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Composite Disturbance-Observer-Based Control and Terminal Sliding Mode Control for Uncertain Structural Systems

Xinjiang Wei * **Huifeng Zhang** ** **Lei Guo** ***

* *Southeastern University, Nanjing, 210096 CHINA (e-mail: weixinjiang@ 163.com).*
** *Ludong University, Yantai, 264025 CHINA (e-mail: zhanghuifeng624@ 163.com)*
*** *Southeastern University, Nanjing, 210096 CHINA (e-mail: l.guo@seu.edu.cn)*

Abstract: A new type of composite control scheme of disturbance-observer-based control and terminal sliding mode control is proposed for uncertain structural systems. The disturbance are supposed to include two parts. One is generated by an exogenous system, which can represent the harmonic signals with modeling perturbations in structural system. The other part is external excitation in H_2-norm context. By combining the disturbance observer with terminal sliding mode control law, the disturbance with the exogenous system can be estimated and compensated, and external excitation can be attenuated in finite time which can be computed by our approach. Especially,the design of disturbance observer and controller can be obtained separately. Finally, simulations for a four-degree-of-freedom building model excited by 1940 El Centro earthquake excitation are given to demonstrate the effectiveness of the approach and compare the proposed results with the previous schemes in accuracy.

1. INTRODUCTION

Since Yao first proposed the concept of structural control for civil engineering applications(See Yao [1972]), considerable progress has been made to reduce effects of undesirable external forces due to extreme events such as earthquakes and strong winds. Among noteworthy progress, feedback control has been widely used in active control of vibration, for which numeral control schemes have been provided based on optimal and robust control, neural network and fuzzy control, nonlinear and adaptive control (See Isidori[1985]; Pirrotta[2006]; Wang[2007]). Their main objective is to use appropriate control input implemented in practical applications to attenuate the effects of structural vibration.

However, studies for structural systems are still insufficient due to their special system structure. Up to now, there are several obstacles in the previous works on vibration control theory and application. One of the main obstacles in applications is the existence of the modeling errors, variations of materials, component nonlinearities, and changing load environments in most structural plants. It has been shown that the model perturbations and exogenous disturbances may cause instability or degradation of a structural system. In such cases, robustness and stability of a control system as well as its performance toward attenuation of external disturbance is important.

Another shortcomings in the existing literature is the absence of higher-accuracy controller. For structural system, it is a significant thing to design the high-accuracy control against the disturbance and perturbation to improve control performance. However, few higher-accuracy controller are proposed in the existing literature owing to absence of the information of disturbance . To improve the ability of anti-disturbance, disturbance-observer-based control (DOBC) strategies have been studied in the late of 1980s and applied in many control areas. Recent development can be seen in (Wang[2004]; Guo [1997]; Guo[2001]). It is shown that linear DOBC strategies have been effectively applied to many mechanical plants (Guo [1997]; Chen[2004]). Instead of the linear DOBC approaches, for a nonlinear system, a nonlinear DOBC law can improve the performance and robustness greatly against noises and unmodelled dynamics(See Guo[2005]; Chen[2004]). A survey on nonlinear DOBC has been presented in (Guo[2001]). In (Guo[2005]), the DOBC approaches for a class of multiple-input-multiple-output (MIMO) nonlinear systems have been considered, where the disturbances were represented by a linear exogenous system, which is not required to be neutral stable as in the output regulation theory.This extended the assumptions of the disturbances, which were limited to be constant, harmonic or neutral stable in (Chen[2004]). However, it has been reported that when the disturbance also has perturbations, the proposed approaches in DOBC (See Guo[2005];Chen[2004]) are unsatisfactory, which has also been verified by the simulations in (Guo[2005]).

For the above purposes, composite DOBC and terminal sliding mode control (TSMC) for structural systems is proposed in this paper. The disturbance includes two parts, One part is the external excitation in H_2-norm context.

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In many cases, system disturbance can be considered as a dynamic system with unknown parameters and initial conditions (See Guo[2005]). So the other part is supposed to be generated by an exogenous system. A novel composite control scheme is presented such that the disturbance with the exogenous system can be estimated and compensated, the external excitation can be attenuated in finite time by TSM control law. A reduced-order observer is structured for the estimation of the disturbance.To most important, the setting time is estimated by our approach. Especially, the disturbance observer and controller design can be obtained separately. Simulations are given to demonstrate the advantages of the proposed combined control scheme in accuracy.

