Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

# Uniformly Observable and Globally Lipschitzian Nonlinear Systems Admit Semi-global Finite-time Observers

**Yanjun Shen** * **Xiaohua Xia** [1] **

* *Institute of Nonlinear Complex System, China Three Gorges University, Yichang, Hubei, 443002, China ( e-mail: shenyj@ctgu.edu.cn).*
** *Department of Electrical, Electronic and Computer Engineering, University of Pretoria, Pretoria 0002, South Africa ( e-mail: xxia@postino.up.ac.za)*

**Abstract:** It is well-known that high gain observers exist for nonlinear systems that are uniformly observable and globally Lipschitzian. Under the same conditions, we show that these systems admit semi-global and finite-time converging observers. This is achieved with a derivation of a new sufficient condition for local finite-time stability, in conjunction with applications of geometric homogeneity and Lyapunov theories.

## 1. INTRODUCTION

The first approach to observer design for nonlinear systems, as already contained in the early works is the extended Kalman and Luenberger observers (Zeitz [1987]) where linear algorithms are applied to the systems linearized around the estimated trajectory.

The research on nonlinear observers has achieved remarkable progress, since the formal introduction of the concept and the Lyapunov approach based results of existence and design in (Thau [1973]). With the advance of the nonlinear observability theory (Hermann and Krener [1977]) in the differential geometric framework (Isidori [1995]), quite some early works are devoted to establishing the linkage between nonlinear observer and nonlinear observability. The existence of exponential observers is closely related the observability of the linearized system (Kou et al. [1975], Xia and Gao [1988]). Uniform observability of a nonlinear system results in a triangular structure useful for observer design (see (Gauther et al. [1992], Gauthier and Kupka [1994]) and their other works). The linearized observability is a standing assumption for both the Lyapunov based approach (Raghavan and Hedrick [1994]) and the observer canonical form approach (Krener and Isidori [1983], Bestle and Zeitz [1983]). High-gain observers are very much associated with the triangular structure derived from uniform observability of nonlinear systems (Gauther et al. [1992], Gauthier and Kupka [1994]). New developments of all three design methods have been carried out in various directions. For instance, the Lyapunov based approach, under the assumption of linearized observability and Lipschitzian nonlinearity, finds its relations to the unobservability distance (Rajamani and Cho [1998]), the multiple output case of the observer canonical form method are studied in (Krener and Respondek [1985], Xia and Gao [1989]). Another method is to seek a direct coordinate transformation and by making use the Lyapunov auxiliary theorem (Kazantzis and Kravaris [1998]). The application of high gain observers in the nonlinear output stabilization problem can be seen in (Teel and Praly [1994], Khalil and Esfandiari [1993], Atassi and Khalil [1999]). More results can be found in recent books (Nijmeijer and Fossen [1999], Besancon [2007]).

Retrospecting these results, there seems to realize the importance of the Lipschitzian condition on the nonlinearities. The most well-known conclusion in this regard is the existence of a global high gain observers for uniformly observable and globally Lipschitzian systems (Gauther et al. [1992], see also a multi-output extension with semi-global convergence in (Shim et al. [2001])).

Observers with finite-time convergence have a certain advantages and therefore are desirable in some situations of control and supervision. There exist a series of methods that achieve finite-time convergence, *e.g.*, sliding mode observers (Haskara et al. [1998]), moving horizon observers (Michalska and Mayne [1995]). Some of these observers, such as the sliding mode observers, are not continuous. The continuity property and its importance in finite-time stability are realized in (Bhat and Bernstein [2000, 2005]). It is also interesting to point out that continuous observers are noted to be different and unique in the nonlinear context (Krener [1986], Xia and Zeitz [1997]). For instance, the linearized observability is no more necessary for the existence of a continuous observer (Xia and Zeitz [1997]).

For linear control systems, it has become now clear that observability implies existence of finite-time continuous observers. A first approach to design such an observer is a dedicated introduction of time-delay in the observers (Engel and Kreisselmeier [2002]). This approach was extended to linear time-varying systems in (Menold et al. [2003]) and to nonlinear systems that can be transformed into the observer canonical form (Menold et al. [2003]). Sauvage *et al* (Sauvage et al. [2007]) also proposed a nonlinear finite-time observers for a class of nonlinear systems, with a time-

[1] Corresponding author

delay in the observers. A finite-time observer for a class of observer error linearizable systems is recently constructed in (Moulay et al. [2007]) by making use of the geometric homogeneity theory of (Bhat and Bernstein [2005]).

The aim of this paper is to prove a general result: uniformly observable and globally Lipschitzian nonlinear systems admit semi-global finite-time observers. This paper reports only the result of SISO systems. Thorough out the paper, $R^n$ denotes $n$-dimension real space and $R_+^n$ denotes $n$-dimension positive real space. Let $\lceil x \rfloor^\alpha = |x|^\alpha \mathrm{sign}(x)$, $\forall x \in R$.

## 2. PRELIMINARIES

Let us begin by introducing some terminologies given in (Bhat and Bernstein [2000, 2005]). Consider the following system

$$\dot{x} = f(x(t)),\ f(0) = 0,\ x \in R^n, x(0) = x_0, \tag{1}$$

where $f : D \to R^n$ is continuous on an open neighbourhood $D$ of the origin $x = 0$.

