Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

On the Tracking Problem for Linear Systems subject to Control Saturation ⋆

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Abstract: This paper addresses the problem of tracking constant references for linear systems subject to control saturation. Considering an unitary output feedback loop, containing an integral action, conditions in LMI form are proposed to compute a state feedback and an integrator anti-windup gain. These conditions ensure that the trajectories of the closed-loop system are bounded in an invariant ellipsoidal set, provided that the initial conditions are taken in this set and the references and the disturbances belong to a certain admissible set. Based on these conditions, optimization problems aiming at the maximization of the invariant set of admissible states and/or the maximization of the set of admissible references/disturbances are proposed.

Keywords: Tracking, Disturbance rejection, Anti-windup, LMIs, Control of constrained systems.

1. INTRODUCTION

Saturation can be source of various well-known problems in control theory. Motivated by this kind of problem, a large amount of works have been published in the last decade (see for instance Kapila and Grigoriadis (Editors) [2002], Hu and Lin [2001], Tarbouriech et al. [2007] and references therein). In particular, we can find approaches to design stabilizing control laws taking into account *a priori* the possibility of saturation occurrence. However, most of these methods are concerned only with the regulation problem. For a given equilibrium point (considered without loss of generality as the origin), the global Sussmann et al. [1994], semiglobal Lin and Saberi [1993] or local (regional) Gomes da Silva Jr. and Tarbouriech [2001] asymptotically stability can be ensured. Hence, in the semi-global and local contexts, the control law is associated to a region of stability which is contained in the actual basin of attraction of the equilibrium point. The reference input signal is in this case considered as zero. On the other hand, we can find the works dealing with the so called anti-windup approach. Considering a pre-computed dynamic output feedback controller whose design neglected the possibility of input saturation, the idea in this case consists of feeding the controller with the difference between the actuator input and output, through a static or dynamic compensator. The aim of the anti-windup compensation is to correct the controller state in order to recover, as much as possible, the nominal performance of the system under saturation (see for instance Kothare et al. [1994], Grimm et al. [2003], Gomes da Silva Jr. and Tarbouriech [2005] and references therein). However, most of the proposed techniques in this context deals with external representations of the system (transfer functions) and the effect of initial states and nonlinear behaviors associated to the nonlinear state trajectories cannot be fully analyzed.

Regarding a practical perspective, the output tracking of reference signals (particularly constant ones) is of major interest in control systems. As pointed in Turner et al. [2000], relatively few works concentrate their attention on the constrained tracking schemes. In particular, in Turner et al. [2000] a control structure, composed by a nonlinear state feedback and a reference feedforward, is proposed to address the set-point tracking problem in the presence of control saturation. On the other hand, from the internal model principle, it is well known that in a unitary output feedback scheme, for "robust" perfect tracking and disturbance rejection, the controller (or the controlled plant) should present the reference unstable modes. Hence, the importance of integration actions for tracking constant reference signals (and/or rejecting constant disturbance) is clear. In the presence of control saturation, however, this is not a sufficient condition to ensure perfect tracking. In this case, additional issues should be taken into account: some reference signals can lead the trajectories to converge to equilibrium points that does not assure zero tracking error or can lead to divergent state trajectories. These particular issues are in part addressed in Tarbouriech et al. [2000] and Cao et al. [2004]. In Tarbouriech et al. [2000], considering the introduction of an integral action and a called "intelligent" windup loop (Krikelis and Barkas [1984]) in a unitary output feedback scheme, both a stabilizing state feedback and an integrator anti-windup gain are synthesized in order to ensure that the closed-loop system trajectories do not leave an ellipsoidal set, provided that the initial condition is taken in this set and the references and the disturbances belong to a certain admissible set. Furthermore, it is ensured that the equilibrium point associated to the perfect tracking (zero

⋆ This work was supported in part by CNPq, Brazil.

error) is inside the linear operating region of the closed-loop system. Following the same ideas, but using a less conservative polytopic differential inclusion to represent the saturation effects, these results are improved in Cao et al. [2004]. The main drawback of those approaches is that the theoretical conditions are in the form of nonlinear matrix inequalities. In order to solve them, iterative LMI problems (where some variables should be fixed at each step) are proposed. Another issue not addressed regards, for the given set of admissible references/disturbances, the possibility of existing other equilibria, not contained in the linearity region and not leading to zero tracking error.

