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Attackability in Games of Pursuit and Evasion with Antagonizing Players

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Abstract: For the games of pursuit evasion with antagonizing players (PEAP), the following three stages have been proposed in [Ge et al., 2007]: detection, attack and engagement, in which the roles of the two players are symmetric and each one is meant to search and attack its opponent. In the general framework established in our previous work, while the fundamental concepts, such as *detectability*, for the first stage have been laid before, this paper is dedicated to the second stage and to develop the associated fundamental concepts including the *attackability*, which describes whether one player (say P_1) could attack its opponent (P_2) before P_2 could see P_1, under the assumptions that (i) each player has a limited range vision zone and a limited range attack zone, and (ii) P_2 would follow a predefined trajectory and P_1 could choose its trajectory so as to attack P_2 since P_1 could see P_2 first. To demonstrate the concepts of *attackability* by detailed analysis, a simple yet typical planar PEAP game is discussed in this contribution, where two players are moving along two straight lines with constant speeds and each player has circular vision zone and attack zone. Sufficient and necessary conditions for all possible cases of attackability are given under several natural assumptions, which yields a complete analysis for the new concepts of attackability.

Keywords: pursuit-evasion game; attackability; antagonizing players; limited range vision zone; limited range attack zone.

1. INTRODUCTION

Pursuit-evasion games arise in numerous situations such as combat games [Isaacs, 1965], hide-and-seek game [LaValle, 2006], art gallery guarding [Rourke, 1987], etc. Pursuit-evasion games were initiated within framework of differential games by Isaacs in his classic work [Isaacs, 1965], where some typical examples including bomber and batter game, homicidal chauffeur game, princess and monster game, and cornered rat game are studied in a systematic manner. From then on, these games and other pursuit-evasion games (such as homicidal chauffeur game [Merz, 1971], princess and monster game [Fitzgerald, 1979], games of combat [Ardema et al., 1987], art gallery problem [Gal, 1979], the lady in the lake game [Basar and Olsder, 1982], lion and man game [Sgall, 2001], the obstacle tag game [Lewin, 1994], etc.) were extensively studied in wide literature. Besides the approach of differential games [Isaacs, 1965, Hájek, 1975, Alpern, 1974, Arkin et al., 1994, Yong, 1986, Zaremba, 1989], other formulations of pursuit-evasion games also emerged, for instance, pursuit-evasion games in graph [Parsons, 1976, Bienstock and Seymour, 1991, Yavin and Pachter, 1987, Mehlmann, 1998, Petrosjan, 1993, Lapaugh, 1993, Adler et al., 2002] take place in an environment defined by a graph; visibility-based pursuit-evasion games [Suzuki and Yamashita, 1992, LaValle et al., 1997, Guibas et al., 1999, LaValle and Hinrichsen, 2001, Lee et al., 2002, Tan, 2000, Gerkey et al., 2004] are characterized by searcher(s) equipped with thin "flashlights" which have unlimited range (but cannot see through the walls).

In [Ge et al., 2007], a class of games of pursuit and evasion with antagonizing players (PEAP) has been proposed, which are different from most previous work on pursuit-evasion games (see [Isaacs, 1965, Hájek, 1975, Gal, 1980, Yavin and Pachter, 1987, Mehlmann, 1998, Petrosjan, 1993, Basar and Olsder, 1982] and the references therein) in the following aspects: both players have possibilities to attack each other, but they may have different ability because there exist different limitations in their resources (such as sensors, radars, engines, etc.); each player has only limited range of view rather than previously investigated ideal cases in which the searcher(s) could see the evader within any distance. Such games are rooted in many practical problems, especially in security issues, thus study on them will be of importance in theory and in practice.

For PEAP games, roughly speaking, the whole process can normally involve three possible stages — S1: detecting, S2: attacking, and S3: engagement. As described before, basic foundations for PEAP games have been laid in our previous work, which also proposed and studied the the fundamental concepts of "detectability" together with other related concepts (including detection time, detectable area) for the detection stage (Stage S1), where players could not see their opponents until one player could detect its opponent; furthermore, to demonstrate the use of these concepts, we have given complete analysis of detectability in a typical planar PEAP game with only two players where both players have circular vision zones and follow two predefined straight lines until one player could be detected by its opponent.

Based on our previous work on the detection stage, this paper is dedicated to the attack stage (Stage S2) and to develop associated fundamental concepts, such as attack zone, attackability, etc. To demonstrate these concepts, the simple yet typical pla-

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nar PEAP game as in [Ge et al., 2007] is investigated in this paper to demonstrate new ideas for Stage S2. The main contributions of this paper are: (i) A concept of "attack zone" is proposed for further study of Stage S2; (ii) Basic concepts of "attackability" are mathematically defined for Stage S2 of a class of PEAP games; (iii) Complete analysis for attackability problem is given for Stage S2 of a simple yet typical two-player planar PEAP game, which classifies all possible cases of this stage by sufficient and necessary conditions.

