can be synthesized.

4.3 Example

To illustrate the above algorithm, consider the problem of temperature control of a nonuniform two-dimensional plate. This example is a modified version of an example given in (D'Andrea and Dullerud, 2003) (to test the above algorithm, a spatial variation of the thermal conductivity has been introduced). Let Q be a heat source, then the multi-dimensional heat transfer in the absence of any convective heat loss, is given by,

$$\rho c \frac{\partial T}{\partial t} = \nabla(K \nabla T) + Q \quad (29)$$

where, T, ρ, c, K are the temperature, density, specific heat and thermal conductivity of the material and $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.

It is assumed that the thermal conductivity of the plate varies linearly along both the dimensions by the following relation.

$$K(x, y) = K_0(1 - \epsilon \frac{x}{L_1} - \epsilon \frac{y}{L_2}) \quad (30)$$

Using the finite difference approximation of the time derivative and two spatial partial derivatives, results in the following discrete-time, discrete-space approximation

Table 1. Worst case $\mathcal{L}_2$-norm (γ) for different values of ϵ.

ϵ	γ	γ_{ro}	γ_{gs}
0.1	1.621	1.92	1.84
0.3	2.425	5.31	3.55
0.7	3.273	10.21	6.43

$$\begin{aligned} \frac{T_{k+1} - T_k}{t_s} &= (K' - \alpha\delta_1 - \alpha\delta_2)(S_1 + S_1^{-1} - 2)T_k \\ &+ (K' - \alpha\delta_1 - \alpha\delta_2)(S_2 + S_2^{-1} - 2)T_k \\ &- \frac{2\alpha}{L_1}(S_1 - S_1^{-1})T_k - \frac{2\alpha}{L_2}(S_2 - S_2^{-1})T_k \\ &+ \frac{1}{\rho c} Q, \end{aligned} \quad (31)$$

where $K' = K_0(1 - \epsilon)$, $\alpha = \epsilon \frac{K_0}{2\rho c}$, $\delta_1 = 2x/L_1 - 1$, $\delta_2 = 2y/L_2 - 1$ and t_s is the time interval, taken as 0.01 sec. The boundary conditions are taken to be simply $T(t, 0, y) = T(t, L_1, y) = T(t, x, 0) = T(t, x, L_2) = 0$. Let d_1 be the input disturbance and r be the reference temperature. The control objective is disturbance rejection with minimum control effort.

For comparison, first the approach presented in (Wu and Yildizoglu, 2005) (the controller shown in Figure 2) is applied to the system for different values of ϵ. The results are summarized in Table 1, γ denotes the achieved worst-case performance. One can see that as the spatial variation in the system increases the worst-case $\mathcal{L}_2$-norm also increases.

Next the fixed-structure controllers shown in Figure. 4 and 3 are designed. The worst-case $\mathcal{L}_2$-norms achieved by these controllers, after 100 generations with 20 individuals, are γ_{ro} and γ_{gs}, respectively. Note that the worst-case performance index has increased, which indicates a deterioration in achieved control objectives - this is the price one has to pay in order to use simply structured controllers. The main advantage is that the communication burden has been reduced and also the connective stability is ensured as discussed in (Siljak, 1978).

5. CONCLUSIONS

This paper presents sufficient LMI conditions to find the worst case $\mathcal{L}_2$-norm of discrete time SIPV systems. These conditions are less conservative than previously presented LMI conditions since they are based on parameter dependent Lyapunov functions and multipliers.

The LMIs are then used in a combined evolutionary-LMI algorithm to design low-order fixed-structure robust or gain-scheduled controllers. The proposed algorithm involves genetic operators to span the solution space and LMI solvers to find the worst case performance.

The efficiency of algorithm is demonstrated by applying it to the problem of controlling the temperature profile of a large non-uniform plate.

REFERENCES

S. S. Chughtai and H. Werner. Synthesis of low-order controllers for discrete LPV systems using LMIs and evolutionary search. In ., Proceedings of ECC07, 2007.

R. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*. Springer-Verlag, New York, 1995. ISBN 0-387-94475-3.

R. D'Andrea and G. E. Dullerud. Distributed control design for spatially interconnected systems. *IEEE Transactions on Automatic Control*, 48(9), 2003.

A. Farag and H. Werner. A Riccati - genetic algorithms approach to fixed-structure controller synthesis. In *Proc. of the American Control Conference*, 2004.

T. Iwasaki and G. Shibata. LPV system analysis via quadratic separator for uncertain implicit systems. *IEEE Transactions on Automatic Control*, 46(8):1195–1208, 2001.

D. Siljak. *Large Scale Dynamic Systems*. system science and engineering. North-Holland, NewYork, USA, 1978.

F. Wu and S. E. Yildizoglu. Distributed paramter dependent modelling and control of flexible structures. *Transactions of the AMSE*, 127, 2005.