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A Delay Decomposition Approach to Stability of Linear Neutral Systems ⋆

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Abstract: This paper is concerned with stability of linear neutral systems. Firstly, a *new* approach, *a delay decomposition approach*, is proposed to deal with the stability issue. The idea of the approach is that the delay interval is uniformly divided into N segments with N a positive integer, and a proper Lyapunov-Krasovskii functional is chosen with different weighted matrices corresponding to different segments in the Lyapunov-Krasovskii functional. Secondly, based on the *delay decomposition approach*, some new delay-dependent stability criteria for linear neutral systems are derived. These criteria are much less conservative and include some existing results as their special cases. Finally, numerical examples show that significant improvement using the delay decomposition approach is achieved over some existing method even for coarse delay decomposition.

1. INTRODUCTION

Time-delays are frequently encountered in many fields of science and engineering, including communication network, manufacturing systems, biology, economy and other areas (Hale and Verduyn Lunel, 1993; Gu *et al.*, 2003). During the last two decades, the problem of stability of retarded and neutral type have been the subject of considerable research efforts. Many significant results have been reported in the literature. For the recent progress, the reader is referred to Gu *et al.* (2003) and the references therein.

For the stability issue of time-delay systems, in the existing literature, there are two kinds of Lyapunov-Krasovskii functionals, i.e. *complete* Lyapunov-Krasovskii functionals and *simple* Lyapunov-Krasovskii functionals, for estimating the maximum time-delay bound the system can tolerate and still retain stability. Using the *complete* Lyapunov-Krasovskii functionals (Gu, 2001; Gu *et al.*, 2003), one can obtain the maximum time-delay bound which is very close to the analytical delay limit for stability. Employing the *simple* Lyapunov-Krasovskii functionals usually yields conservative results. However, the results based on the *simple* Lyapunov-Krasovskii functionals can be easily applied to controller synthesis and filter design. Hence, it is still an attractive topic for finding some simple Lyapunov-Krasovskii functionals, by which one can have less conservative results.

In order to introduce a new approach to stability analysis for time-delay systems, we consider the retarded system

$$\dot{x}(t) = Ax(t) + A_d x(t-r), \tag{1}$$

with the continuous vector valued initial condition

$$x(\theta) = \phi(\theta),\ \forall\theta \in [-r, 0]. \tag{2}$$

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For comparison, we use a numerical example for system (1) with

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix},\ \ A_d = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}.$$

The analytical delay limit for stability for the numerical example is calculated as $r_{analytical} = 6.17258$. In the existing literature, in order to derive a delay-dependent stability criterion, one transforms system (1) into a system with a distributed delay, i.e.

$$\dot{x}(t) = (A + A_d)x(t) - A_d \int_{t-r}^{t} [Ax(\xi) + A_d x(\xi - r)]d\xi. \tag{3}$$

Choose a Lyapunov function

$$V(t, x_t) = x^T(t)Px(t),\ P = P^T > 0, \tag{4}$$

and apply Razumikhin Theorem to obtain $r_{max} = 0.9041$ (Gu *et al.*, 2003). As pointed out by Gu *et al.* (2003) (Example 5.3 in Gu *et al.* (2003)), for this example, the stability of system (1) is equivalent to that of system (3). The conservatism of the result is due to the application of the Razumikhin Theorem. For some system (1) with different system's matrices, the model transformation (3) may induce additional dynamics (Gu *et al.*, 2003). To reduce the conservatism, instead of transforming system (1) into (3), one transforms it into

$$\dot{x}(t) = (A + A_d)x(t) - A_d \int_{t-r}^{t} \dot{x}(\xi)d\xi. \tag{5}$$

Then choosing a Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-r}^{t} x^T(\xi)Qx(\xi)d\xi + \int_{-r}^{0}\int_{t+\theta}^{t} x^T(\xi)A_d^T R A_d x(\xi)d\xi d\theta, \tag{6}$$

where $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, and using the bounding technique for some cross term yield $r_{max} = 4.3588$ (Park, 1999). This result was also derived by decomposing delayed term matrix as $A_d = A_{d1} + A_{d2}$ in Han (2002). In Fridman (2001), the author introduced a descriptor transformation

$$\dot{x}(t) = y(t), \quad (7)$$

$$y(t) = (A + A_d)x(t) - A_d \int_{t-r}^{t} y(\xi)d\xi. \quad (8)$$

A Lyapunov-Krasovskii functional was chosen as

$$V(t, x_t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}^T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ P_2 & P_3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \int_{t-r}^{t} x^T(\xi)Qx(\xi)d\xi + \int_{-r}^{0}\int_{t+\theta}^{t} y^T(\xi)A_d^T RA_d y(\xi)d\xi d\theta, \quad (9)$$

where $P_1 = P_1^T > 0$, $Q = Q^T > 0$ and $R = R^T > 0$. Use this model transformation and bounding technique for cross terms (Park, 1999) to obtain $r_{max} = 4.4721$ in Fridman and Shaked (2002). In He *et al.* (2004a,b), the authors introduce some slack variables (free-weighting matrices) to derive the same result.