2. PROBLEM STATEMENT

Consider the following structural system with nonlinear uncertainties described by

$$M\ddot{d}(t) + C\dot{d}(t) + Kd(t) = U(t) + \omega_0(t) + B_\omega \omega_1(t) \quad (1)$$

where $d(t) \in R^n$ is the displacement, $U(t)$ is the control input, $\omega_0(t)$ is is supposed to be described by an exogenous system, which can represent the constant and the harmonic noises, and $\omega_1(t)$ is the external disturbance or excitation in the H_2-norm. $M \in R^{n\times n}$, $C \in R^{n\times n}$ and $K \in R^{n\times n}$ are the mass, damping and stiffness matrices respectively. $B_\omega \in R^{n\times p}$ is the disturbance matrix. By defining $\dot{x}(t) = \left[\, d^T(t) \;\; \dot{d}^T(t) \,\right]^T$, Eq.(1) can be formulated as

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \dot{x}(t) = & \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} x(t) + \\ & \begin{bmatrix} 0 \\ I \end{bmatrix} (U + \omega_0(t) + B_\omega \omega_1(t)) \end{aligned} \quad (2)$$

and further be transformed to give

$$\dot{x}(t) = Ax(t) + B(U + \omega_0(t) + B_\omega \omega_1(t)) \quad (3)$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix},$$

Assumption A1
The friction force disturbance $\omega_0(t)$ in the control input path can be formulated by the following exogenous system

$$\dot{\xi}(t) = W\xi(t), \qquad \omega_0(t) = V\xi(t)$$

where W and V are known matrices. In many cases, system disturbance can be described as a dynamic system with unknown parameters and initial conditions(See Guo[2005]). Many kinds of disturbances in engineering can be described by this model, for example, unknown constant and harmonics with unknown phase and magnitude (Isidori[1985]). Similarly to (Guo[2005]), the following assumption is a necessary condition for the DOBC formulation.
Assumption A2
(A, B) is controllable and (W, BV) is observable.

In this section, the basic definition of TSM can be briefly summarized as follows

Definition 1 The terminal sliding mode (TSM)can be described by the following first-order nonlinear differential equations

$$s = \dot{x} + \beta|x|^\gamma sign(x) = 0$$

respectively, where $x \in R,\ \beta > 0,\ 0 < \gamma < 1$.
Definition 2 The so-called nonsingular TSM (Feng [2002]) can be expressed as

$$s = x + \beta|\dot{x}|^\gamma sign(\dot{x}) = 0$$

where $x \in R,\ \beta > 0,\ 1 < \gamma < 2$.
TSM function of Definition 2 is continuous and differentiable although the absolute value and sign operators are involved. Its first derivative can be expressed as (Yua[2005])

$$\dot{s} = \dot{x} + \beta\gamma|\dot{x}|^{\gamma-1}\ddot{x} \quad (4)$$

Moreover, the following notions are introduced for simplicity of expression and used in the analysis and design of the TSM controllers.

$$\begin{aligned} sig(x)^\gamma &= [|x_1|^{\gamma_1} sign(x_1), ..., |x_n|^{\gamma_n} sign(x_n)]^T; \\ x^\gamma &= [x_1^{\gamma_1}, ..., x_n^{\gamma_n}]^T; \;\; |x|^\gamma = [|x_1|^{\gamma_1}, ..., |x_n|^{\gamma_n}]^T \end{aligned} \quad (5)$$

where

$$x = [\, x_1 \; x_2 \; \dots \; x_n \,]^T; \qquad \gamma = [\, \gamma_1 \; \gamma_2 \; \dots \; \gamma_n \,]^T$$

3. COMPOSITE CONTROLLER WITH DISTURBANCE OBSERVER AND TERMINAL SLIDING MODE CONTROL(TSMC)

3.1 the design of Disturbance Observer

Suppose that the system state (3) are available. If $\omega(t)$ is considered as a part of the augmented state, then a reduced-order observer is needed. In this paper, we construct the following reduced-order observer for $\omega_0(t)$.