The remainder of this paper is organized as follows. Section 2 provides preliminary notations and definitions used in this paper. Section 3 first introduces three possible stages in the process of PEAP games, and then together with several necessary assumptions for Stage S2, fundamental concepts of "attackability" and related concepts, such as attacking time, feasible attacking trajectory and attackable target set, are mathematically defined for Stage S2. Then in Section 4, to demonstrate the concepts proposed, for Stage S2 of a simple planar PEAP game, all possible cases of the attackability are classified with sufficient and necessary conditions given for each case, which are verified further by several simple computational examples, and consequently complete analysis for the proposed concepts is made. Finally some concluding remarks are given in Section 5.

2. PRELIMINARIES

We first recall the concept of *vision zone* introduced in [Ge et al., 2007]:

Definition 2.1. (Vision zone) The *vision zone* (or V-zone) of Player i at position $\boldsymbol{P}$, denoted by $\mathcal{V}_i(\boldsymbol{P})$, is the set of positions at which its opponent can be seen by this player, i.e. Player i can observe the position of its opponent.

Remark 2.1. The size of V-zone is usually determined by features of sensors, radars or etc. Complex environment may also affect the size or shape of V-zone.

The new concept, *attack zone* of a player at position $\boldsymbol{P}$, is defined as follows:

Definition 2.2. (Attack zone) The *attack zone* (or A-zone) of Player i at position $\boldsymbol{P}$, $\mathcal{A}_i(P)$, is the set of positions at which Player i's opponent can be attacked by Player i. By the word "attack", we mean that Player i's opponent will be out of action due to the attack of Player i when Player i's opponent enters $\mathcal{A}_i(P)$.

Remark 2.2. The reason why the attack zone exists for each player is usually because of physical limitations of players. For example, one player may have no time to avoid being attacked by another player when their distance is too small.

Example 2.1. Circular attack zone: As shown in Figure 1, this type of attack zone is an area surrounded by a circle centered at $\boldsymbol{P}_i$ with fixed radius r_i.

3. PROBLEM FORMULATION

In [Ge et al., 2007], it is mentioned that for a practical two-person pursuit-evasion game, generally speaking, the game may be normally divided into three stages:

S1: Initially no player can observe the opponent, but the players search their opponents until at least one player can observe the opponent.

S2: In this stage, the player (say A) who observes the opponent (say B) first can take the initiative and launch attack on its opponent before its opponent can counterattack.

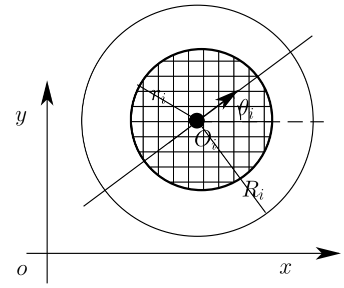

Fig. 1. Circular attack zone of Player i (characterized by radius r_i). It is a subset of Player i's vision zone (characterized by radius R_i).

S3: In this stage, players from both sides could see each other and try their best to attack the other by choosing its own actions (including trajectory, velocity, etc.).

As in [Ge et al., 2007], PEAP games with only two players (Player 1 and Player 2) are denoted by $\text{PEAP}_{(1,1)}$. For Stage S1 of $\text{PEAP}_{(1,1)}$, fundamental concepts of "detectability" have been proposed and studied in our previous work. In this paper, based on the foundations established before, we shall dedicate to Stage S2 of $\text{PEAP}_{(1,1)}$ and develop associate fundamental concepts and basic problems for Stage S2.

The following assumptions are made throughout this paper:

Assumption 3.1. At any time, each player has a limited V-zone and a limited A-zone which is a subset of its V-zone.

Assumption 3.2. Initially Player 2 lies in the V-zone of Player 1 (yet outside of the A-zone of Player 1), but Player 1 does not lie in the V-zone of Player 2.

Assumption 3.3. Player 2 has a predefined fixed trajectory γ_2, and it will follow γ_2 exactly until Player 1 enters its V-zone.

Assumption 3.4. Player 1 can choose its trajectory γ_1 from a set Γ_1 of admissible trajectories so as to attack Player 2 as effectively as possibly.

Remark 3.1. Assumptions 3.2—3.4 are different from those in [Ge et al., 2007] because Stage S2 and Stage S1 have different characteristics and correspondingly different problems should be studied in these two stages. Assumption 3.1 is natural in sense that a player can try to attack its opponent only if it can see its opponent. Assumption 3.2 states the relationship between the initial positions of two players. In Assumption 3.3, only Player 2 will follow a predefined fixed trajectory since it cannot see Player 1 during the Stage S2. In Assumption 3.4, we should remark that Player 1 can only choose its trajectory within some constraints (described by the set Γ_1), e.g. it can choose its heading direction but its speed should not exceed a certain constant.