In Han (2005a), the author avoided using model transformation on system (1) and chose the following Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-r}^{t} x^T(\xi)Qx(\xi)d\xi + \int_{t-r}^{t} (r - t + \xi)\dot{x}^T(\xi)(rR)\dot{x}(\xi)d\xi, \quad (10)$$

which is equivalent to

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-r}^{t} x^T(\xi)Qx(\xi)d\xi + \int_{-r}^{0}\int_{t+\theta}^{t} \dot{x}^T(\xi)(rR)\dot{x}(\xi)d\xi d\theta \quad (11)$$

where $P = P^T > 0$, $Q = Q^T > 0$ and $R = R^T > 0$. Instead of using the bounding technique for some cross term, the author used the following bounding (Han, 2005a)

$$-\int_{t-r}^{t} \dot{x}^T(\xi)(rR)\dot{x}(\xi)d\xi \leq \begin{pmatrix} x(t) \\ x(t-r) \end{pmatrix}^T \begin{pmatrix} -R & R \\ R & -R \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-r) \end{pmatrix} \quad (12)$$

to derive the maximum allowed delay bound as $r_{max} = 4.4721$. Compared with the above mentioned results, the most advantage of the result in Han (2005a) is that the stability condition which was formulated in an LMI form, was very simple and easily applied to controller design, and did only include the Lyapunov-Krasovskii functional matrices variables P, Q and R, which means that no additional matrix variable was involved. From the computation point of view, it is clear to see that testing the result in Han (2005a) is less time-consuming than some existing results in the literature. *However*, the result $r_{max} = 4.4721$ is not close enough to the analytical delay limit for stability $r_{analytical} = 6.17258$ and work needs to be done to arrive at a value much closer to the analytical delay limit for stability. Therefore, the natural question is: How can one improve the result by using *simple* Lyapunov-Krasovskii functionals? Answer to this question will significantly enhance the stability analysis and controller synthesis of time-delay systems. It seems that using the existing *simple* Lyapunov-Krasovskii functionals can not realize the outcome even if one introduces more *additional* matrices variables apart from Lyapunov-Krasovskii functional matrices variables. One way to solve the problem is to choose a *new* Lyapunov-Krasovskii functional. For this purpose, we propose the following *new* simple Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)Px(t) + \sum_{i=1}^{N} \int_{t-ih}^{t-(i-1)h} x^T(\xi)Q_i x(\xi)d\xi + \sum_{i=1}^{N} \int_{-ih}^{-(i-1)h}\int_{t+\theta}^{t} \dot{x}^T(\xi)(hR_i)\dot{x}(\xi)d\xi d\theta \quad (13)$$

where $h = r/N$ and N is the number (a positive integer) of divisions of the interval $[-r, 0]$, and h is the length of each division; and x_t is defined as $x_t = x(t + \theta), \forall\theta \in [-r, 0]$ and $P = P^T > 0$, $Q_i = Q_i^T > 0$, $R_i = R_i^T > 0$ $(i = 1, 2, \cdots, N)$.

It should be pointed out that if $N = 1$, the Lyapunov-Krasovskii functional (13) reduces to (11) employed in Han (2005a) by setting $Q = Q_1$ and $R = R_1$. Based on (13), one can have a delay-dependent stability criterion (see Corollary 5 in this paper). Applying this new criterion, one obtains the maximum allowed delay bound as $r_{max} = 5.7175$ for $N = 2$; $r_{max} = 5.9678$ for $N = 3$; and $r_{max} = 6.0569$ for $N = 4$ and so on, which significantly improve the result $r_{max} = 4.4721$ in the above mentioned references. One can clearly see that we have made very significant steps towards the analytical delay limit for stability of the system.

