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Discrete-Time $\mathcal{H}_\infty$ Gaussian Filter ⋆

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Abstract:
A discrete-time signal estimator for systems subject to both white noise and bounded-power disturbance signals is developed. Sufficient and necessary conditions for the robust optimal filter are proved and the resulting filter gain is characterized by a set of two cross-coupled Riccati equations.

1. INTRODUCTION

It is generally known that signal estimation for the dynamic systems is one of the most important problems in engineering applications [Anderson, 1979, Petersen and Savkin, 1999]. The popular Kalman filter[Anderson, 1989], also known as $\mathcal{H}_2$ filter, is an optimal design that is based on the stochastic noise model with known power spectral densities. However, this technique may be very sensitive to changes in system parameters or other disturbances with unknown spectral densities. For such cases, a better choice is to use an $\mathcal{H}_\infty$ filter, which is developed specifically to address model uncertainty [Fu et al., 1992, Nagpal et al., 1991], and different techniques have been well developed and applied for different systems (see for example [Gao and Wang, 2003, Li and Fu, 1997, Xu and Chen, 2002] and the references therein). Although $\mathcal{H}_\infty$ filter usually provides much better robustness than $\mathcal{H}_2$ filter, it, in general, does not guarantee desired optimal performance if it is applied to a system affected by stochastic noise. Clearly, a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter design scheme that can combine the strengths of these two estimation methods in a systematic way is highly desirable.

Several methods have been proposed to carry out the robust optimal filter design and a few examples are given here for different approaches to this problem. One method is to convert the problem to an auxiliary minimization one [Bernstein et al., 1989]. In this approach, an $\mathcal{H}_2$ estimation error is minimized which is subject to a prespecified $\mathcal{H}_\infty$ constraint. This constraint is introduced in the optimization process by a Riccati equation whose solution leads to an upper bound on the $\mathcal{H}_2$ error variance. In [Rotstein et al., 1996], [Halder et al., 1997] and [Khargonekar et al., 1996], the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters are obtained using convex programming characterization. For systems with norm-bounded parameter uncertainties, the problem is solved in [Wang et al., 2000] and [Wang et al., 1999] by using Riccati-like equations, where the transfer function from the noise inputs to error state outputs meets an $\mathcal{H}_\infty$-norm upper bound constraint. For discrete-time polytopic systems, [Palhares et al., 2001] obtains the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters by solving a set of linear matrix inequalities (LMIs), while [Gao et al., 2005] uses the parameter-dependent stability idea and finds a filter that depends on the parameters, which are assumed to reside in a polytope and be measurable online. A time domain game theoretic approach is proposed in [Theodor et al., 1996] which improves the $\mathcal{H}_2$ performance of the central $\mathcal{H}_\infty$ filter while satisfying the required $\mathcal{H}_\infty$ performance. For systems with polytope-bounded uncertainty, a pole-placement design strategy is proposed in [Goncalves et al., 2006] which utilizes an optimization algorithm in the space of filter parameters.

In [Chen and Zhou, 2002], utilizing the game approach, a new formulation called '$\mathcal{H}_\infty$ Gaussian filter' is proposed, and it is shown that the robust optimal filter can be obtained by solving a set of cross-coupled Riccati equations. The result is a Kalman-type filter for uncertain plants and is characterized by the choice of the disturbance attenuation level γ. One advantage of this approach is that optimal state estimation is achieved at the presence of the worst case model uncertainty. Therefore, it clearly reflects the trade-off between the inherently conflicting $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances.

Motivated by the approach in [Chen and Zhou, 2002], in this paper, the Nash game methodology is adopted to derive a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter in discrete time. The design is based on a constrained optimization problem and is characterized by two cross-coupled Riccati equations. As it can be seen, obtaining the discrete-time counterpart of the continuous procedure is not so straightforward. An optimal filter gain is characterized by an equation consisting of the plant parameters and the solutions to the Riccati equations.

The remaining part of this paper is organized as follows: Section II presents some definitions and preliminary results required for obtaining the main solution; in Section III, the problem formulation and the design of the discrete-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter is provided; the conclusion can be found in Section IV.