$$\hat{\omega}_0(t) = V\hat{\xi}(t), \quad \hat{\xi}(t) = \nu(t) - Lx(t) \quad (6)$$

where $\nu(t)$ is the auxiliary variable generated by

$$\dot{\nu}(t) = (W + LBV)(\nu(t) - Lx(t)) + L(Ax(t) + BU(t)) \quad (7)$$

The estimation error is denoted as $e_\xi = \xi(t) - \hat{\xi}(t)$, Combining plant (3), Assumption A1 with equation (6),(7) yields

$$\begin{aligned} \dot{e_\xi}(t) &= (W + LBV)e_\xi(t) + LBB_\omega \omega_1(t) \\ z(t) &= e_\xi. \end{aligned} \quad (8)$$

where $z(t)$ is the controlled output. Based on regional pole placement and H_∞theory, the following result can be obtained.
Theorem 1. For parameters $\theta_i, i = 1, 2, ..., n$, suppose that there exist the Lyapunov matrix $P > 0$ and matrix T, satisfying

$$\min \lambda < 0 \quad (9)$$

$$\begin{bmatrix} PW + W^TP + TBV + V^TB^TT^T + I & TBB_\omega \\ B_\omega^T B^T T^T & -\lambda^2 I \end{bmatrix} < 0 \quad (10)$$

$$PW + W^TP + TBV + V^TB^TT^T + \Theta P < 0 \quad (11)$$

where $L = P^{-1}T, \Theta = diag(\theta_i)$ such that the error system (8) is asymptotically stable in the absence of $\omega_1(t)$ and satisfies $\|z(t)\|_2 \leq \lambda\|w_1(t)\|_2$, where λ is a given positive constant for the level of disturbance attenuation.

Proof See Appendix 1.
Remark 1: The objective of disturbance rejection can be achieved by designing the observer gain such that (8) satisfies the desired stability and robustness performance.

3.2 Composite Controller

For structural system (1), the terminal sliding mode (TSM) can be defined as

$$s = d(t) + \beta sig(\dot{d}(t))^{\gamma} = 0 \tag{12}$$

where $s = [s_1, ..., s_n]^T \in R^n,\ \beta = diag(\beta_1, ..., \beta_n)$, $\beta_i > 0,\ (i = 1, ...n),\ 1 < \gamma_1, ..., \gamma_n < 2$. A fast-TSM-type reaching law is defined as

$$\dot{s} = -K_1 s - K_2 sig(s)^{\rho} \tag{13}$$

where $K_1 = diag[k_{11}, ..., k_{1n}],\ K_2 = diag[k_{21}, ..., k_{2n}]$, $k_{1i}, k_{2i} > 0,\ (i = 1, ...n),\ 0 < \rho < 1$.
Lemma 1. Suppose $a_1, a_2, ..., a_n$ and $0 < p < 2$ are all positive numbers, then the following inequality holds

$$(a_1^2 + a_2^2 + ... + a_n^2)^p \leq (a_1^p + a_2^p + ... + a_n^p)^2$$

Theorem 2. For structural system (1) under Assumptions A1 and A2, if the TSM manifold is chosen as (12), the reaching law is as (13) and the continuous TSM controller based on disturbance observer is designed as

$$\begin{aligned} U &= U_0 + U_1 + U_2 \\ U_0 &= C\dot{d}(t) + Kd(t) - M\beta^{-1}\gamma^{-1} sig(\dot{d}(t))^{2-\gamma} \\ U_1 &= -\hat{\omega_0}; \\ U_2 &= -M(K_1 s + K_2 sig(s)^{\rho}) \end{aligned} \tag{14}$$

where $\gamma = diag(\gamma_1, ..., \gamma_n)$, then we have
a. closed-loop system composed by (1),(6),(14)is asymptotically stable in the absence of $\omega_1(t)$.
b. the trajectory of system state will converge to the neighborhood of TSM $s = 0$ as

$$\begin{aligned} &\|s\| < \Delta = min(\Delta_1, \Delta_2) \\ &|d_i(t)| \leq 2\Delta_d = \Delta, \quad |\dot{d}_i(t)| \leq \Delta_{\dot{d}_i} = (\frac{\Delta}{\beta_i})^{1/\gamma_i} \end{aligned} \tag{15}$$

in finite time T , where

$$\begin{aligned} \Delta_1 &= \frac{(||\lambda M^{-1}V|| + ||M^{-1}B_{\omega}||)||\omega_1||}{k_1}, \\ \Delta_2 &= (\frac{(||\lambda M^{-1}V|| + ||M^{-1}B_{\omega}||)||\omega_1||}{k_2})^{1/\rho} \end{aligned} \tag{16}$$