In this paper we follow similar ideas to the ones proposed in Tarbouriech et al. [2000] and Cao et al. [2004]. We use, however, the modified sector condition proposed in Gomes da Silva Jr. and Tarbouriech [2005] to consider the saturation effects. Based on this representation, LMI conditions are directly derived, avoiding therefore the necessity of applying iterative schemes. These LMI conditions can be obtained both for the "classical" and "intelligent" windup structure (used in Krikelis and Barkas [1984] Tarbouriech et al. [2000] and Cao et al. [2004]). Based on these conditions, optimization problems aiming at the maximization of the invariant set of admissible states and/or the maximization of the set of admissible references/disturbances are proposed. It is also shown, by means of an example, that the obtained results are less conservative than the ones presented in Tarbouriech et al. [2000] and Cao et al. [2004]. On the other hand, a study about the possibility of existence of other equilibria inside the invariant set is carried out. For the mono-input case, we show that this is only possible if the open-loop system is asymptotically stable. In this case, in order to avoid convergence to these points, additional constraints on the admissible references to be tracked should be considered.

Notations: The ith component of a vector x is denoted by $x_{(i)}$. $A_{(i)}$ denotes the ith row of a matrix $A \in \mathbb{R}^{n\times n}$ and A^T denotes its transpose. $diag\{x\}$ denotes a diagonal matrix obtained from vector x, I_m denotes the m-order identity matrix and $conv\{\}$ denotes the convex hull.

2. PRELIMINARIES

2.1 Control Structure

Consider the continuous-time system subjected to actuator saturation described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bsat_{u_0}(u(t)) + B_d d \\ y(t) &= Cx(t) \\ e(t) &= y(t) - r \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output, $d \in \mathbb{R}^k$ is a vector of constant disturbances, $r \in \mathbb{R}^p$ is a constant reference to be tracked and $e(t) \in \mathbb{R}^p$ is the tracking error. Matrices A, B, B_d and C are constant real matrices of appropriate dimensions. Each component of the saturation term in (1) can be defined, $\forall i = 1, \ldots, m$, as

$$sat_{u_{0(i)}}(u_{(i)}(t)) \triangleq sign(u_{(i)}(t)) \min(|u_{(i)}(t)|, u_{0(i)}). \tag{2}$$

Pairs (A, B) and (C, A) are assumed to be controllable and observable, matrices B, B_d and C are assumed to be full rank, the number of inputs is supposed to be greater than or equal the number of outputs ($m \geq p$) and the relation $rank\left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right) = n + p$ hold.

In order to ensure the output constant reference tracking in steady state, an integral action is considered as follows:

$$\dot{\xi}(t) = e(t) + E_c(sat_{u_0}(u(t)) - u(t)). \tag{3}$$

The term $E_c(sat_{u_0}(u(t)) - u(t))$ corresponds to an integrator anti-windup term introduced in order to reduce any undesirable effects caused by the actuator saturation. $E_c \in \mathbb{R}^{p\times m}$ is the anti-windup gain matrix.

As in Krikelis and Barkas [1984], we introduce the error coordinates representation with a new state vector $z(t) = \begin{bmatrix} e(t)^T & x_2(t)^T & \xi(t)^T \end{bmatrix}^T \in \mathbb{R}^{n+p}$, with $x_2(t) \in \mathbb{R}^{n-p}$ defined by $x_2(t) = M_1 x(t)$ and $M_1 \in \mathbb{R}^{(n-p)\times n}$ being chosen such that $M_2 = \begin{bmatrix} C \\ M_1 \end{bmatrix}$ is non-singular. Thus the system (1) can be re-written as

$$\begin{aligned} \dot{z}(t) = {} & \mathbf{A}z(t) + \mathbf{B}_1 sat_{u_0}(u(t)) \\ & + \mathbf{B}_2(sat_{u_0}(u(t)) - u(t)) + \mathbf{B}_3 q, \end{aligned} \tag{4}$$

with $q = M_2 A M_2^{-1} E r + M_2 B_d d$,

$$\mathbf{A} = \begin{bmatrix} M_2 A M_2^{-1} & 0_{n\times p} \\ E^T & 0_{p\times p} \end{bmatrix}, E = \begin{bmatrix} I_p \\ 0_{(n-p)\times p} \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} M_2 B \\ 0_{p\times m} \end{bmatrix}$$

$$\mathbf{B}_2 = \begin{bmatrix} 0_{n\times m} \\ E_c \end{bmatrix} = \mathbf{V}E_c,\ \mathbf{V} = \begin{bmatrix} 0_{n\times p} \\ I_p \end{bmatrix},\ \mathbf{B}_3 = \begin{bmatrix} I_n \\ 0_{p\times n} \end{bmatrix}.$$

In order to stabilize system (4), we will consider the state feedback $u(t) = Fz(t)$, $F \in \mathbb{R}^{m\times(n+p)}$, which leads to the following closed-loop system:

$$\dot{z} = (\mathbf{A} + \mathbf{B}_1 F)z(t) - (\mathbf{B}_1 + \mathbf{B}_2)\Psi_{u_0}(Fz(t)) + \mathbf{B}_3 q \tag{5}$$

where $\Psi_{w_0}(w(t)) = w(t) - sat_{w_0}(w(t))$ is a decentralized deadzone nonlinearity. The considered control structure is depicted in Figure 1.