Since the delay interval $[-r, 0]$ is decomposed into N uniform segments, for the purpose of distinguishing from the existing methods, in what follows, we refer the approach based on the decomposition idea as *a delay decomposition approach*.

In this paper, we will use *a delay decomposition approach* proposed above to study the stability of linear neutral systems. We will derive *new* stability criteria which include some existing results as their special cases. These new stability criteria will be much less conservative than some existing results.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices. The notation $P > 0$ $(P \geq 0)$ means that P is symmetric and positive definite (positive semi-definite). For symmetric matrices P and Q, the notation $P > Q$ $(P \geq Q)$ means that matrix $P - Q$ is positive definite (positive semi-definite). I is an identity matrix of appropriate dimensions. $\lambda_i(A)$ is the ith eigenvalue of a real matrix A. $\|\cdot\|$ stands for the Euclidean vector norm or the induced matrix 2-norm as appropriate. For an arbitrary matrix U and two symmetric matrices P and Q, the symmetric term in a symmetric matrix is denoted by $*$, i.e. $\begin{pmatrix} P & U \\ * & Q \end{pmatrix} = \begin{pmatrix} P & U \\ U^T & Q \end{pmatrix}$.

2. STABILITY OF LINEAR NEUTRAL SYSTEMS

Consider the following linear neutral system

$$\frac{d}{dt}[x(t) - Cx(t-\tau)] = Ax(t) + A_d x(t-r), \quad (14)$$

with

$$x(\theta) = \phi(\theta), \ \forall\theta \in [-\max\{r, \tau\}, 0]. \quad (15)$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system; $r > 0$ is the constant discrete delay and $\tau > 0$ is the constant neutral delay; $\phi(\cdot)$ is a continuous vector valued initial function; $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times n}$ are constant matrices.

Define x_t as $x_t = x(t+\theta), \forall\theta \in [-max\{r,\tau\}, 0]$, and the operator $\mathcal{D}x_t = x(t) - Cx(t-\tau)$. We assume that

Assumption 1. $|\lambda_i(C)| < 1$ ($i = 1, 2, \cdots, n$).

Choose a Lyapunov-Krasovskii functional candidate as

$$\begin{aligned} V(t,x_t) =& (\mathcal{D}x_t)^T P(\mathcal{D}x_t) + \sum_{i=1}^{N}\int_{t-ih}^{t-(i-1)h} x^T(\xi)Q_i x(\xi)d\xi \\ &+\sum_{i=1}^{N}\int_{-ih}^{-(i-1)h}\int_{t+\theta}^{t}\dot{x}^T(\xi)(hR_i)\dot{x}(\xi)d\xi d\theta \\ &+\int_{t-\tau}^{t} x^T(\xi)S_1x(\xi)d\xi + \int_{t-\tau}^{t}\dot{x}^T(\xi)S_2\dot{x}(\xi)d\xi \quad (16) \end{aligned}$$

where $h = r/N$ and N is the number of divisions of the interval $[-r, 0]$, and h is the length of each division; and $P = P^T > 0$, $Q_i = Q_i^T > 0$, $R_i = R_i^T > 0$ ($i = 1, 2, \cdots, N$), $S_j > 0$ ($j = 1, 2$).

We now state and establish the following result.

Proposition 2. Under Assumption 1, for given scalars $r > 0$ and $\tau > 0$, the system described by (14) and (15) is asymptotically stable if there exist real $n\times n$ matrices $P > 0$, $Q_i > 0$, $R_i > 0$ ($i = 1, 2, \cdots, N$) and $S_j > 0$ ($j = 1, 2$) such that

$$\Pi = \begin{pmatrix} \Pi^{(1)} & \Pi^{(2)} & 0 & \Pi^{(3)} & \Pi^{(4)} \\ * & -S_1 & 0 & 0 & 0 \\ * & * & -S_2 & \Pi^{(5)} & C^TS_2 \\ * & * & * & \Pi^{(6)} & 0 \\ * & * & * & * & -S_2 \end{pmatrix} < 0, \quad (17)$$

where

$$\Pi^{(1)} = \begin{pmatrix} \Pi_{11}^{(1)} & R_1 & 0 & \cdots & 0 & PA_d \\ * & \Pi_{22}^{(1)} & R_2 & \cdots & 0 & 0 \\ * & * & \Pi_{33}^{(1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \Pi_{NN}^{(1)} & R_N \\ * & * & * & \cdots & * & \Pi_{N+1\ N+1}^{(1)} \end{pmatrix}$$