$k_1 = min_i(k_{1i}) > 0$ and $k_2 = min_i(k_{2i}) > 0$ represent the minimum eigenvalues of $K_1$and K_2, respectively. The setting time T is given as,

$$T \leq \frac{1}{\underline{k}_1(1-\rho)} \ln \frac{\underline{k}_1 V^{(1-\rho)/2} + 2^{(1-\rho)/2}\underline{k}_2}{2^{(1-\rho)/2}\underline{k}_2} \tag{17}$$

where $\underline{k}_1 = min_i(\underline{k}_{1i}) > 0$ and $\underline{k}_2 = min_i(\underline{k}_{2i}) > 0$ represent the minimum eigenvalues of $\underline{K}_1, \underline{K}_2$ respectively, and

$$\begin{aligned} \underline{K}_1 &= \beta\gamma diag(|\dot{d}(t)|^{\gamma-1})[K_1 - diag(M^{-1}(\omega(t) - \hat{\omega}(t) + \\ &\quad B_{\omega}\omega_1))diag^{-1}(s)] \\ \underline{K}_2 &= \beta\gamma diag(|\dot{d}(t)|^{\gamma-1})K_2 \end{aligned}$$

Proof See Appendix 2.
Remark 2: The disturbance observer and controller design can be obtained separately.
(i) Based on regional pole placement and H_{∞}theory, compute the observer gain L by theory 1.
(ii) Based on the estimation of disturbance observer, construct combined control law in theory 2 with TSMC.
Remark 3: The setting time in which the system trajectory will converge to the neighborhood of TSM is given in proposed approach, which is a very significant thing in vibration control theory and application.
Remark 4: The control law (14) is continuous and therefore is chattering-free. It does not involve any negative fractional power, hence it is also singularity-free.

4. SIMULATION EXAMPLE

This section gives a numerical example to demonstrate the applicability of the proposed approach in Section 3. The robust stability of the steady state motion of an uncertain four-degree-of-freedom building model is considered. The basic structural system has been used in Refs (See Du[2004]) for vibrational control simulation. Where $x_i, m_i, c_i, k_i, i = 1, ..., 4$ are the relative displacement, mass, damping and stiffness of each storey, respectively, and $m_1 = m_2 = 2 * 1.05(10^6 kg);\ m_3 = m_3 = 1.05(10^6 kg);\ k_1 = k_2 = 2 * 350(10^6 N/m);\ k_3 = k_4 = 350(10^6 N/m);\ c_1 = c_2 = c_3 = c_4 = 1.575(10^6 Ns/m)$. The dynamic equation of the system is given as in Eq. (1) with system matrices given by

$$M = \begin{bmatrix} 2.1 & 0 & 0 & 0 \\ 0 & 2.1 & 0 & 0 \\ 0 & 0 & 1.05 & 0 \\ 0 & 0 & 0 & 1.05 \end{bmatrix} * 10^6; B_w = \begin{bmatrix} 2.1 \\ 2.1 \\ 1.05 \\ 1.05 \end{bmatrix} * 10^6$$

$$K = \begin{bmatrix} 1.4 & -0.7 & 0 & 0 \\ -0.7 & 1.05 & -0.35 & 0 \\ 0 & -0.35 & 0.7 & -0.35 \\ 0 & 0 & -0.35 & 0.35 \end{bmatrix} * (10^9 N/m);$$

$$C = \begin{bmatrix} 3.15 & -1.575 & 0 & 0 \\ -1.575 & 3.15 & -1.575 & 0 \\ 0 & -1.575 & 3.15 & -1.57 \\ 0 & 0 & -1.575 & 3.15 \end{bmatrix} * (10^6 Ns/m);$$

The external disturbance corresponds to 1940 El Centro earthquake excitation force given in Ref. (See Wang[2001]). $\omega_0(t)$ is dry friction force of the contact surface between two solids (see Wang[2007]), assumed to be an unknown disturbance described by Assumption A1 with

$$W = \begin{bmatrix} 0 & 6 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ 3 & 0 & -8 & 0 \\ 1 & 0 & 0 & -6 \end{bmatrix}; \quad V = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} * 10^4.$$

In simulation, we select

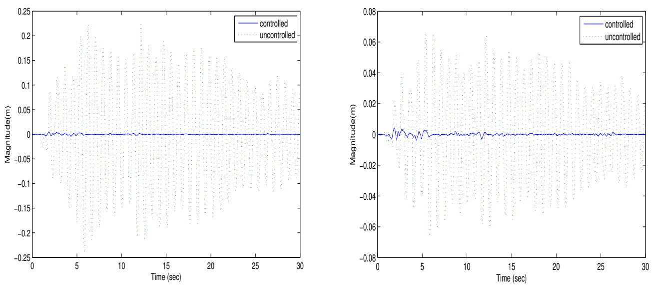

Fig 1,(a) Uncontrolled and Controlled displacements of the fourth storey with DOBC+TSMC method (b) Uncontrolled and Controlled displacements displacements of the first storey with DOBC+TSMC method.