*Definition 1.* (Bhat and Bernstein [2000]). The zero solution of (1) is *finite-time convergent* if there is an open neighborhood $U \subseteq D$ of the origin and a function $T : U \backslash \{0\} \to (0, \infty)$, such that $\forall x_0 \in U$, the solution $\psi(t, x_0)$ of system (1) with $x_0$ as the initial condition is defined and $\psi(t, x_0) \in U \backslash \{0\}$ for $t \in [0, T(x_0))$, and $\lim_{t \to T(x_0)} \psi(t, x_0) = 0$. Then, $T(x_0)$ is called *the settling time*. If the zero solution of (1) is finite-time convergent, the set of point $x_0$ such that $\psi(t, x_0) \to 0$ is called the *domain of attraction of the solution*.

*Definition 2.* (Bhat and Bernstein [2000]). The zero solution of (1) is *finite-time stable* if it is Lyapunov stable and finite-time convergence. When, $U = D = R^n$, the zero solution is said to be *globally finite-time stable*.

To illustrate finite-time stability further, as well as for later use, we consider a scalar system as follows.

*Example 1.* The scalar system

$$\dot{y}(t) = -l \lceil y(t) \rfloor^\alpha + k y(t), y(0) = x, \tag{2}$$

where $l, k > 0$, $\alpha \in (0, 1)$, is continuous everywhere and locally Lipschitzian everywhere except at the origin. Hence every initial condition in $R \backslash \{0\}$ has a unique solution. If $|x|^{1-\alpha} < \frac{l}{k}$. Multiplying (2) by $e^{-kt}$, we can obtain

$$\frac{d(e^{-kt} y(t))}{dt} = -l |y(t) e^{-kt}|^\alpha e^{(\alpha-1)kt} \mathrm{sign}(y(t)),$$

The solution trajectories are unique and described by

$$\mu(t, x) = \begin{cases} \mathrm{sign}(x) e^{kt} \left[ |x|^{1-\alpha} - \frac{l}{k} + \frac{l}{k} e^{k(\alpha-1)t} \right]^{\frac{1}{1-\alpha}}, \\ \qquad t < \dfrac{\ln(1 - \frac{k}{l}|x|^{1-\alpha})}{k(\alpha - 1)},\quad x \neq 0, \\ 0, \qquad t \geq \dfrac{\ln(1 - \frac{k}{l}|x|^{1-\alpha})}{k(\alpha - 1)}, \\ 0, \qquad t \geq 0,\quad x = 0. \end{cases} \tag{3}$$

Clearly, all the solutions converge to the origin in finite time.

*Lemma 1.* Suppose there is a $C^1$ positive definite function $V(x)$ defined on a neighborhood $U \subset R^n$ of the origin, such that

$$\dot{V}(x) \leq -l V(x)^\alpha + k V(x),\ \forall x \in U \setminus \{0\}. \tag{4}$$

Then, the origin of system (1) is finite-time stable. The domain of attraction of the origin is given by

$$\Omega = \left\{ x | V(x)^{1-\alpha} < \frac{l}{k} \right\} \cap U. \tag{5}$$

The settling time satisfies

$$T(x) \leq \frac{\ln(1 - \frac{k}{l} V(x)^{1-\alpha})}{k(\alpha - 1)}, x \in \Omega. \tag{6}$$

**Proof.** Note that the following inequality holds:

$$\dot{V}(x) \leq -l V(x)^\alpha \left( 1 - \frac{k}{l} V(x)^{1-\alpha} \right) < 0,\ \forall x \in U \setminus \{0\}. \tag{7}$$

Since $V$ is positive definite and $\dot{V}$ takes negative values on $U \setminus \{0\}$, $x = 0$ is the unique solution of (1) satisfying $x(0) = 0$. Thus every initial condition in $U$ has a unique solution in forward time. Consider $x \in U \setminus \{0\}$, $\psi(t, x)$ is the unique solution of (1) (it is obviously that $\psi(t, x) \in U$), we have

$$\dot{V}(\psi(t, x)) \leq -l V(\psi(t, x))^\alpha + k V(\psi(t, x)). \tag{8}$$

Next, applying the comparison lemma to the differential inequality (8) and (2) yields

$$V(\psi(t, x)) \leq \mu(t, V(x)), \tag{9}$$

where $\mu$ is given by (3). It follows from (3) and (9), and the positive-definiteness of $V$ that

$$\psi(t, x) = 0, t \geq \frac{\ln(1 - \frac{k}{l} V(x)^{1-\alpha})}{k(\alpha - 1)},\ \forall x \in \Omega. \tag{10}$$

Thus, the conclusion holds.

*Remark 1.* Lemma 1, when choosing $k = 0$, reduces to Theorem 4.2 (Bhat and Bernstein [2000]). Compared with Theorem 4.2 (Bhat and Bernstein [2000]), Lemma 1 is sometimes more handy to test, with a clear indication of the domain of attraction.

## 3. FINITE-TIME OBSERVERS

### 3.1 Finite-time observers

Let the dynamics of a physical system be described by the following equation:

$$\dot{x} = f(x, u), \tag{11}$$

where $x \in R^n$, $u \in R^k$ are the states and inputs of the system, respectively. $f : R^n \times R^k \to R^n$ is assumed to be smooth enough, and $f(0, 0) = 0$. The state variables $x(t)$ are not available for direct measurement, only outputs $y(t) \in R^m$ are available:

$$y = h(x), \tag{12}$$

where $h : R^n \to R^m$ and smooth enough. We give the following definition.