For convenience of discussion, we first present some basic definitions for the pursuit-evasion games in $\mathcal{R}^n$. As in [Ge et al., 2007], let $\boldsymbol{P}_i(t) \in \mathcal{R}^n$ and $\boldsymbol{V}_i(t) = \frac{d}{dt}\boldsymbol{P}_i(t) \in \mathcal{R}^n$ be the position vector and the velocity vector of Player i at time t, respectively. In the planar case, i.e. $n = 2$, let $v_i(t)$ and $\theta_i(t)$ denote the speed and the heading direction of Player i at time t, respectively, then

$$\boldsymbol{V}_i(t) = [v_i(t)\cos\theta_i(t), v_i(t)\sin\theta_i(t)]^{\tau}. \qquad (3.1)$$

Without loss of generality, we take the initial time instant of Stage S2 as $t_0 = 0$. In later parts, $\mathcal{R}[2\pi]$ denotes the quotient space of $\mathcal{R}$ over $[0, 2\pi)$, i.e. $x \sim x + 2k\pi$ for any integer k; $\arctan(x, y)$ denotes the arctangent angle in the correct quadrant determined by the coordination (x, y).

Definition 3.1. (Attackability in PEAP$_{(1,1)}$) If Player 1 can choose a trajectory $\gamma_1 \in \Gamma_1$ and there exists a time instant $T > 0$ such that

(i) $\boldsymbol{P}_2(T) \in \mathcal{A}_1(\boldsymbol{P}_1(T))$, and
(ii) $\boldsymbol{P}_2(t) \notin \mathcal{A}_1(\boldsymbol{P}_1(t))$ and $\boldsymbol{P}_1(t) \notin \mathcal{V}_2(\boldsymbol{P}_2(t))$, for all $t \in [0, T)$

then we say that *Player 1 is able to attack Player 2*. The trajectory γ_1 is called a *feasible attacking trajectory of Player 1*, and the corresponding $T = T(\gamma_1)$ is called the *attacking time of Player 1 w.r.t. Player 2*.

Remark 3.2. For planar PEAP games ($n = 2$), when Γ_1 is a family of straight lines, each trajectory $\gamma \in \Gamma_1$ is characterized by its heading direction. In this case, the heading direction of *feasible attacking trajectory* is called *feasible attacking direction*.

Definition 3.2. If for any trajectory of Player 1, for any time $t > 0$, $\boldsymbol{P}_2(t) \notin \mathcal{A}_1(\boldsymbol{P}_1(t))$ or $\boldsymbol{P}_1(t) \in \mathcal{V}_2(\boldsymbol{P}_2(t))$, then we say that *Player 1 is unable to attack Player 2*.

When Player 1 is unable to attack Player 2, there are two possible cases for each trajectory γ_1 of Player 1:

(a) $\boldsymbol{P}_2(t) \notin \mathcal{A}_1(\boldsymbol{P}_1(t))$ and $\boldsymbol{P}_1(t) \notin \mathcal{V}_2(\boldsymbol{P}_2(t))$ for any time $t > 0$;
(b) Otherwise, we must have $T'(\gamma_1) < \infty$, where

$$T'(\gamma_1) \triangleq \inf\{t > 0 : \boldsymbol{P}_1(t) \in \mathcal{V}_2(\boldsymbol{P}_2(t))\}.$$

In case (a), we denote $T'(\gamma_1) = \infty$. Intuitively, $T'(\gamma_1)$ is the time when Player 1 enters the vision zone of Player 2.

Definition 3.3. If for any trajectory γ_1 of Player 1, $T'(\gamma_1) = \infty$, then we say that *Player 1 is completely unable to attack Player 2*.

Definition 3.4. If there exists a trajectory $\gamma_1 \in \Gamma_1$ of Player 1 such that $T'(\gamma_1) < \infty$, then we say that *Player 1 fails in attacking Player 2 (along trajectory γ_1)*. In this case, γ_1 is called an *unsuccessful attacking trajectory* of Player 1.

Remark 3.3. For planar pursuit-evasion games ($n = 2$), when Γ_1 is a family of straight lines, the heading direction of *unsuccessful attacking trajectory* is called *unsuccessful attacking direction*.

In Definitions 3.1, 3.2, 3.3 and 3.4, generally speaking, the attackability of Player 1 may depend on the trajectory of Player 2. Hence we formulate the following concept:

Definition 3.5. (Attackable target set) The *attackable target set $\mathcal{T}_1$ of Player 1 (w.r.t. Player 2)* is the set of trajectories of Player 2 such that Player 1 is able to attack Player 2.

Remark 3.4. Given initial positions of two players, the set $\mathcal{T}_1$ is determined by admissible trajectories of two players and the parameters of game. The intrinsic relation between them is demonstrated later for a simple planar pursuit-evasion game and correspondingly $\mathcal{T}_1$ becomes a set of heading directions of Player 2.