Notations: Throughout this work, if a is a vector and A is a matrix of arbitrary dimensions, a^T and A^T represent their transposes, respectively. We use the notation $\partial\mathbb{D} := \{z : |z| = 1\}$ to describe the points on the unit circle in the complex plane. Furthermore, when working with a discrete time signal $u_{(k)}$ (scalar- or vector-valued), for simplicity of presentation, we denote $\delta u := u_{(k+1)}$, and then the time variable k will be omitted. $\|\cdot\|$ represents the Euclidian norm of a vector.

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2. PRELIMINARIES

Definition 1. (Bounded power signal). Consider the given discrete-time real vector stochastic signal $u_{(k)}$:

$$u_{(k)} = [u_{1(k)} \quad u_{2(k)} \quad \cdots \quad u_{m(k)}]^T \in \mathbb{R}^m \;\; \forall k \in \mathbb{Z}\,,$$

where $u_{i(k)}$, $i = 1, \ldots, m$ are real stationary discrete random processes, we define the *mean* and *autocorrelation* matrices, respectively, as:

$$E\{u\} := [E\{u_{1(k)}\} \quad E\{u_{2(k)}\} \quad \cdots \quad E\{u_{m(k)}\}]^T\,,$$

$$R_{uu(n)} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{u_{(k+n)} u_{(k)}^T\}\,.$$

where n is an integer. The Fourier transform of $R_{uu(n)}$, or the *power spectral density* of $u_{(k)}$, is:

$$S_{uu} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} R_{uu(k)} e^{-j\omega k}\,.$$

A stationary stochastic vector signal is said to have *bounded power* if:

- both R_{uu} and S_{uu} exist;
- $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\| u_{(k)} \|_2^2\} < \infty$.

Definition 2. ($\mathcal{P}$-norm). Let $\mathcal{P}$ be the set of all signals with bounded power, we define the seminorm:

$$\| u \|_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\| u_{(k)} \|^2\}\,. \tag{1}$$

Definition 3. (Mutually uncorrelated signals). Two stochastic vector signals u_1 and u_2 are said to be mutually uncorrelated if:

$$E\{u_{1(k_1)} u_{2(k_2)}^T\} = 0, \quad \forall k_1, k_2 \in \mathbb{Z}\,.$$

Constrained Optimization

The constrained optimization problem presented in this section plays an important role in the main derivations of this work.

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times r}$ and $R = DD^T > 0$, let L be any consistent matrix such that $A + LC$ is stable in discrete time sense and define the index function:

$$J(L) = \text{trace}(QPQ^T)\,, \tag{2}$$

where Q is any constant weighting matrix, and $P = P^T \geq 0$ satisfies:

$$P = (A + LC)P(A + LC)^T + (B + LD)(B + LD)^T\,. \tag{3}$$

The constrained optimization problem is stated as follows: *find* $(L_*\,, P_*)$ *where* $A + L_*C$ *is stable, such that* $J(L)$ *is minimized at* L_**, i.e.:*

$$\min_L J(L) = \min_P \textit{trace}(QPQ^T) = \textit{trace}(QP_*Q^T)\,,$$

where (L, P) *and* (L_*, P_*) *are all subject to constraint (3).*

The following theorem presents the solution to this problem. Only the proof of the sufficiency part is given here. The necessity part of the proof can also be carried out, however, it consists of more derivations that could not be included due to page limit.

Theorem 4. For the constrained optimization problem stated above, suppose (C, A) is detectable. If there is a solution $P_* \geq 0$ for:

$$P_* = AP_*A^T - (BD^T + AP_*C^T)(R + CP_*C^T)^{-1} \\ .(DB^T + CP_*A^T) + BB^T\,, \tag{4}$$

i.e., $A - (AP_*C^T + BD^T)(R + CP_*C^T)^{-1}C$ is stable, where $R + CP_*C^T > 0$, then $J(L)$ achieves the minimum value at:

$$L_* = -(AP_*C^T + BD^T)(R + CP_*C^T)^{-1}\,.$$

Conversely, let (C, A) be detectable. If there are L_1 and $P_1 \geq 0$, where $A + L_1C$ is stable and P_1 solves:

$$P_1 = (A + L_1C)P_1(A + L_1C)^T + (B + L_1D)(B + L_1D)^T\,,$$

such that $J(L)$ is minimized at L_1, then there is a $P_* \geq 0$ solving (4), where $R + CP_*C^T > 0$.