$$\Theta = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}; \quad \beta = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}; \quad K_1 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix};$$

$$K_2 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}; \quad \gamma = 1.5; \quad \rho = 0.5;$$

Based on Theorem 1, it can be solved that

$$P = \begin{bmatrix} 0.0988 & -0.0630 & -0.2417 & -0.1178 \\ -0.063 & 0.0508 & 0.1636 & 0.0914 \\ -0.2417 & 0.1636 & 0.9258 & -0.0172 \\ -0.1178 & 0.0914 & -0.0172 & 1.2282 \end{bmatrix} * 10^8;$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & -431.9288 & 26.5993 & 195.2961 & 210.0333 \\ 0 & 0 & 0 & 0 & -342.3032 & -227.2587 & 300.9874 & 268.5745 \\ 0 & 0 & 0 & 0 & -54.2162 & 48.8866 & -2.2926 & 7.6221 \\ 0 & 0 & 0 & 0 & -17.1827 & 20.5443 & -3.7212 & 0.3596 \end{bmatrix}$$

When DOBC+TSMC is used to control the vibration system, the uncontrolled and controlled displacements of the fourth storey and the first storey are shown in Fig. 1(a),(b) respectively. Figures 2 shows Uncontrolled and Controlled displacements of the storey with DOBC+TSMC method and -25% parametric perturbations. Figures 3 demonstrates Uncontrolled and Controlled displacements of the storey with DOBC+TSMC method and +50% parametric perturbations. Figures 4 (a) shows the responses of error dynamics of the disturbance observer, and illustrates our disturbance observer can estimate the unknown disturbances very well in a short time. The trajectory of continuous TSM and TSMC+DOBC are shown in Figure 4 (b)and Figures 5, respectively.

Fig. 1 illustrates although there are exogenous disturbances in the system, the DOBC+TSMC controller can still attenuate the vibration due to the seismic disturbance very well. Moreover, Figure 1 depicts that the displacements of the fourth storey and the first storey will converge to the small regions shown by Figure 4(b) in 2.12 second, which testify the validity and effectiveness of theorem 2. Figures 2 and 3 show that the satisfying robustness performance can also be guaranteed with DOBC+TSMC method and parametric perturbations

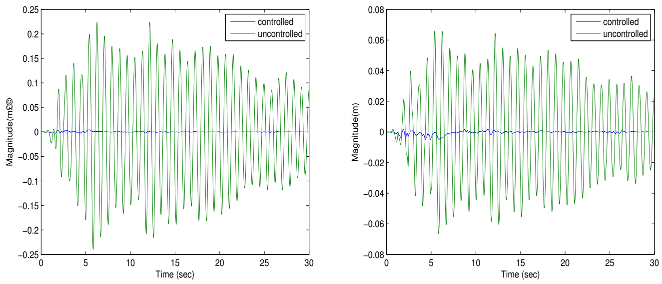

Fig 2,(a) Uncontrolled and Controlled displacements of the fourth storey with DOBC+TSMC method and -25% parametric perturbations. (b)Uncontrolled and Controlled displacements of the first storey with DOBC+TSMC method and -25% parametric perturbations.

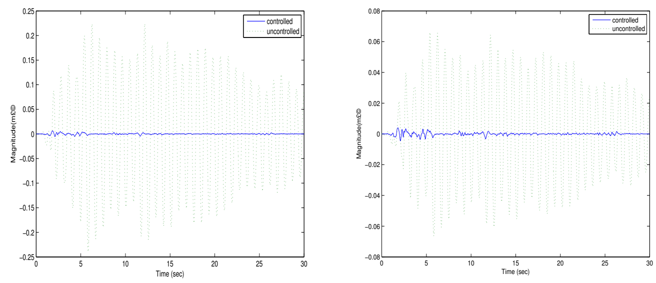

Fig 3,(a) Uncontrolled and Controlled displacements of the fourth storey with DOBC+TSMC method and +50% parametric perturbations. (b)Uncontrolled and Controlled displacements of the first storey with DOBC+TSMC method and +50% parametric perturbations.