4. COMPLETE ANALYSIS ON ATTACKABILITY

In this section, to demonstrate the concepts of attackability proposed in last section, we shall give complete analysis on attackability for one simple yet typical planar PEAP game with only two players, whose detection stage (Stage S1) has been analyzed in [Ge et al., 2007]. In this game, two players can move on the plane along two straight lines. Each player has a circular V-zone with radius R_i and a circular A-zone with radius r_i. For $i = 1, 2$, let $[x_i(0), y_i(0)]^\tau$, θ_i, v_i denote the initial position, moving direction and speed of Player i, respectively. In this case, $\boldsymbol{V}_i(t) = [v_i \cos\theta_i, v_i \sin\theta_i]^\tau$ and $\boldsymbol{P}_i(t) = [x_i(t), y_i(t)]^\tau \in \mathcal{R}^2$ is given by

$$\gamma_i : \begin{cases} x_i(t) = x_i(0) + v_i t \cos\theta_i, \\ y_i(t) = y_i(0) + v_i t \sin\theta_i. \end{cases}$$

By the assumptions given before, Player 2's trajectory is fixed and Player 1 can choose its trajectory, that is to say, θ_2 is fixed and θ_1 can be chosen by Player 1. Here v_1, v_2, R_1, R_2, r_1, r_2 and θ_2 are given constants. The set Γ_1 of admissible trajectories of Player 1 is

$$\Gamma_1 = \{\gamma_1 : \gamma_1 = \gamma_1(\theta_1; v_1, x_1(0), y_1(0)), \quad \theta_1 \in \mathcal{R}[2\pi]\}.$$

In this section, our main theorem stating sufficient and necessary conditions for detectability will be given first, and later a theorem classifying all possible outcomes of Stage S2 will be presented. Then, by using these theorems, several simple examples will be illustrated and discussed. And in the last subsection, we shall present a theorem on attackable target set and give some related discussions.

4.1 Sufficient and Necessary Conditions

Theorem 4.1. Let $I_0 \subseteq \mathcal{R}[2\pi]$, $I_1 \subseteq \mathcal{R}[2\pi]$ be defined as

$$I_1 \triangleq \begin{cases} \mathcal{R}[2\pi] & v_1 > v_2 \\ [\theta_2 + \pi - \arcsin\dfrac{v1}{v2}, \theta_2 + \pi + \arcsin\dfrac{v_1}{v_2}] & v_1 < v_2 \\ (\theta_2 + \dfrac{\pi}{2}, \theta_2 + \dfrac{3\pi}{2}) & v_1 = v_2 \end{cases} \tag{4.1}$$

$$I_0 \triangleq [\alpha_0 - \delta_0, \alpha_0 + \delta_0] \tag{4.2}$$

where

$$\begin{aligned} \alpha_0 &= \arctan(x_2(0) - x_1(0), y_2(0) - y_1(0)) \\ \delta_0 &= \arcsin\frac{r_1}{d_0} \\ d_0 &= \sqrt{[x_2(0) - x_1(0)]^2 + [y_2(0) - y_1(0)]^2} \\ v_R(\theta_1, \theta_2, v_1, v_2) &= \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos(\theta_1 - \theta_2)} \\ \gamma_R(\theta_1, \theta_2, v_1, v_2) &= \arctan(v_1\cos\theta_1 - v_2\cos\theta_2, \\ &\qquad v_1\sin\theta_1 - v_2\sin\theta_2) \end{aligned} \tag{4.3}$$

Then the following results can be obtained:

(a) Player 1 is able to attack Player 2 if and only if $R_1 > r_1 \geq R_2 > r_2$ and $I_0 \cap I_1 \neq \emptyset$.
(b) Under conditions of (a), any $\theta_1 \in \Lambda_1$ is a feasible heading direction of Player 1, where

$$\Lambda_1 \triangleq \{\theta_1 \in \mathcal{R}[2\pi] : \gamma_R(\theta_1, \theta_2, v_1, v_2) \in I_0 \cap I_1\} \tag{4.4}$$

(c) Under conditions of (a), for any $\theta_1 \in \Lambda_1$, the corresponding attacking time T is

$$T = \frac{d_0\cos\delta_1 - \sqrt{r_1^2 - d_0^2\sin^2\delta_1}}{v_R} \tag{4.5}$$

where

$$\delta_1 \triangleq \gamma_R(\theta_1, \theta_2, v_1, v_2) - \alpha_0.$$

(d) The relative speed v_R can be also given by

$$v_R = a(\alpha; \theta_2, v_1, v_2) \tag{4.6}$$

where $\alpha = v_R(\theta_1, \theta_2, v_1, v_2)$ and

$$a(\alpha; \theta_2, v_1, v_2) \triangleq \begin{cases} -v_2\cos(\alpha - \theta_2) + \nu(v_1, v_2, \alpha, \theta_2) \\ \qquad \text{in case of (i)} \\ -v_2\cos(\alpha - \theta_2) \pm \nu(v_1, v_2, \alpha, \theta_2) \\ \qquad \text{in case of (ii)} \\ -2v_2\cos(\alpha - \theta_2) \\ \qquad \text{in case of (iii)} \end{cases} \tag{4.7}$$

with

$$\nu(v_1, v_2, \alpha, \theta_2) \triangleq \sqrt{v_1^2 - v_2^2 \sin^2(\alpha - \theta_2)}.$$