Moreover, the optimal L_* can be found as:

$$L_* = -(AP_*C^T + BD^T)(R + CP_*C^T)^{-1}\,,$$

if $A + L_*C$ is Hurwitz.

proof(***Sufficiency***) For any L for which $A + LC$ is stable, there is a $P \geq 0$ solving:

$$P = (A + LC)P(A + LC)^T + (B + LD)(B + LD)^T\,.$$

On the other hand, since P_* is a stabilizing solution, so $A + L_*C$ is stable, for $L_* = -(AP_*C^T + BD^T)(R + CP_*C^T)^{-1}$. Using (4) , (3) can be rewritten as:

$$\begin{aligned} P =& APA^T + APC^TL^T + LCPA^T + LCPC^TL^T \\ &- L(R + CP_*C^T)L_*^T - LCP_*A^T - AP_*C^TL^T \\ &+ L_*(R + CP_*C^T)L_*^T - L_*(R + CP_*C^T)L^T \\ &- AP_*A^T + P_* + LRL^T\,. \end{aligned}$$

If we define $\Delta P = P - P_*$, then the above expression can be simplified into:

$$\begin{aligned} \Delta P =& (A + LC)\Delta P(A + LC)^T \\ &+ (L - L_*)(R + CP_*C^T)(L - L_*)^T\,. \end{aligned}$$

From this Lyapunov equation, it is obvious that $\Delta P \geq 0$ and also $\Delta P = 0$ if and only if $L = L_*$. Hence:

$$J(L) - J(L_*) = \text{trace}(Q\Delta PQ^T) \geq 0\,,$$

or in other words, $J(L)$ achieves the minimum value at L_*, which concludes the proof for the sufficiency condition.

3. DISCRETE-TIME $\mathcal{H}_\infty$ FILTER DESIGN

Consider the filter design problem in Fig. 1. For the plant G, described by:

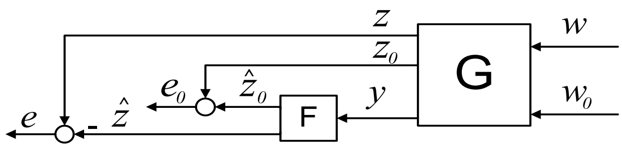

Fig. 1. $\mathcal{H}_\infty$ Gaussian filter

$$\begin{aligned} \delta x &= Ax + B_0 w_0 + B_1 w\,, \quad x_{(0)} = 0\,,\\ z_0 &= C_0 x\,,\\ z &= C_1 x\,,\\ y &= C_2 x + D_{20} w_0\,, \end{aligned} \tag{5}$$

where w is a bounded power signal and w_0 is a white noise signal. The following standard assumptions are made:

(A1) (C_2, A) is detectable;

(A2) $R_0 := D_{20} D_{20}^T > 0$;

(A3) $\begin{bmatrix} A - \lambda I & B_0 \\ C_2 & D_{20} \end{bmatrix}$ has full row rank, $\forall \lambda \in \partial\mathbb{D}$.

The goal is to find a filter:

$$F: \quad y \rightarrow \begin{bmatrix} \hat{z} \\ \hat{z_0} \end{bmatrix}, \tag{6}$$

where $\hat{z}$ and $\hat{z_0}$ are estimates of z and z_0, respectively. The filter is to be designed as:

$$\begin{aligned} \delta\hat{x} &= A\hat{x} + L(C_2\hat{x} - y)\,,\\ \hat{z_0} &= C_0\hat{x}\,,\\ \hat{z} &= C_1\hat{x}\,, \end{aligned} \tag{7}$$

where L is the filter gain to be calculated. Define the following error variables:

$$e := z - \hat{z}, \quad e_0 := z_0 - \hat{z}_0, \quad e_x := x - \hat{x}\,, \tag{8}$$

and the cost functions as:

$$J_1(F, w, w_0) = \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel e \parallel_{\mathcal{P}}^2\,, \tag{9}$$

$$J_2(F, w, w_0) = \parallel e_0 \parallel_{\mathcal{P}}^2\ . \tag{10}$$

The discrete-time $\mathcal{H}_\infty$ Gaussian filter design problem is stated as follows:

Find a filter F_* in the form (7) and a worst disturbance signal w_* such that they achieves:

$$\begin{aligned} J_1(F_*, w_*, w_0) &\le J_1(F_*, w, w_0)\,,\\ J_2(F_*, w_*, w_0) &\le J_2(F, w_*, w_0)\,. \end{aligned}$$

for any bounded power signal $w \neq w_*$ and any other admissible F.

Combining the equations of the plant and the filter and implementing $e_x := x - \hat{x}$, derive:

$$\begin{aligned} \delta e_x &= (A + LC_2)e_x + (B_0 + LD_{20})w_0 + B_1 w\,,\\ e_0 &= C_0 e_x\,,\\ e &= C_1 e_x\,. \end{aligned} \tag{11}$$

The design is presented in the following theorem.

Theorem 5. Let the plant G be described by the equation set (5), where w and w_0 are assumed to be uncorrelated, and the cost functions J_1 and J_2 are defined as (9) and (10). If there are stabilizing solutions $P_1 \ge 0$ and $P_2 \ge 0$ to:

$$\begin{aligned} P_1 =& \bar{A}^T P_1 \bar{A} + \gamma^{-2}\bar{A}^T P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 \bar{A}\\ &+ C_1^T C_1, \end{aligned} \tag{12}$$

$$\begin{aligned} P_2 =& A_F P_2 A_F^T - (B_0 D_{20}^T + A_F P_2 C_2^T)(R_0 + C_2 P_2 C_2^T)^{-1}\\ &.(D_{20} B_0^T + C_2 P_2 A_F^T) + B_0 B_0^T\,, \end{aligned} \tag{13}$$

where $(I - \gamma^{-2} B_1^T P_1 B_1) > 0$, $R_0 + C_2 P_2 C_2^T > 0$ and:

$$\bar{A} = A + L_* C_2\,, \qquad \Delta_1 = I - \gamma^{-2} B_1 B_1^T P_1\,,$$

$$A_F = (I + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1})A + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1} L_* C_2\,.$$

Then by choosing L_* that satisfies:

$$L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1}\,, \tag{14}$$

the filter F_*:

$$\begin{aligned} \delta\hat{x} &= (A + L_* C_2)\hat{x} - L_* y\,,\\ \hat{z_0} &= C_0 \hat{x}\,,\\ \hat{z} &= C_1 \hat{x}\,, \end{aligned}$$

and the worst disturbance signal:

$$w_* = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} \bar{A} e_x\,,$$

achieve:

$$\begin{aligned} J_1(F_*, w_*, w_0) &\le J_1(F_*, w, w_0)\,,\\ J_2(F_*, w_*, w_0) &\le J_2(F, w_*, w_0)\,. \end{aligned}$$

Conversely, if there exists a filter F_*, with a worst disturbance signal w'_*, such that for the system without white noise, we have:

$$0 < J_1(F_*, w'_*, 0) \le J_1(F_*, w, 0)\,, \qquad \forall w \neq w'_*\,,$$

and a worst disturbance signal w_* at the presence of white noise, such that:

$$\begin{aligned} J_1(F_*, w_*, w_0) &\le J_1(F_*, w, w_0)\,,\\ J_2(F_*, w_*, w_0) &\le J_2(F, w_*, w_0)\,, \end{aligned}$$

then, there exist stabilizing solutions $P_1 \ge 0$ and $P_2 \ge 0$ to:

$$\begin{aligned} P_1 =& \bar{A}^T P_1 \bar{A} + \gamma^{-2}\bar{A}^T P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 \bar{A}\\ &+ C_1^T C_1,\\ P_2 =& A_F P_2 A_F^T - (B_0 D_{20}^T + A_F P_2 C_2^T)(R_0 + C_2 P_2 C_2^T)^{-1}\\ &.(D_{20} B_0^T + C_2 P_2 A_F^T) + B_0 B_0^T\,, \end{aligned}$$

where $(I - \gamma^{-2} B_1^T P_1 B_1) > 0$ and $R_0 + C_2 P_2 C_2^T > 0$.