*Definition 3.* Let a dynamical system be described by

$$\dot{z} = g(z, y, u), \tag{13}$$

in which $z \in R^n$. $g : R^n \times R^m \times R^k \to R^n$ is continuously differentiable. Denote the solution of (11), (13) with respect to the corresponding input functions and passing through $x_0$ and $z_0$ respectively as $x(t, x_0, u)$ and $z(t, z_0, y, u)$, respectively. If no confusion arises, we denote $x(t, x_0, u)$ simply by $x(t)$, and $z(t, z_0, h(t, x_0, u), u)$ by $z(t)$. If

(i) $z_0 = x_0$ implies $z(t) = x(t)$, $t \geq 0$, for all admissible $u$;

(ii) there exist an open neighborhood $U \subset R^n$ of the origin such that $e_0 = z_0 - x_0 \in U$ implies $z(t) - x(t) \in U$ and a function $T : U \setminus \{0\} \to (0, \infty)$, such that

$$\|z(t) - x(t)\| \to 0, \quad \text{as } t \to T(e_0), \tag{14}$$

Then, the system (13) is called a *finite-time observer* of the system (11) and (12). In this case, all points $e_0 = z_0 - x_0$ such that (14) holds constitute *a domain of observer attraction.* If the open set $U$ can be chosen as the whole space $R^n$, then (13) is called a *global finite-time observer.* If for any given compact $W \subset R^n$ containing the origin, there exists a finite-time observer of the form (13), such that $W$ is contained in the domain of observer attraction, then the system (11) and (12) is said to admit *semi-global finite-time observers.*

### *3.2 Review of high gain observers design*

Consider an SISO nonlinear system on $R^n$

$$\Gamma : \begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x). \end{cases} \tag{15}$$

If $(\Gamma)$ is uniformly observable for any input (Gauther et al. [1992]). Then, a coordinate change can be found to transform the system (15) into the form

$$\begin{cases} \dot{x}_1 &= x_2 + g_1(x_1)u, \\ \dot{x}_2 &= x_3 + g_2(x_1, x_2)u, \\ \vdots & \quad \vdots \\ \dot{x}_{n-1} &= x_n + g_{n-1}(x_1, \cdots, x_{n-1})u, \\ \dot{x}_n &= \varphi(x_1, \cdots, x_n) + g_n(x_1, \cdots, x_{n-1}, x_n)u, \\ y &= x_1 = C_0 x, \end{cases} \tag{16}$$

where $C_0 = [1 \ 0 \ \cdots \ 0]$, $\varphi$ and $g_i$ $(i = 1, \cdots, n)$ are continuous functions with $\varphi(0) = 0$, $g_i(0, \cdots, 0) = 0$. If in addition, $g_i (i = 1, \cdots, n)$ and $\varphi$ satisfy the global Lipschitz condition with Lipschitz constants $L$, high gain observers of the system (16) can be designed as follows:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + s_1 e_1 + g_1(\hat{x}_1)u, \\ \dot{\hat{x}}_2 &= \hat{x}_3 + s_2 e_1 + g_2(\hat{x}_1, \hat{x}_2)u, \\ \vdots & \quad \vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + s_{n-1} e_1 + g_{n-1}(\hat{x}_1, \cdots, \hat{x}_{n-1})u, \\ \dot{\hat{x}}_n &= s_n e_1 + \varphi(\hat{x}_1, \cdots, \hat{x}_n) + g_n(\hat{x}_1, \cdots, \hat{x}_n)u, \end{cases} \tag{17}$$

where $[s_1 \ s_2 \ \cdots \ s_n]^T = S^{-1}(\theta) C_0^T$. $S(\theta)$ has the following properties.

*Lemma 2.* (Gauther et al. [1992]). $S(\theta)$ satisfies

$$-\theta S(\theta) - A_0^T S(\theta) - S(\theta) A_0 + C_0^T C_0 = 0, \tag{18}$$

$$S(\theta) \geq \delta I, \tag{19}$$

and

$$[S(\theta)]_{i,j} = [S(1)]_{i,j} \frac{1}{\theta^{i+j-1}}, \tag{20}$$

where $A_0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$.

### *3.3 Linear finite-time observers*

For observable linear systems, the above high gain observers design technique plus some homogeneity result in finite-time observers. The construction of homogeneity and proof are similar to those in (Moulay et al. [2007]), which are actually rooted in (Bhat and Bernstein [2005]). Without loss of generality, consider the following observable linear systems:

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{21}$$

where $A \in R^{n \times n}$, $B \in R^{n \times 1}$, $C \in R^{1 \times n}$ and

$$A = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ a_n & \cdots & \cdots & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \ C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T.$$

Construct the following observer for the linear system (21)

$$\begin{cases} \dot{\hat{x}}_1 = a_1 y_1 + \hat{x}_2 + s_1 \lceil e_1 \rfloor^{\alpha_1} + b_1 u, \\ \dot{\hat{x}}_2 = a_2 y_1 + \hat{x}_3 + s_2 \lceil e_1 \rfloor^{\alpha_2} + b_2 u, \\ \vdots \\ \dot{\hat{x}}_n = a_n y_1 + s_n \lceil e_1 \rfloor^{\alpha_n} + b_n u. \end{cases} \tag{22}$$

where $\alpha_i = i\alpha - (i-1)$, $e_i = x_i - \hat{x}_i$, $i = 1, \cdots, n$ and $[s_1, \cdots, s_n]^T = S^{-1}(\theta) C^T$. Let $e_i = x_i - \hat{x}_i$, the error dynamics is then given by

$$\begin{cases} \dot{e}_1 &= e_2 - s_1 \lceil e_1 \rfloor^{\alpha_1}, \\ \dot{e}_2 &= e_3 - s_2 \lceil e_1 \rfloor^{\alpha_2}, \\ \vdots & \quad \vdots \\ \dot{e}_{n-1} &= e_n - s_{n-1} \lceil e_1 \rfloor^{\alpha_{n-1}}, \\ \dot{e}_n &= -s_n \lceil e_1 \rfloor^{\alpha_n}. \end{cases} \tag{23}$$