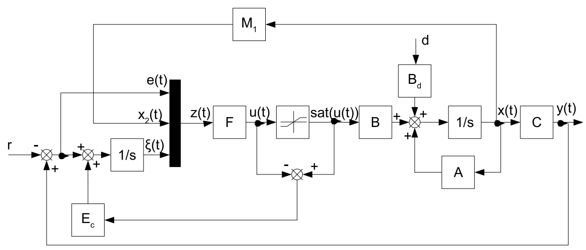

Fig. 1. Controller structure

2.2 Equilibrium point

We suppose that the equilibrium point associated to perfect tracking is inside the region of linearity for the closed-loop system (5), i.e. $z_e \in S(F, u_0)$ with

$$S(F, u_0) \triangleq \{z \in \mathbb{R}^{n+p}; |F_{(i)}z| \leq u_{0(i)},\ i = 1, \ldots, m\}. \tag{6}$$

In the region of linearity, system (5) admits the following linear model:

$$\dot{z}(t) = (\mathbf{A} + \mathbf{B}_1 F)z(t) + \mathbf{B}_3 q. \tag{7}$$

Hence, if $(\mathbf{A} + \mathbf{B}_1 F)$ is Hurwitz,

$$z_e = -(\mathbf{A} + \mathbf{B}_1 F)^{-1}\mathbf{B}_3 q \tag{8}$$

is the unique equilibrium point in $S(F, u_0)$(see Tarbouriech et al. [2000]).

In order to have $z_e \in S(F, u_0)$, relation

$$\left|-F_{(i)}((\mathbf{A} + \mathbf{B}_1 F)^{-1}\mathbf{B}_3 q)\right| \le u_{0(i)},\ i = 1, \ldots, m \tag{9}$$

must be verified. Note that relation (9) imposes a constraint on the admissible signals q.

From the integral action and asymptotic stability of the system (7), the equilibrium point $z_e \in S(F, u_0)$ assumes the following form

$$z_e = \begin{bmatrix} 0 \\ x_{2e} \\ \xi_e \end{bmatrix}, \tag{10}$$

which results in perfect reference tracking since it is ensured $e(t) = 0$.

2.3 Problem Statement

Let $\mathcal{Z}_0$ be a set of admissible initial conditions and $\mathcal{Q}_0$ a set of admissible references/disturbances for the system (5). Based on these sets, the following problem can be formulated:

Problem 1. Compute the gain matrices F and E_c such that, $\forall z(0) \in \mathcal{Z}_0$ and $\forall r, d$ such that $q \in \mathcal{Q}_0$ the equilibrium point $z_e \in S(F, u_0)$ is locally asymptotically stable and $y(t) \to r$ when $t \to \infty$.

An implicit optimization problem regarding Problem 1 concerns the determination of F and E_c in order to maximize the size of the admissible sets $\mathcal{Z}_0$ and $\mathcal{Q}_0$ for which is possible to ensure that the reference will be tracked. In this work, we will consider $\mathcal{Z}_0$ and $\mathcal{Q}_0$ as ellipsoidal sets defined as follows:

$$\mathcal{Z}_0 = \Omega(P, 1) = \{z \in \mathbb{R}^{n+p}; z^T P z \le 1\}, P = P^T > 0,$$
$$\mathcal{Q}_0 = \Omega(R, 1) = \{q \in \mathbb{R}^{n}; q^T R q \le 1\}, R = R^T > 0.$$

3. MAIN RESULT

Consider a matrix $G \in \mathbb{R}^{m\times(n+p)}$ and the polyhedral set

$$S(F - G, u_0) \triangleq \{z \in \mathbb{R}^{n+p}; \left|(F_{(i)} - G_{(i)})z\right| \le u_{0(i)}, \forall i = 1, \ldots, m\} \tag{11}$$

The following Lemma can be stated.

Lemma 1. (Gomes da Silva Jr. and Tarbouriech [2005]) If $z(t) \in S(F - G, u_0)$ then relation

$$\Psi_{u_0}(Fz(t))^T T[\Psi_{u_0}(Fz(t)) - Gz(t)] \le 0 \tag{12}$$

is verified for any matrix $T \in \mathbb{R}^{m\times m}$ diagonal positive-definite.