Moreover, the optimal value of the filter gain L_* satisfies:

$$L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1}\,.$$

proof ***(Sufficiency)*** If we choose the filter gain L_*, that satisfies:

$$L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1}\,,$$

completing the squares, using (12), we have:

$$
\begin{aligned}
J_1(F_*, w, w_0) =& \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel e \parallel_{\mathcal{P}}^2 \\
=& \gamma^2 \parallel w - w_* \parallel_{\mathcal{P}}^2 \\
& - \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \text{trace}\big((B_0 + L_* D_{20})^T \\
& \qquad .P_1 A E\{x w_0^T\}\big) \\
=& \gamma^2 \parallel w - w_* \parallel_{\mathcal{P}}^2 ,
\end{aligned}
$$

where

$$w_* = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} \bar{A} e_x ,$$

which is bounded since $A + L_* C_2 + B_1 w_*$ is stable.

Next it is shown that J_1 achieves the minimum value at the given L_*. Let L_1 be any filter gain such that both $A + L_1 C$ and $A + L_1 C_2 + B_1 w_*$ are stable. Substituting the above w_* in the plant-filter equations (11), we get:

$$
\begin{aligned}
\delta e_x =& \big(A + L_1 C_2 + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1} (A + L_1 C_2)\big) e_x \\
& + (B_0 + L_1 D_{20}) w_0 \\
=& A_L e_x + B_L w_0 , \\
e_0 =& C_0 e_x .
\end{aligned}
$$

The first difference equation above can be solved as:

$$e_{x(k)} = \sum_{j=0}^{k-1} A_L^{k-j-1} B_L w_{0(j)} ,$$

and

$$
\begin{aligned}
J_2(F, w_*, w_0) &= \parallel e_0 \parallel_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{e_x^T C_0^T C_0 e_x\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\Big\{ \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} w_{0(i)}^T B_L^T \\
&\qquad .(A_L^T)^{k-i-1} C_0^T C_0 A_L^{k-j-1} B_L w_{0(j)} \Big\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \text{trace}[C_0 A_L^{k-i-1} B_L \\
&\qquad . \, \delta_{(i-j)} B_L^T (A_L^T)^{k-j-1} C_0^T] \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{k-1} \text{trace}[C_0 A_L^{k-i-1} B_L B_L^T \\
&\qquad .(A_L^T)^{k-i-1} C_0^T] \\
&= \text{trace}(C_0 Y C_0^T)
\end{aligned}
$$

where

$$Y = \sum_{i=0}^{\infty} A_L^i B_L B_L^T (A_L^T)^i ,$$

which is the solution of the Lyapunov equation $A_L Y A_L^T - Y + B_L B_L^T = 0$. Then, by Theorem 4, the solution to this constrained optimization problem, L_* is to satisfy:

$$L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1} ,$$

where P_2 is the solution to (13).

(Necessity) First, for the system without white noise ($w_0 = 0$), suppose there exists a filter F_* and a signal w'_* such that they achieve:

$$0 < J_1(F_*, w'_*, 0) \leq J_1(F_*, w, 0) , \qquad \forall w \neq w'_* .$$

In other words, for the linear operator $R_{e'w}$, defined as:

$$
\begin{aligned}
\delta e'_x &= \bar{A} e'_x + B_1 w , \\
e &= C_1 e'_x ,
\end{aligned}
$$

it holds that $\parallel R_{e'w} \parallel_\infty < \gamma$. Then, by the bounded real lemma [de Souza and Xie, 1992], there exists a $P_1 \geq 0$, solving (12) and the worst disturbance signal is

$$w'_* = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} \bar{A} e'_x .$$

Next, including the white noise signal into the system, it can be seen that:

$$
\begin{aligned}
J_1(F_*, w'_*, w_0) =& \gamma^2 \parallel w - w'_* \parallel_{\mathcal{P}}^2 \\
& - \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \text{trace}\big[(B_0 + L_* D_{20})^T \\
& \qquad .P_1 A E\{x w_0^T\}\big] \\
=& \gamma^2 \parallel w - w'_* \parallel_{\mathcal{P}}^2 ,
\end{aligned}
$$

which means that the worst disturbance signal at the presence of the white noise is $w_* = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} \bar{A} e_x$.