*Lemma 3.* (Moulay et al. [2007]). For $\alpha > 1 - \frac{1}{n-1}$, the system (23) is homogeneous of degree $\alpha - 1$ with respect to the weights $\{(i-1)\alpha - (i-2)\}_{1 \leq i \leq n}$.

*Lemma 4.* There exists $\varepsilon_1 \in (1 - \frac{1}{n-1}, \ 1]$ such that for all $\alpha \in (1 - \varepsilon_1, 1]$, the system (23) is globally finite-time stable.

**Proof.** Consider the following differentiable positive definite function

$$V_\alpha(e) = y^T S(\theta) y, \tag{24}$$

where $y = (\lceil e_1 \rfloor^{\frac{1}{r}} \ \lceil e_2 \rfloor^{\frac{1}{\alpha_1 r}} \ \cdots \ \lceil e_n \rfloor^{\frac{1}{\alpha_{n-1} r}})^T$, $e = (e_1 \ e_2 \ \cdots \ e_n)^T$, $r = \prod_{i=1}^{n-1}[(i-1)\alpha - (i-2)]$ is the product of the weights. It is obvious that $V_\alpha(e)$ is homogeneous of degree $1/r^2$ with respect to the weights $\{(i-1)\alpha-(i-2)\}_{1\le i\le n}$. Let $f_\alpha$ denote the vector field of system (23). For each $\alpha > 0$, the vector field $f_\alpha$ is continuous. When, $\alpha = 1$, system (23) can be rewritten as follows:

$$\dot{e} = (A - S^{-1}(\theta)C^T C)e. \tag{25}$$

Then,

$$\frac{d}{dt}V_1(e) = 2e^T S(\theta)\dot{e} = 2e^T S(\theta)Ae - 2e^T C^T Ce$$

By the fact that $S(\theta)$ is the solution of (18), we have

$$\frac{d}{dt}V_1(e) = -\theta e^T S(\theta)e - e^T C^T Ce < 0. \tag{26}$$

Let $\mathcal{A} = V_\alpha^{-1}([0,1])$ and $\mathcal{S} = V_1^{-1}(\{1\})$. Then, $\mathcal{A}$ and $\mathcal{S}$ are compact. Define $\varphi : (0,1] \times \mathcal{S} \to R$ by $\varphi(\alpha, e) = L_{f_\alpha}V_\alpha(e)$. Then $\varphi$ is continuous and satisfies $\varphi(1,e) < 0$ for all $e \in \mathcal{S}$, that is, $\varphi(\{1\}\times\mathcal{S}) \subset (-\infty, 0)$. Since $\mathcal{S}$ is compact, there exists $\varepsilon_1 > 0$ such that $\varphi((1-\varepsilon_1, 1]\times\mathcal{S}) \subset (-\infty, 0)$. Thus, for $\alpha \in (1-\varepsilon_1, 1]$, $L_{f_\alpha}V_\alpha$ takes negative values on $\mathcal{S}$. Therefore, $\mathcal{A}$ is strictly positive invariant under $f_\alpha$ for every $\alpha \in (1-\varepsilon_1, 1]$. By Theorem 6.1 in (Bhat and Bernstein [2005]), the origin is a globally asymptotically stable equilibrium under $f_\alpha$ for every $\alpha \in (1-\varepsilon_1, 1]$. By Theorem 7.1 (Bhat and Bernstein [2005]), we can obtain that the origin is a globally finite-time stable equilibrium by noting that $\alpha - 1 < 0$. Moreover, by Lemma 4.2 (Bhat and Bernstein [2005]), we have

$$\begin{aligned} -c_1(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}} &\le L_f V_\alpha(e) \\ \le -c_2(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}}&, \end{aligned} \tag{27}$$

where $c_1(\alpha,\theta) = -\min_{\{z:V_\alpha(z)=1\}} L_f V_\alpha(z)$ and $c_2(\alpha,\theta) = -\max_{\{z:V_\alpha(z)=1\}} L_f V_\alpha(z)$. Thus, the system (23) is globally finite-time stable.

The above proof is independent of $\theta$. However, $c_2(\alpha,\theta)$ in (27) is a function of $\theta$ and has the following property.