Consider now a quadratic Lyapunov candidate function

$$V(z(t)) = z(t)^T P z(t). \tag{13}$$

The following Theorem can therefore be stated.

Theorem 1. If there exist matrices $W \in \mathbb{R}^{(n+p)\times(n+p)}$ and $R \in \mathbb{R}^{n\times n}$ symmetric positive definite, matrices $Y \in \mathbb{R}^{m\times(n+p)}$, $X \in \mathbb{R}^{(n+p)\times m}$, $M \in \mathbb{R}^{p\times m}$, a diagonal positive-definite matrix $L \in \mathbb{R}^{m\times m}$ and a scalar $\lambda > 0$ satisfying [1]

$$\begin{bmatrix} \Lambda(W, Y) & * & * \\ -L\mathbf{B}_1^T - M^T\mathbf{V}^T + X^T & -2L & * \\ \mathbf{B}_3^T & 0 & -\lambda R \end{bmatrix} < 0 \tag{14}$$

$$\begin{bmatrix} W & * \\ Y_{(i)} - (X^T)_{(i)} & u_{0(i)}^2 \end{bmatrix} \ge 0,\ i = 1, \ldots, m \tag{15}$$

$$\begin{bmatrix} \begin{pmatrix} -\mathbf{B}_3\mathbf{B}_3^T(\mathbf{A}W + \mathbf{B}_1 Y) \\ -(\mathbf{A}W + \mathbf{B}_1 Y)^T\mathbf{B}_3\mathbf{B}_3^T \end{pmatrix} & * & * \\ \mathbf{B}_3^T & \lambda R & * \\ Y_{(i)} & 0 & \lambda^{-1}u_{0(i)}^2 \end{bmatrix} \ge 0, \quad i = 1, \ldots, m \tag{16}$$

with $\Lambda(W, Y) = W\mathbf{A}^T + \mathbf{A}W + \mathbf{B}_1 Y + Y^T\mathbf{B}_1^T + \lambda W$, then gains $F = YW^{-1}$ and $E_c = ML^{-1}$ are such that, $\forall q \in \Omega(R, 1)$, the ellipsoid $\Omega(P, 1)$ is a positively invariant region for system (5) and (8) is the only equilibrium point inside $S(F, u_0)$.

Proof. Computing the time-derivative of (13) along the trajectories of the system (5), we get

$$\dot{V}(z(t)) = z(t)^T((\mathbf{A} + \mathbf{B}_1 F)^T P + P(\mathbf{A} + \mathbf{B}_1 F))z(t) + 2z(t)^T P\mathbf{B}_3 q - 2z(t)^T P(\mathbf{B}_1 + \mathbf{B}_2)\Psi_{u_0}(Fz(t)).$$

From Lemma 1, it follows that

$$\dot{V}(z(t)) \le \dot{V}(z(t)) - 2\Psi_{u_0}(Fz(t))^T T\Psi_{u_0}(Fz(t)) + 2\Psi_{u_0}(Fz(t))^T TGz(t), \forall z \in S(F - G, u_0).$$

The set $\Omega(P, 1)$ will be invariant if $\dot{V}(z(t)) < 0$ is ensured, for all $z(t)$ and q such that $z(t)^T P z(t) \ge 1$ and $q^T R q \le 1$. Using the S-procedure (Boyd et al. [1994]), this is accomplished if there exists a positive scalar $\lambda > 0$, such that

$$\begin{aligned} &z(t)^T((\mathbf{A} + \mathbf{B}_1 F)^T P + P(\mathbf{A} + \mathbf{B}_1 F))z(t) \\ &\quad - 2z(t)^T(P(\mathbf{B}_1 + \mathbf{B}_2) - G^T T)\Psi_{u_0}(Fz(t)) \\ &\quad - 2\Psi_{u_0}(Fz(t))^T T\Psi_{u_0}(Fz(t)) + 2z(t)^T P\mathbf{B}_3 q \\ &\quad + \lambda z(t)^T P z(t) - q^T\lambda R q < 0 \end{aligned} \tag{17}$$

and provided that $\Omega(P, 1) \subset S(F - G, u_0)$.