Now, substituting the w_* into the equation set (11), we get:

$$
\begin{aligned}
\delta e_x &= \big(\bar{A} + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1} \bar{A}\big) e_x + (B_0 + L_1 D_{20}) w_0 \\
&= A_L e_x + B_L w_0 \\
e_0 &= C_0 e_x
\end{aligned}
$$

Similar to the proof of sufficiency, we can write:

$$J_2(F, w_*, w_0) = \parallel e_0 \parallel_{\mathcal{P}}^2 = \text{trace}(C_0 Y C_0^T)$$

where

$$Y = \sum_{i=0}^{\infty} A_L^i B_L B_L^T (A_L^T)^i$$

and by Theorem 4, L_* is to satisfy:

$$L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1} ,$$

and P_2 is the solution to (13).

4. ILLUSTRATIVE EXAMPLE

Consider the following dynamic system:

$$
\begin{aligned}
\delta x &= \begin{bmatrix} 1 & -0.1 \\ 0.12 & 0.95 \end{bmatrix} x + \begin{bmatrix} 0.05 & 0.1 \\ 0.1 & 0.01 \end{bmatrix} w_0 + \begin{bmatrix} -0.12 \\ 0.03 \end{bmatrix} w , \\
z &= [\, 0.6 \;\; 0.4 \,]\, x , \\
y &= [\, 0.5 \;\; -0.65 \,]\, x + [\, 1.2 \;\; 1.6 \,]\, w_0 .
\end{aligned}
$$

First, considering only the $\mathcal{H}_\infty$ performance, assume the filter gain $L = [2\ 3]^T$. Fixing $\gamma = 1.5$, there exists a solution $P_1 \geq 0$ to:

$$P_1 = (A + LC_2)^T P_1 (A + LC_2) + C_1^T C_1 + \gamma^{-2} (A + LC_2)^T .P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 (A + LC_2) .$$

This filter achieves $\| T_{ew} \| = 0.9663 \leq 1.5$, where T_{ew} represents the transfer function from w to e, and therefore satisfying the $\mathcal{H}_\infty$ requirement. In this case, the worst disturbance signal is characterized by $w_* = 0.444 B_1^T P_1 \Delta_1^{-1} (A + LC_2) e_x = K_w e_x$. However, when the noise signal w_0 is added, the optimal performance of the system in the worst case is then calculated by:

$$J_2 = \text{trace}(C_1 C_1^T P_2) = 35.463 ,$$

where P_2 is the solution to the discrete-time Lyapunov equation:

$$P_2 = (A + LC_2 + B_1 K_w) P_2 (A + LC_2 + B_1 K_w)^T + (B_0 + LD_{20})(B_0 + LD_{20})^T .$$

It is clear that the performance of this filter in presence of noise is not desirable. On the other hand, a Kalman filter can be calculated that satisfies the $\mathcal{H}_2$ optimal performance requirement. This filter can be found by solving the Riccati equation:

$$P_2 = A P_2 A^T - (B_0 D_{20}^T + A P_2 C_2^T)(R_0 + C_2 P_2 C_2^T)^{-1} .(D_{20} B_0^T + C_2 P_2 A^T) + B_0 B_0^T ,$$

leading to a filter gain:

$$L_* = -(A P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1} = \begin{bmatrix} -0.0622 \\ -0.0248 \end{bmatrix} .$$

The optimal performance of the system with this filter then becomes:

$$J_2 = \text{trace}(C_1 C_1^T P_2) = 0.0472 ,$$

which in fact is much lower than 35.463 obtained when only the $\mathcal{H}_\infty$ performance was considered.