*Lemma 5.* $c_2(\alpha,\theta)$ satisfies

$$\lim_{\alpha\to 1} c_2(\alpha,\theta) = \theta. \tag{28}$$

**Proof.** If follows from (26) that

$$\max_{\{e:V_1(e)=1\}} L_f V_1(e) = -\theta. \tag{29}$$

It is obvious that $L_f V_1(e^*) = -\theta$, where $e^* = [0 \ 0 \ \cdots \ 0 \ \frac{1}{\sqrt{s_{nn}}}]^T$ and $s_{nn} = [S(\theta)]_{n,n}$. Since $V_\alpha(e)$ is continuous, $\forall z \in \{e : V_\alpha(e) = 1\}$, $1 = \lim_{\alpha\to 1} V_\alpha(z) = V_1(z)$. *i.e.*, $z \in \{e : V_1(e) = 1\}$. Then, we have $\{e : V_\alpha(e) = 1\} \subset \{e : V_1(e) = 1\}$

$$\max_{\{e:V_\alpha(e)=1\}} L_f V_\alpha(e) \le \max_{\{e:V_1(e)=1\}} L_f V_\alpha(e).$$

Then,

$$\overline{\lim_{\alpha\to 1}} \max_{\{e:V_\alpha(e)=1\}} L_f V_\alpha(e) \le \lim_{\alpha\to 1} \max_{\{e:V_1(e)=1\}} L_f V_\alpha(e) = -\theta.$$

On the other hand, let $e_0 = \left[0 \ 0 \ \cdots \ 0 \ s_{nn}^{\frac{-\alpha_{n-1} r}{2}}\right]^T$, then $e_0 \in \{e : V_\alpha(e) = 1\}$, and $\lim_{\alpha\to 1} L_f V_\alpha(e_0) = L_f V_1(e^*) = -\theta$. Then, $\max_{\{e:V_\alpha(e)=1\}} L_f V_\alpha(e) \ge L_f V_\alpha(e_0)$. Therefore,

$$\underline{\lim_{\alpha\to 1}} \max_{\{e:V_\alpha(e)=1\}} L_f V_\alpha(e) \ge \lim_{\alpha\to 1} L_f V_\alpha(e_0) = -\theta.$$

Then, we have $\lim_{\alpha\to 1} \max_{\{e:V_\alpha(e)=1\}} L_f V_\alpha(e) = -\theta$. Thus, the proof is completed.

### 3.4 Nonlinear finite-time observers

Now we are ready to state our main result.

*Theorem 1.* Assume that the nonlinear system (15) is uniformly observable for any input and is globally Lipschitzian. Then, it admits semi-global finite-time high gain observers.

Without loss of generality, assume that the system takes the triangular structure of the form (16). An observer is designed as follows:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + s_1\lceil e_1 \rfloor^{\alpha_1} + g_1(\hat{x}_1)u, \\ \dot{\hat{x}}_2 = \hat{x}_3 + s_2\lceil e_1 \rfloor^{\alpha_2} + g_2(\hat{x}_1, \hat{x}_2)u, \\ \vdots \\ \dot{\hat{x}}_{n-1} = \hat{x}_n + s_{n-1}\lceil e_1 \rfloor^{\alpha_{n-1}} + g_{n-1}(\hat{x}_1, \cdots, \hat{x}_{n-1})u, \\ \dot{\hat{x}}_n = \varphi(\hat{x}_1, \cdots, \hat{x}_n) + s_n\lceil e_1 \rfloor^{\alpha_n} + g_n(\hat{x}_1, \cdots, \hat{x}_n)u, \end{cases} \tag{30}$$

where $[s_1 \ s_2 \ \cdots \ s_n]^T = S^{-1}(\theta)C_0^T$ and $\alpha_1, \cdots, \alpha_n$ are chosen as in (22).

The dynamics of the observation error $e = x - \hat{x}$ is given by

$$\begin{cases} \dot{e}_1 &= e_2 - s_1\lceil e_1 \rfloor^{\alpha_1} + \tilde{g}_1, \\ \dot{e}_2 &= e_3 - s_2\lceil e_1 \rfloor^{\alpha_2} + \tilde{g}_2, \\ \vdots & \quad \vdots \\ \dot{e}_{n-1} &= e_n - s_{n-1}\lceil e_1 \rfloor^{\alpha_{n-1}} + \tilde{g}_{n-1}, \\ \dot{e}_n &= -s_n\lceil e_1 \rfloor^{\alpha_n} + \tilde{\varphi} + \tilde{g}_n, \ e(0) = e_0, \end{cases} \tag{31}$$

where $\tilde{g}_1 = g_1(x_1)u - g_1(\hat{x}_1)u$, $\tilde{g}_2 = g_2(x_1, x_2)u - g_2(\hat{x}_1, \hat{x}_2)u$, $\cdots$, $\tilde{g}_{n-1} = g(x_1, \cdots, x_{n-1})u - g_{n-1}(\hat{x}_1, \cdots, \hat{x}_{n-1})u$, $\tilde{g}_n = g_n(x_1, \cdots, x_n)u - g_n(\hat{x}_1, \cdots, \hat{x}_n)u$, $\tilde{\varphi} = \varphi(x_1, \cdots, x_n) - \varphi(\hat{x}_1, \cdots, \hat{x}_n)$.

The proof the Theorem (1) is divided into the following several parts.

*Lemma 6.* (Gauther et al. [1992]). When $\alpha = 1$, for inputs $u$ uniformly bounded by some $u_0 \ge 0$, there exists a large enough $\theta_1 \ge 1$, such that if $\theta \ge \theta_1$, then system (31) is exponentially stable.