Now, write the left hand side of (17) in the form

$$\left[\, z(t)^T\ \ \Psi_{u_0}(Fz(t))^T\ \ q^T \,\right] \mathcal{M} \begin{bmatrix} z(t) \\ \Psi_{u_0}(Fz(t)) \\ q \end{bmatrix},$$

[1] $*$ denotes symmetric elements.

$$\mathcal{M} = \begin{bmatrix} \bar{\Gamma}(P,F) & -P(\mathbf{B}_1 + \mathbf{B}_2) + G^T T & P\mathbf{B}_3 \\ * & -2T & 0 \\ * & * & -\lambda R \end{bmatrix} \quad (18)$$

with $\bar{\Gamma}(P,F) = (\mathbf{A} + \mathbf{B}_1 F)^T P + P(\mathbf{A} + \mathbf{B}_1 F) + \lambda P$. Recalling that $\mathbf{B}_2 = \mathbf{V}E_c$, pre- and post-multiplying (18) by $diag\{P^{-1}, T^{-1}, I\}$ and considering $W = P^{-1}$, $X = P^{-1}G^T$, $Y = FP^{-1}$, $L = T^{-1}$ and $M = E_c T^{-1}$, it follows that (14) is equivalent to $\mathcal{M} < 0$, which implies that (17) holds. On the other hand, the satisfaction of relation (15) implies that $\Omega(P,1)$ is included in $S(F - G, u_0)$ as needed to the satisfaction of the sector condition (12).

In Tarbouriech et al. [2000] is proved that (16) implies the satisfaction of (9). Due to space restrictions, this proof will not be presented here. ■

Theorem 1 ensures that, provided $q \in \Omega(R,1)$ and $z(0) \in \Omega(P,1)$, the closed-loop trajectories do not leave $\Omega(P,1)$. Furthermore, it is ensured that the equilibrium point associated to zero tracking error is contained in the linearity region. However, it should be pointed out, that it is not eliminated the possible existence of other equilibria associated to $q \in \Omega(R,1)$. Note that this possibility was not considered in the previous related results presented in Tarbouriech et al. [2000] and Cao et al. [2004].

In the sequel, considering the mono-input case, we focus on the analysis of possible existence of other equilibria in $\Omega(P,1)$. With this aim, consider the system (4) and let z_{eo} be a possible equilibrium point outside the linearity region. Then, the following equations must be verified:

$$0 = \hat{A}\begin{bmatrix} e_e \\ x_{2e} \end{bmatrix} + M_2 B sat_{u_0}(F z_{eo}) + q, \quad (19)$$

$$0 = E^T \begin{bmatrix} e_e \\ x_{2e} \end{bmatrix} + E_c(sat_{u_0}(F z_{eo}) - F z_{eo}), \quad (20)$$

where $\hat{A} = M_2 A M_2^{-1}$. Note that equation (20) is equivalent to

$$e_e = E_c \Psi_{u_0}(F z_{eo}). \quad (21)$$

Define now the following open-loop transfer functions:

$$G_v(s) = \frac{y(s)}{v(s)} \quad G_d(s) = \frac{y(s)}{d(s)}$$

with $v(t) = (sat_{u_0}(u(t))$. For a sake of simplicity, we suppose that $G_v(0) > 0$.

The analysis of existence of z_{eo} is carried out bellow considering 3 cases related to the eigenvalues of matrix A. We suppose that the conditions of Theorem 1 are satisfied.

Case 1: Matrix A have all the eigenvalues with negative real part (i.e. the open-loop system is asymptotically stable).

Corollary 1. If the open-loop system is asymptotically stable and the reference r and the disturbance d are constant and verifies

$$\left| [I \;\; -G_d(0)] \begin{bmatrix} r \\ d \end{bmatrix} \right| \leq G_v(0) u_0, \quad (22)$$

then the equilibrium point $z_e \in S(F, u_0)$ is the unique stable equilibrium point inside the ellipsoid $\Omega(P,1)$.

Proof. Suppose that $r > 0$ and $E_c < 0$. Considering $G_v(0) > 0$, if the equilibrium point z_{eo} exists, by construction, the control must be saturated in the upper bound, i.e., $\psi(F z_{eo}) > 0$ and $e_{eo} = E_c \psi(F z_{eo}) = y_{eo} - r < 0$. Note that in this case $y_{eo} = G_v(0) u_0 + G_d(0) d$. Hence, it follows that $r - G_d(0) d > G_v(0) u_0$ which contradicts (22).

Suppose now that $r < 0$ and $E_c < 0$. It follows that $\psi(F z_{eo}) < 0$ and $e_{eo} = E_c \psi(F z_{eo}) = y_{eo} - r > 0$. Note that in this case $y_{eo} = -G_v(0) u_0 + G_d(0) d$. Hence it follows that $r - G_d(0) d < -G_v(0) u_0$ which also contradicts (22).