Consequently, T can be explicitly expressed as a function of $\alpha \in I_0 \cap I_1, \theta_2, d_0, r_1, v_1, v_2$ without solving θ_1 first ($\alpha \in I_0 \cap I_1$):

$$T = \begin{cases} \dfrac{d_0 \cos(\alpha - \alpha_0) - \sqrt{r_1^2 - d_0^2 \sin^2(\alpha - \alpha_0)}}{-v_2 \cos(\alpha - \theta_2) + \sqrt{v_1^2 - v_2^2 \sin^2(\alpha - \theta_2)}} \\ \quad \text{if } v_1 > v_2 \\ \dfrac{d_0 \cos(\alpha - \alpha_0) - \sqrt{r_1^2 - d_0^2 \sin^2(\alpha - \alpha_0)}}{-v_2 \cos(\alpha - \theta_2) \pm \sqrt{v_1^2 - v_2^2 \sin^2(\alpha - \theta_2)}} \\ \quad \text{if } v_1 < v_2 \\ \dfrac{d_0 \cos(\alpha - \alpha_0) - \sqrt{r_1^2 - d_0^2 \sin^2(\alpha - \alpha_0)}}{-2v_2 \cos(\alpha - \theta_2)} \\ \quad \text{if } v_1 = v_2 \end{cases}$$

Proof: See [Ma et al.].

Remark 4.1. Necessity of condition $R_1 > r_1 \geq R_2 > r_2$ is obvious for attackability.

Based on the proof of Theorem 4.1, we obtain the following results, which classify all possible cases in Stage S2:

Theorem 4.2. Let I_0 and I_1 be defined as in Theorem 4.1. Define

$$I'_0 \triangleq [\alpha_0 - \delta'_0, \alpha_0 + \delta'_0]$$

where

$$\delta'_0 \triangleq \arcsin \frac{\max(r_1, R_2)}{d_0}$$

and α_0, d_0 are defined in Eq. (4.3). Then the attackability is completely characterized as follows:

(i) Player 1 is able to attack Player 2 if and only if $R_1 > r_1 \geq R_2 > r_2$ and $I_0 \cap I_1 \neq \emptyset$.
(ii) Player 1 is completely unable to attack Player 2 if and only if $I'_0 \cap I_1 = \emptyset$.
(iii) Player 1 fails in attacking Player 2 if and only if $r_1 < R_2$ and $I'_0 \cap I_1 \neq \emptyset$. In this case, the set Δ_1 of unsuccessful attacking directions of Player 1 is

$$\Delta_1 \triangleq \{\theta_1 \in \mathcal{R}[2\pi] : \gamma_R(\theta_1, \theta_2, v_1, v_2) \in I'_0 \cap I_1\} \tag{4.8}$$

Proof: See [Ma et al.].

By Theorem 4.2, whether Player 1 is completely unable to attack Player 2 is determined by the set $I'_0 \cap I_1$, and whether Player 1 is able to attack Player 2 or fails in attacking Player 2 is determined by the relationship between r_1 and R_2.

4.2 Computational Examples

Now we discuss several special cases by using Theorem 4.1.

Example 4.1. In this example, we consider case of $v_1 > v_2 = 0$, i.e. Player 2 does not move at all. For this example, we get $v_R = v_1$, $\gamma_R = \theta_1$, and

$$I_1 = \mathcal{R}[2\pi]$$
$$I_0 \cap I_1 = I_0 = [\alpha_0 - \delta_0, \alpha_0 + \delta_0]$$
$$I'_0 \cap I_1 = I'_0 = [\alpha_0 - \delta'_0, \alpha_0 + \delta'_0]$$