with

$$\begin{aligned} \Pi_{11}^{(1)} &\triangleq A^TP + PA + Q_1 - R_1 + S_1, \\ \Pi_{22}^{(1)} &\triangleq -Q_1 - R_1 + Q_2 - R_2, \\ \Pi_{33}^{(1)} &\triangleq -Q_2 - R_2 + Q_3 - R_3, \\ &\vdots \\ \Pi_{NN}^{(1)} &\triangleq -Q_{N-1} - R_{N-1} + Q_N - R_N, \\ \Pi_{N+1\ N+1}^{(1)} &\triangleq -Q_N - R_N, \end{aligned}$$

and

$$\Pi^{(2)} = \begin{pmatrix} -A^TPC \\ 0 \\ 0 \\ \vdots \\ 0 \\ -A_d^TPC \end{pmatrix}, \ \Pi^{(4)} = \begin{pmatrix} -A^TS_2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -A_d^TS_2 \end{pmatrix},$$

$$\Pi^{(3)} = \begin{pmatrix} hA^TR_1 & hA^TR_2 & \cdots & hA^TR_N \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ hA_d^TR_1 & hA_d^TR_2 & \cdots & hA_d^TR_N \end{pmatrix},$$

$$\Pi^{(5)} = \begin{pmatrix} hC^TR_1 & hC^TR_2 & \cdots & hC^TR_N \end{pmatrix},$$

$$\Pi^{(6)} = diag\begin{pmatrix} -R_1 & -R_2 & \cdots & -R_N \end{pmatrix}.$$

In order to prove Proposition 2, we need the following lemma.

Lemma 3. (Han, 2005a) For any constant matrix $W \in \mathbb{R}^{n\times n}$, $W = W^T > 0$, scalar $r > 0$, and vector valued function $\dot{x} : [-r, 0] \to \mathbb{R}^n$ such that the following integration is well defined, then

$$\begin{aligned} &-\int_{t-r}^{t}\dot{x}^T(\xi)(rW)\dot{x}(\xi)d\xi \\ &\le \begin{pmatrix} x^T(t) & x^T(t-r) \end{pmatrix}\begin{pmatrix} -W & W \\ W & -W \end{pmatrix}\begin{pmatrix} x(t) \\ x(t-r) \end{pmatrix}. \end{aligned}$$

Proof of Proposition 2. Taking the derivative of $V(t,x_t)$ with respect to t along the trajectory of (14) yields

$$\begin{aligned} \dot{V}(t,x_t) =& x^T(t)(A^TP + PA + S_1)x(t) \\ &+2x^T(t)PA_dx(t-Nh) - 2x^T(t)A^TPCx(t-\tau) \\ &-2x^T(t-Nh)A_d^TPCx(t-\tau) \\ &-x^T(t-\tau)S_1x(t-\tau) - \dot{x}^T(t-\tau)S_2\dot{x}(t-\tau) \\ &+\sum_{i=1}^{N}x^T(t-(i-1)h)Q_ix(t-(i-1)h) \\ &-\sum_{i=1}^{N}x^T(t-ih)Q_ix(t-ih) \\ &+\sum_{i=1}^{N}\dot{x}^T(t)(h^2R_i)\dot{x}(t) + \dot{x}^T(t)S_2\dot{x}(t) \\ &-\sum_{i=1}^{N}\int_{t-ih}^{t-(i-1)h}\dot{x}^T(\xi)(hR_i)\dot{x}(\xi)d\xi. \end{aligned} \quad (18)$$

Rewrite system (14) as

$$\dot{x}(t) = Ax(t) + Bx(t-Nh) + C\dot{x}(t-\tau). \quad (19)$$

Then we have

$$\dot{x}^T(t)(h^2R_i)\dot{x}(t) = \eta^T(t)\mathcal{F}^T(h^2R_i)\mathcal{F}\eta(t), \quad (20)$$

and

$$\dot{x}^T(t)S_2\dot{x}(t) = \eta^T(t)\mathcal{F}^TS_2\mathcal{F}\eta(t), \quad (21)$$

where

$$\eta^T(t) = \begin{pmatrix} \eta_1^T(t) & \cdots & \eta_2^T(t) & \eta_3^T(t) \end{pmatrix},$$

with

$$\begin{aligned} \eta_1^T(t) &= \begin{pmatrix} x^T(t) & x^T(t-h) & x^T(t-2h) \end{pmatrix}, \\ \eta_2^T(t) &= \begin{pmatrix} x^T(t-(N-1)h) & x^T(t-Nh) \end{pmatrix}, \\ \eta_3^T(t) &= \begin{pmatrix} x^T(t-\tau) & \dot{x}^T(t-\tau) \end{pmatrix}, \end{aligned}$$

and

$$\mathcal{F} = \begin{pmatrix} A & 0 & 0 & \cdots & 0 & A_d & 0 & C \end{pmatrix}.$$