Note that if $E_c > 0$, by construction there is no possibility of existence of z_{eo}, since (21) does not have possible solution in this case. ■

Case 2: Matrix A is nonsingular and has at least one eigenvalue with positive real part.

Corollary 2. If the open-loop system (1) is exponentially unstable, then the equilibrium point $z_e \in S(F, u_0)$ is the unique stable equilibrium point inside the ellipsoid $\Omega(P,1)$.

Proof. The augmented system (4) can be re-written as

$$\dot{z}(t) = \tilde{A} z(t) + \bar{B} sat_{u_0}(F z(t)) + \mathbf{B}_3 q \quad (23)$$

where $\tilde{A} = \mathbf{A} - \mathbf{B}_2 F$ and $\bar{B} = \mathbf{B}_1 + \mathbf{B}_2$. At a supposed equilibrium point z_{eo} outside the linear region we must have

$$z_{eo} = \tilde{A}^{-1}(-\bar{B} u_e - \mathbf{B}_3 q),$$

where u_e is the lower or upper saturation level. Considering the variable change $\bar{z}(t) = z(t) - z_{eo}$ one gets

$$\begin{aligned} \dot{\bar{z}}(t) &= \tilde{A}\bar{z}(t) + \tilde{A} z_{eo} + \bar{B} sat_{u_0}(F\bar{z} + F z_{eo}) + \mathbf{B}_3 q, \\ \dot{\bar{z}}(t) &= \tilde{A}\bar{z}(t). \end{aligned} \quad (24)$$

From the structure of $\mathbf{A}$ and $\mathbf{B}_2$, it is easy to see that, if A is unstable then $\mathbf{A} - \mathbf{B}_2 F$ is also unstable. Hence, if z_{eo} exists, it is an unstable equilibrium point. ■

Case 3: Matrix A has at least one null eigenvalue

Corollary 3. If the open-loop system (1) has eigenvalues at zero, then the equilibrium point $z_e \in S(F, u_0)$ is the unique stable equilibrium point inside the ellipsoid $\Omega(P,1)$.

Proof. In this case, we have an open-loop integrator system. It is well known from the linear systems theory that, for a system with a null eigenvalue subjected to a constant input, $y(t) \to \infty$ when $t \to \infty$. Suppose now that the system is in equilibrium at z_{eo} with $|F z_{eo}| > u_0$ and $B u_e + B_d d = \tilde{B}\tilde{d} \neq 0$, with $u_e = u_0$ or $u_e = -u_0$ Hence, it follows that

$$\dot{x}(t) = A x(t) + \tilde{B}\tilde{d}$$

Hence, since (A, B) and (A, C) are supposed to be controllable and observable respectively, if $B u_e + B_d d = \tilde{B}\tilde{d}$ the system output $y(t) = C x(t)$ diverges, and in consequence, $e(t) = y(t) - r$ also diverges, which contradicts the fact that there exists the equilibrium. Note that, otherwise, there is enough control to cancel the disturbance effect, and the equilibrium point is in the linear region, since it is achieved with $B F z_e = -B_d d$. ■

4. CONVEX OPTIMIZATION PROBLEM

Based on Theorem 1, we can propose a convex optimization problem to obtain the feedback gain F and the anti-windup gain E_c in order to maximize the size of sets $\Omega(P,1)$ and $\Omega(R,1)$.

The size of $\Omega(P,1)$ and $\Omega(R,1)$ can be evaluated with respect to polyhedral shape sets $\mathcal{X}_R = conv\{\mathtt{x}_1, \mathtt{x}_2, \ldots, \mathtt{x}_l\}$ and $\mathcal{D}_R = conv\{\mathtt{d}_1, \mathtt{d}_2, \ldots, \mathtt{d}_s\}$, where the vertices $\mathtt{x}_i \in \mathbb{R}^{n+p},\ i = 1,\ldots,l$ and $\mathtt{d}_j \in \mathbb{R}^n,\ j = 1,\ldots,s$ correspond to directions in which the sets should be maximized and are known a priori. Hence, our objective is to maximize scalars $\alpha > 0$ and $\beta > 0$ in order to ensure $\alpha\mathcal{X}_R \subset \Omega(P,1)$ and $\beta\mathcal{D}_R \subset \Omega(R,1)$. The solution of this problem can be addressed by solving the following optimization problem:

PO1: $\min\ (1-\epsilon)\gamma + \epsilon\delta$

subject to

$$\begin{bmatrix} \gamma & \mathtt{x}_i' \\ \mathtt{x}_i & W \end{bmatrix} \geq 0, \quad i = 1,\ldots,l \tag{25}$$

$$\mathtt{d}_j' R \mathtt{d}_j \leq \delta, \quad j = 1,\ldots,s \tag{26}$$

Relations (14), (15) and (16).