*Lemma 7.* For system (31) there exists $\varepsilon_2 \in [1 - \frac{1}{n-1}, 1)$ such that for all $\alpha \in (1-\varepsilon_2, 1]$, the following inequalities hold

$$V_\alpha(e) \le S e_0^T e_0, \ \forall t > 0, \tag{32}$$

and

$$y^T y \le \frac{S}{\delta} e_0^T e_0, \ \forall t > 0, \tag{33}$$

where $V_\alpha(e)$ and $y$ are given by (24), $S = \max_{i,j} |S(1)_{i,j}|$. Moreover, for $i = 2, \cdots, n, k = 1, \cdots, i$, there exists $\theta_2 > 1$ such that if $\theta > \theta_2$, the following inequalities hold

$$\frac{|e_k(t)|^{\frac{1}{\alpha_{i-1}r}}}{\theta^i} \le \frac{|e_k(t)|^{\frac{1}{\alpha_{k-1}r}}}{\theta^k}, \tag{34}$$

**Proof.** Let $d = e_0^T S(\theta)e_0$, $\mathcal{A}' = V_\alpha^{-1}([0,d])$, $\mathcal{S}' = V_1^{-1}(\{d\})$. Let $f'_\alpha$ denote the vector field of system (31). Then, $\mathcal{A}'$ and $\mathcal{S}'$ are compact. Define $\varphi' : (0,d] \times \mathcal{S}' \to R$ by $\varphi'(\alpha, e) = L_{f'_\alpha} V_\alpha(e)$. Then $\varphi'$ is continuous and by Lemma 6 satisfies $\varphi'(1,e) < 0$ for all $e \in \mathcal{S}'$, that is, $\varphi(\{1\} \times \mathcal{S}') \subset (-\infty, 0)$. Since $\mathcal{S}'$ is compact, then there exists $\varepsilon_2 > 0$ such that $\varphi((1-\varepsilon_2, 1] \times \mathcal{S}') \subset (-\infty, 0)$. Thus, for $\alpha \in (1-\varepsilon_2, 1]$, $L_{f'_\alpha} V_\alpha$ takes negative values on $\mathcal{S}'$. Therefore, $\mathcal{A}'$ is strictly positive invariant under $f_\alpha$ for every $\alpha \in (1-\varepsilon_2, 1]$, then,

$$y^T S(\theta) y \le e_0^T S(\theta) e_0, \tag{35}$$

Moreover, Form (19), (20) and (35), we have

$$\delta y^T y \le y^T S(\theta) y \le e_0^T S(\theta) e_0$$
$$= \sum_{i,j} \frac{e_{0i} S(1)_{i,j} e_{0j}}{\theta^{i+j-1}} \le S e_0^T e_0.$$

Thus, the inequalities (32) and (33) hold. If $e_0^T e_0 \le 1$, since $1 \le \frac{1}{r} \le \frac{1}{\alpha_1 r} \le \cdots \le \frac{1}{\alpha_{n-1} r}$ and $\theta \ge 1$, it is obvious that inequalities (34) hold. If $e_k^T(t) e_k(t) > 1$, it follows from (33) that $e(t)$ is bounded. Then, there exists $\theta_2$ such that if $\theta \ge \theta_2$, the inequalities (34) hold.

Now, calculating the derivative of $V_\alpha(e)$ as defined in (24) along the solution of system (31) by noting that $\frac{d}{dt}\lceil e_i \rfloor^{\alpha_i} = \alpha_i |e_i|^{\alpha_i - 1}$ (Hong [2002]), we can obtain

$$\frac{d}{dt} V_\alpha(e)_{(31)} = \frac{d}{dt} V_\alpha(e)_{(23)}$$
$$+2y^T S(\theta) \begin{bmatrix} \frac{1}{r}|e_1|^{\frac{1}{r}-1}\tilde{g}_1 \\ \frac{1}{\alpha_1 r}|e_2|^{\frac{1}{\alpha_1 r}-1}\tilde{g}_2 \\ \vdots \\ \frac{1}{\alpha_{n-1} r}|e_n|^{\frac{1}{\alpha_{n-1} r}-1}(\tilde{g}_n + \tilde{\varphi}) \end{bmatrix}$$
$$\le -c_2(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}} + 2\left[y^T S(\theta) y\right]^{\frac{1}{2}}$$
$$\times \Bigg[\sum_{i,j} \frac{|S(1)_{i,j}|}{\theta^{i+j-1}} \frac{|e_i|^{\frac{1}{\alpha_{i-1} r}-1} L(u_0+1)\sum_{k=1}^i |e_k|}{\alpha_{i-1} r}$$
$$\times \frac{|e_j|^{\frac{1}{\alpha_{j-1} r}-1} L(u_0+1)\sum_{k=1}^j |e_k|}{\alpha_{j-1} r}\Bigg]^{\frac{1}{2}}.$$

By Lemma 2.2 of (Qian and Lin [2001]), there exist constants $\bar{c}_i$ $(1 \le i \le n)$, such that the following inequalities hold.