Use Lemma 3 to obtain

$$-\int_{t-ih}^{t-(i-1)h} \dot{x}^T(\xi)(hR_i)\dot{x}(\xi)d\xi \le \zeta^T(t)\begin{pmatrix} -R_i & R_i \\ R_i & -R_i \end{pmatrix}\zeta(t), \quad (22)$$

where $\zeta(t) = \begin{pmatrix} x(t-(i-1)h) \\ x(t-ih) \end{pmatrix}$. Then from (18), (20), (21) and (22), we have

$$\dot{V}(t,x_t) \le \eta^T(t)\begin{pmatrix} \Pi^{(1)} & \Pi^{(2)} & 0 \\ * & -S_1 & 0 \\ * & * & -S_2 \end{pmatrix}\eta(t) + \eta^T(t)\left(\sum_{i=1}^{N} \digamma^T(h^2R_i)\digamma + \digamma^T S_2 \digamma\right)\eta(t). \quad (23)$$

According to Schur complement, (17) is equivalent to $\begin{pmatrix} \Pi^{(1)} & \Pi^{(2)} & 0 \\ * & -S_1 & 0 \\ * & * & -S_2 \end{pmatrix} + \sum_{i=1}^{N} \digamma^T(h^2R_i)\digamma + \digamma^T S_2 \digamma < 0$. Therefore, there exists a scalar $\lambda_1 > 0$ such that $\dot{V}(t,x_t) \le -\lambda_1 x^T(t)x(t) < 0$ for $x(t) \ne 0$. Notice that Assumption 1 guarantees the operator $\mathcal{D}x_t$ is stable. Hence, by Theorem 8.1 (pp. 292-293, Hale and Verduyn Lunel (1993)), the system described by (14) and (15) is asymptotically stable. This completes the proof. □

If $N = 1$, then (16) becomes the Lyapunov-Krasovskii functional employed in Han (2005b)

$$V^{(1)}(t,x_t) = (\mathcal{D}x_t)^T P(\mathcal{D}x_t) + \int_{t-r}^{t} x^T(\xi)Q_1x(\xi)d\xi + \int_{-r}^{0}\int_{t+\theta}^{t} \dot{x}^T(\xi)(rR_1)\dot{x}(\xi)d\xi d\theta + \int_{t-\tau}^{t} x^T(\xi)S_1x(\xi)d\xi + \int_{t-\tau}^{t} \dot{x}^T(\xi)S_2\dot{x}(\xi)d\xi$$

By Proposition 2 we have the following corollary.

Corollary 4. (Han, 2005b) Under Assumption 1, for given scalars $r > 0$ and $\tau > 0$, the system described by (14) and (15) is asymptotically stable if there exist real $n \times n$ matrices $P > 0$, $Q_1 > 0$, $R_1 > 0$ and $S_j > 0$ $(j = 1, 2)$ such that

$$\begin{pmatrix} (1,1) & (1,2) & -A^TPC & 0 & rA^TR_1 & A^TS_2 \\ * & (2,2) & -A_d^TPC & 0 & rA_d^TR_1 & A_d^TS_2 \\ * & * & -S_1 & 0 & 0 & 0 \\ * & * & * & -S_2 & rC^TR_1 & C^TS_2 \\ * & * & * & * & -R_1 & 0 \\ * & * & * & * & * & -S_2 \end{pmatrix} < 0, \quad (24)$$

where

$$(1,1) \triangleq A^TP + PA + S_1 + Q_1 - R_1,$$
$$(1,2) \triangleq PA_d + R_1, \ (2,2) \triangleq -Q_1 - R_1.$$

If $\tau = 0$, then system (14) becomes system (1). The following corollary is immediately implied by using Proposition 2.