Now, designing a multi-objective filter using Theorem 5 and fixing $\gamma = 2.5$, results to solutions to Riccati equations (12) and (13) as:

$$P_1 = \begin{bmatrix} 8.900 & 0.012 \\ 0.012 & 2.684 \end{bmatrix} > 0 , \quad P_2 = \begin{bmatrix} 0.066 & -0.004 \\ -0.004 & 0.086 \end{bmatrix} > 0 ,$$

and a filter gain:

$$L_* = \begin{bmatrix} -0.0647 \\ -0.0210 \end{bmatrix} , \qquad (15)$$

that satisfies (14).

For the closed-loop system consisting this filter, the cost functions are:

$$J_1 = 1.8927 , \quad J_2 = 0.0788 .$$

Note that although the index J_2 is worse than the Kalman filter, but is still much improved compared to the system with only an $\mathcal{H}_\infty$ filter. On the other hand, as can be seen in Figure 2, the error performance of the closed-loop system is much better in the presence of the white noise signal w_0 when the multi-objective filter is used, compared to the filter that only satisfies the $\mathcal{H}_\infty$ performance.

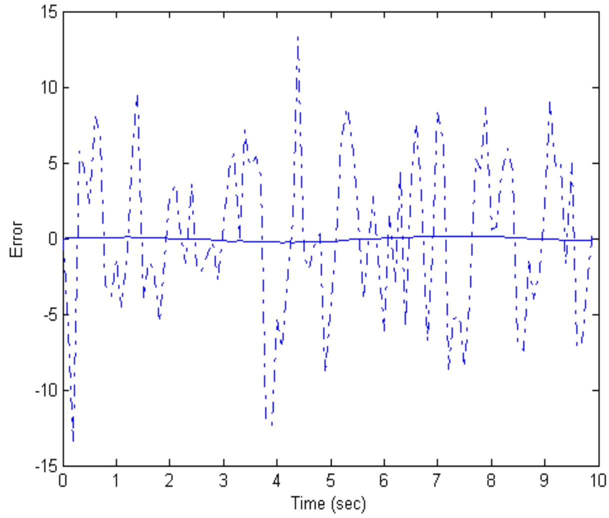

Fig. 2. Error behavior at the presence of white noise signal w_0 for the closed -loop system with filter L_∞ (dashed) and with the multi-objective filter (solid).

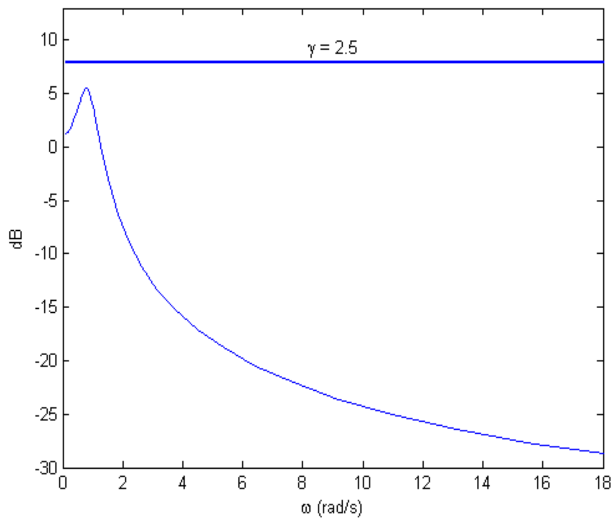

Fig. 3. Singular value diagram for T_{ew} for the system with multi-objective filter.

Figure 3 shows the singular value diagram for the transfer function T_{ew} of the system with filter gain (15) which, as expected, meets the disturbance attenuation of $\gamma = 2.5$.

5. CONCLUSIONS

In this paper, we have developed a direct design method for a robust optimal signal estimator in discrete-time domain. This method provides a filter that can be capable of achieving robust performance against system model uncertainties, as well as optimal performance against white noise. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter can be obtained by solving a set of cross-coupled Riccati equations.

REFERENCES

B.D.O. Anderson, and J.B. Moore. *Optimal Filtering*. Englewood Cliffs, Prentice-Hall, 1979.

B.D.O. Anderson and J.B. Moore. *Optimal Control - Linear Quadratic Methods*. Englewood Cliffs, Prentice-Hall, 1989.