$$\sum_{k=1}^i |e_i|^{\frac{1}{\alpha_{i-1} r}-1} |e_k|$$
$$\le \sum_{k=1}^i \left[\bar{c}_i |e_i|^{\frac{1}{\alpha_{i-1} r}} + \alpha_{i-1} r \left(\frac{1-\alpha_{i-1} r}{\bar{c}_i}\right)^{\frac{1}{\alpha_{i-1} r}-1} |e_k|^{\frac{1}{\alpha_{i-1} r}}\right]$$
$$\triangleq \sum_{k=1}^i b_{i,k} |e_k|^{\frac{1}{\alpha_{i-1} r}},\ 1 \le i \le n,$$

where $b_{i,k}$ are some positive scalars. Let $b = \max_{i,k} b_{i.k}$. Then,

$$\frac{d}{dt} V_\alpha(e)_{(31)} \le -c_2(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}} + 2bL(u_0+1)S^{\frac{1}{2}}$$
$$\times \left[y^T S(\theta) y\right]^{\frac{1}{2}} \left[\sum_{i,j} \theta \frac{\sum_{k=1}^i |e_k|^{\frac{1}{\alpha_{i-1} r}}}{\alpha_{i-1} r \theta^i} \frac{\sum_{k=1}^j |e_k|^{\frac{1}{\alpha_{j-1} r}}}{\alpha_{j-1} r \theta^j}\right]^{\frac{1}{2}}$$
$$\le -c_2(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}} + \frac{2bL(u_0+1)S^{\frac{1}{2}}}{\alpha_{n-1} r}\left[y^T S(\theta) y\right]^{\frac{1}{2}}$$
$$\times \left[\sum_{i,j} \theta \sum_{k=1}^i \frac{|e_k|^{\frac{1}{\alpha_{k-1} r}}}{\theta^k} \sum_{k=1}^j \frac{|e_k|^{\frac{1}{\alpha_{k-1} r}}}{\theta^k}\right]^{\frac{1}{2}}$$
$$\le -c_2(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}} + \frac{2nbL(u_0+1)S^{\frac{1}{2}}\theta^{\frac{1}{2}}}{\alpha_{n-1} r}[y^T S(\theta)$$
$$y]^{\frac{1}{2}} \times \left[\sum_{i,j}\left(\sum_{k=1}^i \frac{e_k^{\frac{2}{\alpha_{k-1} r}}}{\theta^{2k}}\right)^{\frac{1}{2}}\left(\sum_{k=1}^j \frac{e_k^{\frac{2}{\alpha_{k-1} r}}}{\theta^{2k}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{36}$$

Let $\xi_k = \frac{\lceil e_k \rfloor^{\frac{1}{\alpha_{k-1} r}}}{\theta^k}$, for $\theta \ge \max\{\theta_1, \theta_2\} > 1$, we have

$$\frac{d}{dt} V(\alpha, e)_{(31)} \le -c_2(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}}$$
$$+\frac{2n^2 L(u_0+1) b S^{\frac{1}{2}}\theta^{\frac{1}{2}}}{\alpha_{n-1} r}\left[y^T S(\theta) y\right]^{\frac{1}{2}}\left(\sum_{k=1}^n \xi_k^2\right)^{\frac{1}{2}}. \tag{37}$$

On the other hand, let $\xi = [\xi_1, \xi_2, \cdots, \xi_n]^T$, note that $S(\theta) \ge \delta I$, then,

$$\sum_{k=1}^n \xi_k^2 \le \frac{1}{\delta}\xi^T S(1)\xi = \frac{1}{\theta\delta}\sum_{i,j}\left[\lceil e_i \rfloor^{\frac{1}{\alpha_{i-1} r}} \frac{S(1)_{i,j}}{\theta^{i+j-1}} \lceil e_j \rfloor^{\frac{1}{\alpha_{j-1} r}}\right]_{i,j}$$
$$= \frac{1}{\theta\delta}\sum_{i,j}\left[\lceil e_i \rfloor^{\frac{1}{\alpha_{i-1} r}} S(\theta)_{i,j} \lceil e_j \rfloor^{\frac{1}{\alpha_{j-1} r}}\right]_{i,j} = \frac{1}{\theta\delta} y^T S(\theta) y. \tag{38}$$

It follows from (37) and (38) that

$$\dot{V}(\alpha, e) \le -c_2[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}} + c_3 V_\alpha(e), \tag{39}$$

where $c_3 = \frac{2n^2 L(u_0+1) b S^{\frac{1}{2}}}{\alpha_{n-1} r \delta^{\frac{1}{2}}}$.

Now, we can summarize the proof for our main theorem.

*Proof of Theorem 1:* For any given compact set $U \subset R^n$ containing origin, we can choose $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ such that for all $\alpha \in [1-\varepsilon, 1)$, $c_2(\alpha,\theta)$ satisfies

$$c_2(\alpha,\theta) \ge \frac{\theta}{2}. \tag{40}$$

By (39) and Lemma 1, "the domain of the observer attraction", by an abuse of terminology, is given by

$$\Omega = \left\{e : V_\alpha(e)^{r^2(1-\alpha)} < \frac{c_2}{c_3}\right\}. \tag{41}$$

We can choose sufficiently large $\theta \geq \max\{\theta_1, \theta_2\}$, such that $U \subset \left\{ e : (Se^T e)^{r^2(1-\alpha)} < \frac{c_2}{c_3} \right\}$. By (32) and (41), we have $U \subset \Omega$. Thus, the system (15) admits semi-global finite-time observers.

*Remark 2.* In Theorem (1), if we set $\alpha = 1$, we can obtain the results in (Gauther et al. [1992]).

## 4. CONCLUSION

It is well-known that high gain observers exist for nonlinear systems that are uniformly observable and globally Lipschitzian. Under the same conditions, we showed that these systems admit semi-global and finite-time converging observers. This was achieved with a derivation of a new sufficient condition for local finite-time stability, in conjunction with applications of geometric homogeneity and Lyapunov theories.

## REFERENCES

A. N. Atassi and H.K. Khalil. A separation principle for the stabilization of a class of nonlinear systems. *IEEE Trans. Automat. Contr.*, 44:1672-1687, 1999.