Considering $\alpha = \frac{1}{\sqrt{\gamma}}$ and $\beta = \frac{1}{\sqrt{\delta}}$, the minimization of γ and δ causes the maximization of α e β. The scalar $0 \leq \epsilon \leq 1$ can be tuned accordingly our priority in maximizing $\Omega(P,1)$ or $\Omega(R,1)$.

Note that, for a given λ, relations (14) and (16) are LMIs. Problem PO1 can then be easily solved considering the solutions of LMI problems on a grid in λ.

5. ILLUSTRATIVE EXAMPLES

5.1 Example 1

Our first illustrative example is the double integrator considered in Tarbouriech et al. [2000] and Cao et al. [2004], i.e.,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_d = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$C = [\,1\ \ 0\,],\ u_0 = 0.3.$$

In order to compare our results with those from Cao et al. [2004] and Tarbouriech et al. [2000], we will split our example in three different control objectives. We consider $M_1 = [0\ 1]$ such that $M_2 = I_2$.

Is important to notice that $M_2 A M_2^{-1} E = 0$. This implies that the reference r does not appear in q neither in $\Omega(R,1)$. In this case, r is only determined by the size of the ellipsoid $\Omega(P,1)$.

a) Reference Maximization. In this case, the control objective is to maximize the reference the system output can track without disturbances and considering that $x_2(0) = 0$ and $\xi(0) = 0$. Since $M_2 A M_2^{-1} E = 0$, this can be accomplished by maximizing $\Omega(P,1)$ considering $\mathcal{X}_R = conv\{\mathtt{x}_1\}$ with $\mathtt{x}_1 = [1\ 0\ 0]^T$ and $\epsilon = 0$. Note that in this case we maximize $|e(0)| = |r|$.

Different from what has been presented in Cao et al. [2004], for the convex optimization problem PO1, we do not have any a priori controller to compare with, so we have the freedom to redesign the controller. The results obtained are

$$E_c = -1.4020,\ F = [\,-0.4168\ \ -21.6875\ \ -0.0064\,]$$

and $r_{max} = 891.3698$. This result is much less conservative than Cao et al. [2004] and Tarbouriech et al. [2000].

If we set $E_c = 1$, as done in Cao et al. [2004] and Tarbouriech et al. [2000], optimization PO1 leads to $\alpha_{opt} = 803.5$ which is also less conservative than the value obtained in those previous works.

c) Constant Disturbance Maximization. If we set the maximum admissible reference $r_{max} = 1$, the new control objective is to find the largest constant disturbance $\|d\| < d_0$ that our system still tracks the reference. As in Cao et al. [2004], we set $\mathcal{X}_R = conv\{\mathtt{x}_1\}$ with $\mathtt{x}_1 = [1\ 0\ 0]^T$, $\mathcal{D}_R = conv\{\mathtt{d}_1\}$ with $\mathtt{d}_1 = [1\ 0.1]^T$ and $\epsilon = 1$. Again, we consider E_c as a free parameter in optimization PO1. The results are presented in Table 1.

Table 1. Results for Maximum Constant Disturbance.

	β_{opt}	λ_{opt}
Cao et al. [2004]	1.7275	0.0238
Optimization PO1	5.8349	0.0340

It is easy to see that our results are less conservative than the ones presented in Cao et al. [2004]. If we set $E_c = 1$ and solve optimization PO1, the obtained results are $\beta_{opt} = 4.1852$ and $\lambda_{opt} = 0.0290$.

5.2 Example 2

In this example we aim at showing the possibility of existence of an equilibrium point inside the ellipsoid $\Omega(P,1)$ but outside the linear region, if condition (22) is not taken into account. Consider an open-loop system defined by:

$$A = \begin{bmatrix} -0.6 & 1 \\ 0 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = [\,1\ \ 0\,],\ u_0 = 0.3.$$

In this case, we consider $M_1 = [0\ 1]$ such that $M_2 = I_2$. It is important to notice that this example does not present $M_2 A M_2^{-1} E = 0$. This cause the maximum reference admissible to be determined by the size of the ellipsoids $\Omega(P,1)$ and $\Omega(R,1)$.

a) Reference Maximization. As in the previous example, the control objective is to maximize the reference the system output can track assuming $x_2(0) = 0$ and $\xi(0) = 0$. We set $\mathcal{X}_R = conv\{\mathtt{x}_1\}$ with $\mathtt{x}_1 = [1\ 0\ 0]^T$ and $\mathcal{D}_R = conv\{\mathtt{d}_1\}$ with $\mathtt{d}_1 = [(M_2 A M_2^{-1} E)^T\ 0\ 0]^T$. The reference r_{max1} is obtained from the ellipsoid $\Omega(P,1)$ and r_{max2} from the ellipsoid $\Omega(R,1)$.