D.S. Bernstein and W.M. Haddad. "Steady-State Kalman Filtering With an Error Bounded", *Systems & Control Letters*, vol. 12, 1989, pp 9-16.

X. Chen and K. Zhou. "$\mathcal{H}_\infty$ Gaussian Filter on Infinite Time Horizon", *IEEE Transactions on Circuits and Systems - I*, vol. 49(5), 2002, pp 674-679.

C.E. de Souza and L. Xie, "On the discrete-time bounded real lemma with application in the characterization of static feedback $\mathcal{H}_\infty$ controllers", *Systems & Control Letters*, vol. 18, 1992, pp 61-71.

M. Fu, C.E. de Souza and L. Xie, "$\mathcal{H}_\infty$ Estimation for Uncertain Systems", *International Journal of Robust and Nonlinear Control*, vol. 2(1), 1992, pp 87-105.

H. Gao, J. Lam, L. Xie and C. Wang, "New Approach to Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Filtering for Polytopic Discrete-Time Systems", *IEEE Transactions on Signal Processing*, vol. 53(8), 2005, pp 3183-3192.

H. Gao and C. Wang, "Delay-Dependent Robust $\mathcal{H}_\infty$ and $L_2 - L_\infty$ Filtering for a Class of Uncertain Nonlinear Time-Delay Systems", *IEEE Transactions on Automatic Control*, vol. 48(9), 2003, pp 1661-1666.

E.N. Goncalves, R.M. Palhares, and R.H.C. Takahashi, "$\mathcal{H}_2$/$\mathcal{H}_\infty$ filter designs for systems with polytope-bounded uncertainty", *IEEE Transactions on Signal Processing*, vol. 54(9), 2006, pp 3620-3626.

B. Halder, B. Hassibi and T. Kailath, "State-Space Structure of Finite Horizon Optimal Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Filters", *Proceedings of American Control Conference*, 1997.

P.P. Khargonekar, M.A. Rotea and E. Baeyens, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Filtering", *International Journal of Robust and Nonlinear Control*, vol. 6(6), 1996, pp 313-330.

H. Li and M. Fu, "A Linear Matrix Inequality Approach to Robust $\mathcal{H}_\infty$ Filtering", *IEEE Transactions on Signal Processing*, vol. 45(9), 1997, pp 2338-2350.

K.M. Nagpal and P.P. Khargonekar, "Filtering and Smoothing in an $\mathcal{H}_\infty$ Setting", *IEEE Transactions on Automatic Control*, vol. 36, 1991, pp 152-166.

R.M. Palhares and P.L.D. Peres, "LMI Approach to the Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Filtering Design for Discrete-Time Systems", *IEEE Transactions on Aerospace Electronic Systems*, vol. 37(1), 2001, pp 292-296.

I.R. Petersen and A.V. Savkin, *Robust Kalman Filtering for Signals and Systems with Large Uncertainties*, Boston, Birkhauser, 1999.

H. Rotstein, M. Sznaier and M. Idan, "$\mathcal{H}_2/\mathcal{H}_\infty$ Filtering Theory and an Aerospace Application", *International Journal of Robust Nonlinear Control*, vol. 6, 1996, pp 347-366.

Y. Theodor and U. Shaked, "A Dynamic Game Approach to Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Estimation", *International Journal of Robust Nonlinear Control*, vol. 6(4), 1996, pp 331-345.

Z. Wang and B. Huang, "Robust $\mathcal{H}_2/\mathcal{H}_\infty$ Filtering For Linear Systems with Error Variance Constraints", *IEEE Transactions on Signal Processing*, vol. 48(8), 2000, pp 2463-2467.

Z. Wang and H. Unbehauen, "Robust $\mathcal{H}_2/\mathcal{H}_\infty$ State Estimation for Systems with Error Variance Constraints: The Continuous-Time Case", *IEEE Transactions on Automatic Control*, vol. 44(5), 1999, pp 1061-1065.

S. Xu and T. Chen, "Reduced-Order $\mathcal{H}_\infty$ Filtering for Stochastic Systems", *IEEE Transactions on Signal Processing*, vol. 50(12), 2002, pp 2998-3007.