G. Besancon. Nonlinear Observers and Applications. *Lecture Notes in Control and Information Sciences*, vol. 363, 2007.

D. Bestle and M. Zeitz. Cannonical form observer design for non-linear time-variable. *Int. J. of Contr.*, 38:419-431, 1983.

S. Bhat and D. Bernstein. Finite-time stability of continuous autonomous systems. *SIAM J. Control Optim.* 38:751-766, 2000.

S. Bhat and D. Bernstein. Geometric homogeneity with applications to finite-time stability. *Math. Control Signals Systems*, 17:101-127, 2005.

R. Engel and G. Kreisselmeier. A continuous-time observer which converges in finite time. *IEEE Trans. Automat. Contr.*, 47:1202-1204, 2002.

J. P. Gauthier, H. Hammouri and S. Othman. A simple observer for nonlinear systems applications to Bioreactors. *IEEE Trans. Automat. Contr.*. 37:875-880, 1992.

J. P. Gauthier and I. A. K. Kupka. Observability and observers for nonlinear systems. *SIAM J. Control Optim.*, 32:975-994, 1994.

I. Haskara, U. Ozgiiner, and V. Utkin. On sliding mode observers via equivalent control approach. *Int. J. Contr.*, 71:1051-1067, 1998.

R. Hermann and A. J. Krener. Nonlinear controllability and observability. *IEEE Trans. on Automat. Contr.*, 22:728-740, 1977.

Y. Hong. Finite-time stabilization and stabilizability of a class of controllable systems. *Syst. Control Lett.*, 46:231-236, 2002.

A. Isidori. *Nonlinear Control Systems, 3rd ed.*. New York: Springer-Verlag, 1995.

N. Kazantzis and C. Kravaris. Nonlinear observer design using Lyapunov's auxiliary theorem. *Syst. Control Lett.*. 34:241-247, 1998.

H. K. Khalil and F. Esfandiari. Semiglobal stabilization of a class of nonlinear systems using output feedback, *IEEE Trans. Automat. Contr.*, 38:1412-1415, 1993.

S. R. Kou, D. L. Elliott and T. J. Tarn. Exponential observers for nonlinear dynamic systems. *Inform. Control.* 29: 204–216, 1975.

A. J. Krener and A. Isidori. Linearization by output injection and nonlinear observers. *Syst. Control Lett.*. 3:47-52, 1983.

A. J. Krener, Nonlinear stabilizability and detectability, in *Systems and Networks: Mathematical Theory in Nonlinear Control Theory* (Reidel, Dordrecht), 89-98, 1986.

A.J. Kerner and W. Respondek. Nonlinear observers with linear dynamics. *SIAM J. Control Optim.*. 23: 197-216, 1985.

P.H. Menold, R. Findeisen and F. Allgöwer. Finite time convergent observers for linear time-varying systems. *Proceedings of the 11th Mediterranean Conference on Control and Automation (MED'03)*, Rhodos, Greece, T7-078, 2003.

P.H. Menold, R. Findeisen and F. Allgöwer. Finite time convergent observers for nonlinear systems. *Proceedings of the 42nd IEEE Conference on Decision and Control*, Hyatt Regency Maui, Hawaii, USA, 5673-5678, 2003.

H. Michalska and D. Mayne. Moving horizon observers and observer-based control. *IEEE Trans. Automat. Contr*, 40:995-1006, 1995.

E. Moulay, W. Perruquetti and T. Floquet. Finite time observers and secure communication. *preprint*, 2007.

H. Nijmeijer and T. I. Fossen. New directions in nonlinear observer design, Springer, Berlin, *lecture notes in control and information sciences*, Vol. 244, 1999.

C. Qian and W. Lin. Non-lipschitz continuous stabilizer for nonlinear systems with uncontrollable unstable linearization. *Syst. Control Lett.*, 42:185-200, 2001.

S. Raghavan and J. K. Hedrick. Observer design for a class of nonlinear systems. *Int. J. Control*, 59:515-528, 1994.

R. Rajamani, Y. Cho. Existence and design of observers for nonlinear systems: relation to distance to unobservability. *Int. J. Control*, 69:717-731, 1998.

F. Sauvage, M. Guay and D. Dochain. Design of a nonlinear finite-time converging observer for a class of nonlinear systems. *J. of Control Science and Engineering*, 2007:1-9, 2007.

H. Shim, Y. I. Son and J. H. Seo. Semi-global observer for multi-output nonlinear systems. *Syst. Control Lett.*, 42:233-244, 2001.

A. Teel and L. Praly. Global stabilizability and observability imply semi-global stabilizability by output feedback. *Syst. Control Lett.*, 22:313 - 325, 1994.

F.E. Thau. Observing the state of nonlinear dynamic systems, *Int. J. Control*, 18:471-479, 1973.

X-H. Xia and W-B. Gao. Nonlinear observer design by observer error linearization. *SIAM J. Control Optim.*, 27:199-216, 1989.

X-H. Xia and W-B. Gao. On exponential observers for nonlinear systems. *Syst. Control Lett.*, 11:319-325, 1988.

X. Xia and M. Zeitz. On nonlinear continuous observers. *Int. J. Control*, 6:943-954, 1997.

M. Zeitz. The extended Luenberger observer for nonlinear systems. *Syst. Control Lett.*, 9:149-156, 1987.