If the condition (22) is not taken into account, the maximal admissible reference that ensures the invariance of $\Omega(P,1)$ in the case that $\epsilon = 0.5$ is $r = r_{max2} = 245.1580$. In Fig. 2 we show the output and control signals for $r = 50$ at $t = 0$ and $r = -50$ at $t = 30$, far enough from the limit r_{max2}.

Note that there exist a tracking error and the equilibrium points are outside the linear region, reached when the

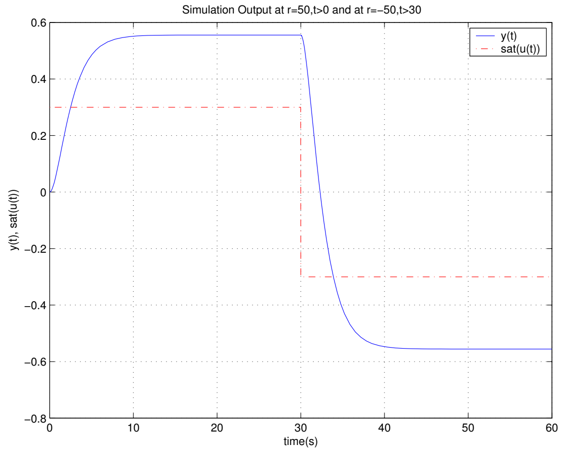

Fig. 2. Time response of the output $y(t)$ and the plant input $sat_{u_0}(u(t))$

control output remains saturated at both upper and lower levels. In this case,

$$P = 10^{-6}\begin{bmatrix} 0.0532 & 0.1135 & 0.0760 \\ 0.1135 & 0.2522 & 0.1675 \\ 0.0760 & 0.1675 & 0.1135 \end{bmatrix},\ z_{eo} = \begin{bmatrix} -49.4444 \\ 0.3333 \\ 32.6793 \end{bmatrix}$$

for the reference $r = 50$, and $z_{eo} = \begin{bmatrix} 49.4444 \\ 0.3333 \\ 32.6793 \end{bmatrix}$, for $r = -50$. In both cases $z_{eo}^T P z_{eo} = 5.5843 \times 10^{-6}$, which shows that z_{eo} is indeed inside the ellipsoid $\Omega(P,1)$.

If we consider the condition (22), the perfect tracking is ensured in fact for $r_{max} = 0.5556$. In Fig. 3 we present the simulation output and the constrained control signal for $r = 0.5$ at $t = 0$ and $r = -0.5$ at $t = 30$.

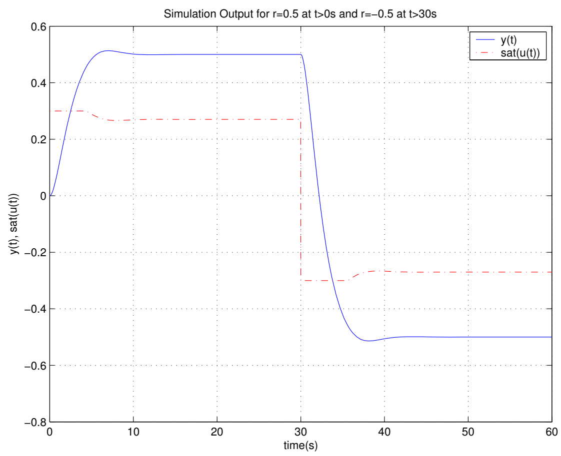

Fig. 3. Time response of the output $y(t)$ and the plant input $sat_{u_0}(u(t))$

6. CONCLUSION

The main advantage of the proposed methodology with respect to similar previous works is the fact that the theoretical conditions are given directly in LMI form, avoiding the necessity of applying iterative schemes to solve the optimization problems. It is shown, by means of examples, that less conservative results are therefore obtained. On the other hand, an analysis concerning the unicity of the equilibrium point inside the invariant domain has been carried out. It is worth to emphasize that such kind of analysis has not been performed in the previous works. Indeed, if some additional conditions are not considered, it is shown that, in the case of asymptotically stable open-loop system, other equilibria which does not lead to zero error tracking can appear inside the invariant domain. The extension of this equilibria analysis to the multi-input case and to open-loop systems presenting imaginary eigenvalues are under investigation.

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