Corollary 5. For a given scalar $r > 0$, the system described by (1) and (2) is asymptotically stable if there exist real $n \times n$ matrices $P > 0$, $Q_i > 0$, $R_i > 0$ $(i = 1, 2, \cdots, N)$ such that

$$\Xi = \begin{pmatrix} \Xi^{(1)} & \Xi^{(2)} \\ \Xi^{(2)T} & \Xi^{(3)} \end{pmatrix} < 0, \quad (25)$$

where

$$\Xi^{(1)} = \begin{pmatrix} \Xi_{11}^{(1)} & R_1 & 0 & \cdots & 0 & PA_d \\ * & \Xi_{22}^{(1)} & R_2 & \cdots & 0 & 0 \\ * & * & \Xi_{33}^{(1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \Xi_{NN}^{(1)} & R_N \\ * & * & * & \cdots & * & \Xi_{N+1\ N+1}^{(1)} \end{pmatrix}$$

with $\Xi_{11}^{(1)} = \Pi_{11}^{(1)} - S_1$, $\Xi_{ii}^{(1)} = \Pi_{ii}^{(1)}$ $(i = 2, \cdots, N+1)$, and

$$\Xi^{(2)} = \begin{pmatrix} hA^TR_1 & hA^TR_2 & \cdots & hA^TR_{N-1} & hA^TR_N \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ hA_d^TR_1 & hA_d^TR_2 & \cdots & hA_d^TR_{N-1} & hA_d^TR_N \end{pmatrix},$$

$$\Xi^{(3)} = diag\,(\,-R_1\ -R_2\ \cdots\ -R_{N-1}\ -R_N\,).$$

When $\tau = r$, system (14) becomes

$$\frac{d}{dt}[x(t) - Cx(t-r)] = Ax(t) + Bx(t-r), \quad (26)$$

with

$$x(\theta) = \phi(\theta), \ \forall\theta \in [-r, 0]. \quad (27)$$

The corresponding Lyapunov-Krasovskii functional candidate is

$$\tilde{V}(t,x_t) = (\tilde{\mathcal{D}}x_t)^T P(\tilde{\mathcal{D}}x_t) + \sum_{i=1}^{N}\int_{t-ih}^{t-(i-1)h} x^T(\xi)Q_ix(\xi)d\xi + \sum_{i=1}^{N}\int_{-ih}^{-(i-1)h}\int_{t+\theta}^{t} \dot{x}^T(\xi)(hR_i)\dot{x}(\xi)d\xi d\theta + \int_{t-Nh}^{t} \dot{x}^T(\xi)S_3\dot{x}(\xi)d\xi \quad (28)$$

where $\tilde{\mathcal{D}}x_t = x(t) - Cx(t-Nh)$. Then we have the following result.

Proposition 6. Under Assumption 1, for a given scalar $r > 0$, the system described by (26) and (27) is asymptotically stable if there exist real $n \times n$ matrices $P > 0$, $Q_i > 0$, $R_i > 0$ $(i = 1, 2, \cdots, N)$ and $S_3 > 0$ such that

$$\Omega = \begin{pmatrix} \Omega^{(1)} & \Omega^{(2)} \\ \Omega^{(1)T} & \Omega^{(3)} \end{pmatrix} < 0, \quad (29)$$

where

$$\Omega^{(1)} = \begin{pmatrix} \Omega_{11}^{(1)} & R_1 & 0 & \cdots & 0 & PA_d - A^TPC & 0 \\ * & \Omega_{22}^{(1)} & R_2 & \cdots & 0 & 0 & 0 \\ * & * & \Omega_{33}^{(1)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \Omega_{NN}^{(1)} & R_N & 0 \\ * & * & * & \cdots & * & \Omega_{N+1\ N+1}^{(1)} & 0 \\ * & * & * & \cdots & * & * & -S_3 \end{pmatrix},$$

with $\Omega_{11}^{(1)} = \Pi_{11}^{(1)} - S_1$, $\Omega_{ii}^{(1)} = \Pi_{ii}^{(1)}$ $(i = 2, \cdots, N)$ and $\Omega_{N+1\ N+1}^{(1)} = \Pi_{N+1\ N+1}^{(1)} - A_d^TPC - C^TPA_d$, and

$$\Omega^{(2)} = \begin{pmatrix} hA^TR_1 & hA^TR_2 & \cdots & hA^TR_N & A^TS_3 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ hA_d^TR_1 & hA_d^TR_2 & \cdots & hA_d^TR_N & A_d^TS_3 \\ hC^TR_1 & hC^TR_2 & \cdots & hC^TR_N & C^TS_3 \end{pmatrix},$$