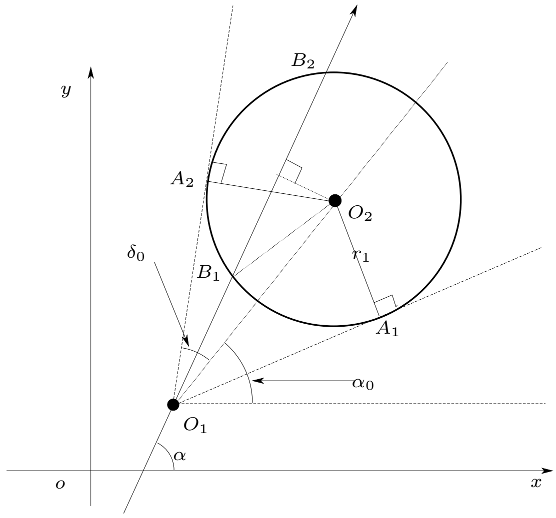

Fig. 2. Player 2 does not move — $v_2 = 0$. In this example $\alpha = \theta_1$. This figure depicts case of $r_1 > R_2$. The directions between O_1A_1 and O_1A_2 are feasible heading directions of Player 1. When Player 1 moves to B_1, Player 2 enters the A-zone of Player 1. Note that $O_2A_1 = O_2A_2 = O_2B_1 = r_1$. The V-zones and the A-zones of both players are not drawn here for the sake of simplicity.

(i) When $r_1 \geq R_2$, by Theorem 4.1, Player 1 is able to attack Player 2 since $I_0 \cap I_1 \neq \emptyset$ and $r_1 \geq R_2$. In this case, by Eq. (4.4), we easily obtain that

$$\Lambda_1 = \{\theta_1 : \theta_1 \in I_0 \cap I_1\} = I_0$$

therefore any $\theta_1 \in [\alpha_0 - \delta_0, \alpha_0 + \delta_0]$ is a feasible heading direction of Player 1, which is consistent with intuitive knowledge.

(ii) When $r_1 < R_2$, by Theorem 4.1, Player 1 fails in attacking Player 2 since $I'_0 \cap I_1 \neq \emptyset$ and $r_1 < R_2$. And obviously the set Δ_1 of unsuccessful directions of Player 1 is

$$\Lambda_1 = \{\theta_1 : \theta_1 \in I'_0 \cap I_1\} = I'_0.$$

Example 4.2. In this example, we consider the case of $v_1 = v_2$, i.e. neither Player 1 nor Player 2 has superior speed to the other one. We take $x_1(0) = 0$, $y_1(0) = 0$, $x_2(0) = 6$, $y_2(0) = 6$, $R_1 = 10$, $r_1 = 3\sqrt{2}$, $R_2 = 4$, $r_2 = 3$, $v_1 = v_2 = 1$. By Theorem 4.1, we obtain that $d_0 = 6\sqrt{2}, \alpha_0 = \frac{\pi}{4}, \delta_0 = \frac{\pi}{6}$, and $I_0 = [\alpha_0 - \delta_0, \alpha_0 + \delta_0] = [\frac{\pi}{12}, \frac{5\pi}{12}]$, $I_1 = (\theta_2 + \frac{\pi}{2}, \theta_2 + \frac{3\pi}{2})$. Consequently

$$I_0 \cap I_1 = \begin{cases} \emptyset & \theta_2 \in [0, \frac{7\pi}{12}) \cup [\frac{23\pi}{12}, 2\pi) \\ [\frac{\pi}{12}, \theta_2 - \frac{\pi}{2}) & \theta_2 \in [\frac{7\pi}{12}, \frac{11\pi}{12}) \\ [\frac{\pi}{12}, \frac{5\pi}{12}) & \theta_2 = \frac{11\pi}{12} \\ [\frac{\pi}{12}, \frac{5\pi}{12}] & \theta_2 \in (\frac{11\pi}{12}, \frac{19\pi}{12}) \\ (\frac{\pi}{12}, \frac{5\pi}{12}] & \theta_2 = \frac{19\pi}{12} \\ (\theta_2 - \frac{3\pi}{12}, \frac{5\pi}{12}] & \theta_2 \in (\frac{19\pi}{12}, \frac{23\pi}{12}) \end{cases}$$

Thus by Theorem 4.1, when $\theta_2 \in [\frac{7\pi}{12}, \frac{23\pi}{12})$, Player 1 is able to attack Player 2, and the set Λ_1 of feasible attacking directions of Player 1 is given by

$$\Lambda_1 \triangleq \{\theta_1 \in \mathcal{R}[2\pi] : \Delta(\theta_1, \theta_2) \in I_0 \cap I_1\}$$

where $\Delta(\theta_1, \theta_2) \stackrel{\Delta}{=} \arctan(\cos\theta_1 - \cos\theta_2, \sin\theta_1 - \sin\theta_2)$. Now we validate this result by investigating two typical values of θ_2.