$$\Omega^{(3)} = diag\,(\,-R_1\;-R_2\;\cdots\;-R_N\;-S_3\,)\,.$$

Notice that system (14) can be rewritten as

$$\dot{x}(t) = Ax(t) + Bx(t-r) + C\dot{x}(t-\tau). \tag{30}$$

For this form, choosing a Lyapunov-Krasovskii functional candidate as

$$\hat{V}(t,x_t) = x^T(t)Px(t) + \sum_{i=1}^{N}\int_{t-ih}^{t-(i-1)h} x^T(\xi)Q_i x(\xi)d\xi + \sum_{i=1}^{N}\int_{-ih}^{-(i-1)h}\int_{t+\theta}^{t}\dot{x}^T(\xi)(hR_i)\dot{x}(\xi)d\xi d\theta + \int_{t-\tau}^{t}\dot{x}^T(\xi)S_4\dot{x}(\xi)d\xi \tag{31}$$

we can conclude that

Proposition 7. For given scalars $r > 0$ and $\tau > 0$, the system (30) is asymptotically stable if there exist real $n\times n$ matrices $P > 0$, $Q_i > 0$, $R_i > 0$ $(i = 1, 2, \cdots, N)$ and $S_4 > 0$ such that

$$\Phi = \begin{pmatrix} \Phi^{(1)} & \Phi^{(2)} \\ \Phi^{(2)T} & \Phi^{(3)} \end{pmatrix} < 0, \tag{32}$$

where

$$\Phi^{(1)} = \begin{pmatrix} \Phi^{(1)}_{11} & R_1 & 0 & \cdots & 0 & PA_d & PC \\ * & \Phi^{(1)}_{22} & R_2 & \cdots & 0 & 0 & 0 \\ * & * & \Phi^{(1)}_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \Phi^{(1)}_{NN} & R_N & 0 \\ * & * & * & \cdots & * & \Phi^{(1)}_{N+1\;N+1} & 0 \\ * & * & * & \cdots & * & * & -S_4 \end{pmatrix}$$

with $\Phi^{(1)}_{11} = \Pi^{(1)}_{11} - S_1$, $\Phi^{(1)}_{ii} = \Pi^{(1)}_{ii}$ $(i = 2, \cdots, N+1)$, and

$$\Phi^{(2)} = \begin{pmatrix} hA^TR_1 & hA^TR_2 & \cdots & hA^TR_N & A^TS_4 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ hA_d^TR_1 & hA_d^TR_2 & \cdots & hA_d^TR_N & A_d^TS_4 \\ hC^TR_1 & hC^TR_2 & \cdots & hC^TR_N & C^TS_4 \end{pmatrix}$$

$$\Phi^{(3)} = diag\,(\,-R_1\;-R_2\;\cdots\;-R_N\;-S_4\,)\,.$$

Remark 8. Notice that for $N = 1$, the Lyapunov-Krasovskii functionals (28) and (31) for systems (26) and (30) reduce to the ones used in Han (2005b). Similar to Corollary 4, we can have the corresponding results which recover the results in Han (2005b).

Remark 9. Similar to Han (2005b), one can address robust stability issue for neutral systems subject to parameter uncertainty. Due to page limit, it is omitted.

3. EXAMPLES

In order to show significant improvements over some existing results in the literature, we employ a Partial Element Equivalent Circuit (PEEC) model, which is recurring an increasingly important role in many practical applications, especially in combined electromagnetic and circuit analysis. Figure 1 shows a small PEEC model for metal strip (Bellen *et al.*, 1999), in which there are delayed circuit elements, that is, the partial inductances of the form $Lp_{ij}\frac{d}{dt}(i_j(t-r))$ are coupled by the retarded dependent current sources of the form $p_{ij}/p_{ii}i_{cj}(t-r)$. This model usually results in a neutral delay differential equation. The mathematical model for the PEEC shown in Figure 1 can be given as

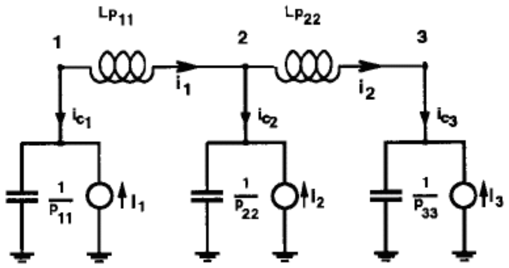

Fig. 1. Small PEEC model for metal strip

$$\begin{cases} C_0\dot{y}(t) + G_0y(t) + C_1\dot{y}(t-r) + G_1y(t-r) \\ \qquad\qquad = Bu(t, t-r),\; t \ge t_0 \\ \qquad y(t) = g(t), \qquad\quad t \le t_0 \end{cases} \tag{33}$$

where C_0 is a diagonal matrix. The associated delay differential equation of neutral type is

$$\begin{cases} \dfrac{d}{dt}[y(t) - Cy(t-r)] = Ay(t) + A_dy(t-r),\; t \ge t_0 \\ \qquad\qquad y(t) = g(t),\; t \le t_0 \end{cases} \tag{34}$$