(i) Take $\theta_2 = 0$. In this case, by the discussion above and Theorem 4.2, we know that Player 1 is completely unable to attack Player 2. In fact, obviously $x_1(t) = t\cos\theta_1$, $y_1(t) = t\sin\theta_1$, $x_2(t) = 6 + t$, $y_2(t) = 6$. Let

$$\begin{aligned} d(t) &\stackrel{\Delta}{=} [x_1(t) - x_2(t)]^2 + [y_1(t) - y_2(t)]^2 \\ &= (2 - 2\cos\theta_1)t^2 - 12(\cos\theta_1 + \sin\theta_1 - 1)t + 72. \end{aligned}$$

If $\theta_1 = 0$, we get $d(t) = 72$ for any $t \geq 0$; otherwise, for any $t \geq 0$, we obtain that

$$\begin{aligned} d(t) &\geq 2(1 - \cos\theta_1)t^2 - 12(\sin\theta_1)t + 72 \\ &= 2(1 - \cos\theta_1)[t - \frac{3\sin\theta_1}{1 - \cos\theta_1}]^2 + 72 - \frac{18\sin^2\theta_1}{1 - \cos\theta_1} \\ &\geq 72 - 18(1 + \cos\theta_1) \\ &\geq 36. \end{aligned} \tag{4.9}$$

Thus we always have $d(t) \geq 36 > r_1^2$, which means Player 2 cannot enter Player 1's A-zone no matter which heading direction Player 1 chooses.

(ii) Take $\theta_2 = \pi$. In this case, $I_0 \cap I_1 = [\frac{\pi}{12}, \frac{5\pi}{12}]$. For any $\alpha \in I_0 \cap I_1$, we get

$$\begin{aligned} \alpha &= \arctan(\cos\theta_1 + 1, \sin\theta_1) \\ &= \arctan(2\cos^2\frac{\theta_1}{2}, 2\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}) \end{aligned}$$

consequently $\theta_1 = 2\alpha$ is the feasible heading direction of Player 1. Therefore $\Lambda_1 = [\frac{\pi}{6}, \frac{5\pi}{6}]$ is the set of feasible heading directions of Player 1. Furthermore, the speed v_R of Player 1 relative to Player 2 is

$$\begin{aligned} v_R &= \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos(\theta_1 - \theta_2)} \\ &= \sqrt{2 - 2\cos(2\alpha - \pi)} \\ &= 2\cos\alpha \end{aligned}$$

and the attacking time corresponding to $\theta_1 \in \Lambda_1$ is

$$\begin{aligned} T &= \frac{d_0\cos\delta_1 - \sqrt{r_1^2 - d_0^2\sin^2\delta_1}}{v_R} \\ &= \frac{6\sqrt{2}\cos(\alpha - \frac{\pi}{4}) - \sqrt{(3\sqrt{2})^2 - (6\sqrt{2})^2\sin^2(\alpha - \frac{\pi}{4})}}{2\cos\alpha} \\ &= \frac{3\sqrt{2}[\cos(\alpha - \frac{\pi}{4}) - \sqrt{\frac{1}{4} - \sin^2(\alpha - \frac{\pi}{4})}]}{\cos\alpha}. \end{aligned}$$

When $\alpha = \frac{\pi}{4}$, we get $T = 3$; when $\alpha = \frac{\pi}{12}$, we get $T = \frac{3\sqrt{6}}{2\cos\frac{\pi}{12}} \approx 3.8038$; when $\alpha = \frac{5\pi}{12}$, we get $T = \frac{3\sqrt{6}}{2\cos\frac{5\pi}{12}} \approx 14.1962$. We only check $T = 3$ is true when $\alpha = \frac{\pi}{4}$: in fact we get $\theta_1 = 2\alpha = \frac{\pi}{2}$, consequently $x_1(t) = 0$, $y_1(t) = t$, $x_2(t) = 6 - t$, $y_2(t) = 6$, thus at time $T = 3$, the distance between two players is

$$\sqrt{(0-3)^2 + (3-6)^2} = 3\sqrt{2} = r_1.$$

The graph of T w.r.t. α is plotted in Fig. 3. From this graph, we can see that it is a nontrivial work to find out the optimal attacking time of Player 1. In this case, by numerical methods, we know that optimal attacking time of Player 1 is $T \approx 2.6360$ which can be reached when $\alpha \approx 0.5299$(rad), correspondingly $\theta_1 \approx 1.0598$(rad). In this example, from Fig. 3, we can clearly see that the attacking time $T(\alpha)$ will increase quickly when the heading direction $\theta_1 = 2\alpha_1$ of Player 1 tends to $\frac{5}{6}\pi$, hence in practice Player 1 should not choose such direction so as to avoid unnecessary long attacking time. Therefore, it is important for Player 1 to make calculations before launching its attack to its opponent.

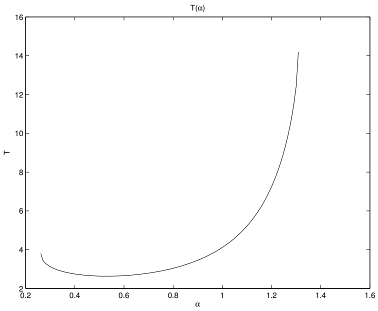

Fig. 3. Attacking time $T(\alpha)$ of Player 1 in Example 4.2(ii). The x-axis is α, and the y-axis is $T(\alpha)$.

4.3 Attackable Target Set

Based on Theorem 4.1, we can explicitly work out the attackable target set, $\mathcal{T}_1 \subseteq \mathcal{R}[2\pi]$, of Player 1 (w.r.t. Player 2), which is given in the following theorem:

Theorem 4.3. Given v_i, R_i, r_i and $(x_i(0), y_i(0))$ for $i = 1, 2$, when $R_1 > r_1 \geq R_2 > r_2$, the attackable target set can be calculated from

$$\mathcal{T}_1 \stackrel{\Delta}{=} \begin{cases} \mathcal{R}[2\pi] & \text{if } v_1 > v_2 \\ (\alpha_0 - \delta_0 - \frac{3\pi}{2}, \alpha_0 + \delta_0 - \frac{\pi}{2}) & \text{if } v_1 = v_2 \\ [\alpha_0 - \delta_0 - \arcsin\frac{v_1}{v_2} - \pi, \alpha_0 + \delta_0 + \arcsin\frac{v_1}{v_2} - \pi] & \text{if } v_1 < v_2 \end{cases} \tag{4.10}$$

where α_0,δ_0 and d_0 are defined in Eq. (4.3).

Proof: See [Ma et al.].

From Theorem 4.3, we can draw the following conclusions:

- If $v_1 > v_2$, Player 1 is always able to attack Player 2 no matter which direction Player 2 is moving in.
- If $v_1 < v_2$, though Player 2's speed is larger than Player 1, it is still possible for Player 1 to attack Player 2 since Player 1's attack zone is even larger than Player 2's vision zone. In this case, whether Player 1 is able to attack Player 2 depends on whether the heading direction θ_2 of Player 2 is in $\mathcal{T}_1$. By Eq. (4.10), the larger the ratio $\frac{v_1}{v_2}$ is, the bigger the size of $\mathcal{T}_1$ is.
- The case of $v_1 = v_2$ is similar to the case of $v_1 < v_2$, and the attackable target set in former case is larger than that in latter case. By taking $v_1 \to v_2$, since $\arcsin\frac{v_1}{v_2} \to \frac{\pi}{2}$, we know that the results in these cases are consistent.

By Eq. (4.10), we know also that when $v_1 \leq v_2$, the size of $\mathcal{T}_1$ depends on δ_0. In fact in this case, the larger the ratio $\frac{r_1}{d_0}$ is, the bigger the set $\mathcal{T}_1$ is. Consequently, the closer the initial distance is, the bigger attackable target set Player 1 has. When $v_1 = v_2$, the size of $\mathcal{T}_1$ is

$$(\alpha_0 + \delta_0 - \frac{\pi}{2}) - (\alpha_0 - \delta_0 - \frac{3\pi}{2}) = 2\delta_0 + \pi$$

which tends to 2π as $d_0 \to r_1$. All these discussions above are consistent with our intuitive knowledge.

Corollary 4.1. In the game studied, if Player 1 can choose its speed under the constraint $v_1 \leq c_1$, then c_1 is the optimal speed of Player 1 which maximizes $\mathcal{T}_1$:

$$\mathcal{T}_1(v_1) \subseteq \mathcal{T}_1(c_1), \forall v_1 \leq c_1.$$

5. CONCLUSION

Based on observations of some practical pursuit-evasion games, new games of pursuit evasion with antagonizing players (PEAP), which may normally be divided into three different stages, were proposed in [Ge et al., 2007]. In such games, two players have symmetric roles, but because they may have different limitations in resources (sensors, weapons, etc.), their abilities in detecting or attacking their opponent are different, which influences the process of the whole game.

In the framework established in our previous work, based on fundamental concepts of *detectability* proposed for the detection stage (Stage S1), this paper was dedicated to the attack stage (Stage S2) further and we have developed the associated fundamental concepts including *attackability* and related concepts, such as attack zone, feasible attacking trajectory, and attacking time. Based on these concepts, we have formulated basic problems for Stage S2 mathematically, under assumptions that (i) each player has a limited range vision zone and a limited range attack zone, and (ii) one player (Player 1) has detected its opponent (Player 2) initially and Player 1 could choose its trajectory from a family of predefined trajectories so as to attack Player 2 as effectively as possible.

To demonstrate the concepts proposed, in this paper, we have given detailed analysis for the attack stage of a simple yet typical planar PEAP game with two players, where both players are moving along two straight lines with constant speeds and each player has a limited circular vision zone and a limited circular attack zone. Under several natural assumptions, sufficient and necessary conditions have been given to classify all possible outcome of the attack stage, and consequently a complete analysis for the new concepts of attackability proposed have been made. Furthermore, several computational examples have verified the validity of our results. More challenging study on much complex PEAP games will be conducted in our future work.

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