Consider the system (34) with

$$A = 100\times\begin{pmatrix} \beta & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{pmatrix},\; A_d = 100\times\begin{pmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{pmatrix},$$

$$C = \frac{1}{72}\times\begin{pmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{pmatrix}.$$

For $\beta = -7$, Bellen *et al.* (1999) studied the stability problem of the system and concluded that the system is asymptotically stable. However, the result is independent of the delay. In Han (2005c), the author claimed that the system is asymptotically stable independent of the delay for $\beta \le -2.106$.

Applying the Proposition 6 in this paper and the criteria in Han (2005c) and Yue and Han (2004), Table 1 lists the maximum allowed time-delay $r_{\max}$ for asymptotic stability for different β. From this table, one can clearly see that for $N = 1$, the results in Han (2005c) are recovered; for $N \ge 2$, the results using Proposition 6 can provide much less conservative results than the criteria in Han (2005c); Yue and Han (2004).

Table 1. Comparison of r_{max} using different methods

Method	$\beta = -2.105$	$\beta = -2.103$	$\beta = -2.1$
Han (2005c)	1.0874	0.3709	0.2433
Yue and Han (2004)	1.1413	0.3892	0.2553
Proposition 6 ($N = 1$)	1.0874	0.3709	0.2433
Proposition 6 ($N = 2$)	1.5318	0.5185	0.3381
Proposition 6 ($N = 3$)	1.6238	0.5490	0.3577
Proposition 6 ($N = 4$)	1.6567	0.5599	0.3647

We now consider another example to show significant improvements over some existing results in the literature.

Example 10. Consider the system (26) with

$$A = \begin{pmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{pmatrix}, \; A_d = \begin{pmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{pmatrix},$$
$$C = \begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix}.$$

The analytical delay limit for stability can be calculated as $r_{analytical} = 2.2255$. Using some existing criteria and Proposition 6 in this paper the maximum allowed time-delay $r_{\max}$ is compared in Table 2 . It is clear to see that for this example the criterion in this paper can provide much less conservative results than some existing ones mentioned in Table 2. From this table, one can also see that as N increases, the results converge to the analytical delay limit for stability.

Table 2. Comparison of $r_{\max}$ using different methods

Methods	$r_{\max}$
Lien *et al.* (2000)	0.3
Chen *et al.* (2001)	0.5658
Fridman (2001)	0.74
Lien and Chen (2003)	0.8844
Han (2004)	1.61
He *et al.* (2004a)	1.6527
Fridman and Shaked (2002)	1.7191
Han (2005b)	1.7858
Proposition 6 ($N = 1$)	1.7858
Proposition 6 ($N = 2$)	2.1077
Proposition 6 ($N = 3$)	2.1708
Proposition 6 ($N = 4$)	2.1932
Proposition 6 ($N = 5$)	2.2036
Proposition 6 ($N = 6$)	2.2093
Proposition 6 ($N = 7$)	2.2121
Proposition 6 ($N = 8$)	2.2149
Proposition 6 ($N = 9$)	2.2164
Proposition 6 ($N = 10$)	2.2175
Proposition 6 ($N = 15$)	2.2201
Proposition 6 ($N = 20$)	2.2210

4. CONCLUSION

We have considered the problem of stability of linear neutral systems. The main contribution of this paper is that we have proposed a *delay decomposition approach* to address the problem. The delay decomposition approach has opened a door in the area of time-delay systems how to use simple Lyapunov-Krasovskii functionals to derive much less conservative results for stability analysis and controller design. For a stability issue, employing the delay decomposition approach, we have obtained some new stability criteria which are much less conservative than some existing ones using simple Lyapunov-Krasovskii functionals. The stability limit can be approached with fine delay decomposition.

It should be pointed out that one can use the delay decomposition approach to address controller design and filtering for time-delay systems. Due to page limit, we have left these issues in other papers.

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