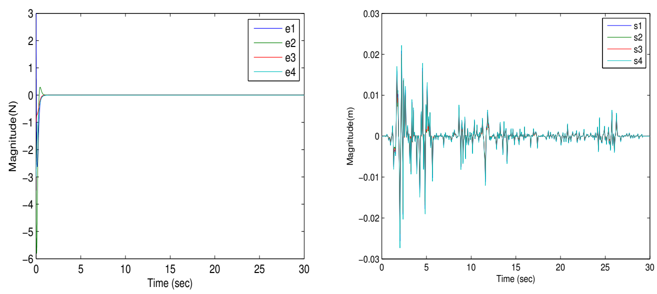

Fig 4,(a) the responses of error dynamics of the disturbance observer (b)the trajectory of Continuous TSM

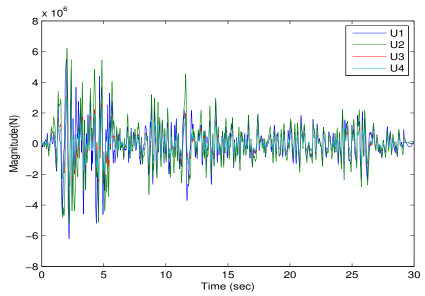

Fig 5,the trajectory of TSMC+DOBC

5. CONCLUSIONS

The DOBC and TSMC are effective robust control schemes against the external disturbances and the modeling perturbations. DOBC has been used to reject the influence of the disturbance with some known information. TSMC is completely robust against load variations and external disturbances. In this paper, a new type of control scheme combined DOBC and TSMC is proposed for structural systems with exogenous disturbances. The unknown external disturbances are supposed to include two parts. One is generated by an exogenous system, which can represent the harmonic signals with modeling perturbations. The other part is external excitation. By combining the disturbance observers with TSMC law, the disturbance with the exogenous system can be estimated and compensated, earthquake excitation can be attenuated by TSMC law. Simulations show that with the proposed integrated control scheme the system performance can be improved well against the external disturbances and the modeling perturbations.

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Appendix 1: Proof of theorem 1

Proof: Consider the following Lyapunov function

$$V(t) = e_\xi^T(t)Pe_\xi(t) \tag{18}$$

In order to testify that system (8) is asymptotically stable with a disturbance attenuation$\lambda > 0$, it is required that the associated function

$$J(t) = \int_0^\infty [z^T(t)z(t) - \lambda^2 w_1^T(t)w_1(t) + \dot{V}(t)]$$

Defining $H(t) = z^T(t)z(t) - \lambda^2 w_1^T(t)w_1(t) + \dot{V}(t)$, then we obtain that

$$\begin{aligned} H(t) &= z^T(t)z(t) - \lambda^2 w_1^T(t)w_1(t) + e_\xi^T(t)P\dot{e}_\xi(t) \\ &\quad + \dot{e}_\xi^T(t)Pe_\xi^T(t) \\ &= z^T(t)z(t) - \lambda^2 w_1^T(t)w_1(t) + e_\xi^T(t)P[(W+LBV)e_\xi(t) \\ &+LBB_\omega\omega_1(t)] \\ &+[(W+LBV)e_\xi(t) + LBB_\omega\omega_1(t)]^T Pe_\xi(t) \\ &= [e_\xi^T \;\; \omega_1^T] \begin{bmatrix} \Upsilon & PLBB_\omega \\ B_\omega^T B^T L^T P & -\lambda^2 I \end{bmatrix} \begin{bmatrix} e_\xi \\ \omega_1 \end{bmatrix} \end{aligned} \tag{19}$$

Where $\Upsilon = P(W+LBV) + (W+LBV)^T P + I$, under the zero initial condition, it can be seen that

$$J(t) = \int_0^\infty (z^T(t)z(t) - \lambda^2 w_1^T(t)w_1(t))dt$$

It is shown that $J(t) < 0$, and $\|z(t)\|_2 \le \lambda\|w_1(t)\|_2$ holds, if $H(t) < 0$. Based on (19) after letting $T = PL$, we can get (9,10). According to regional pole placement and D-stability theory(See Yu[2002]), choose regional pole Θ, we have (11). This completes the proof.

Appendix 2: Proof of theorem 2

Proof. Consider the Lyapunov function

$$V(t) = ||s||^2/2.$$

By differentiating $V(t)$ with respect to time, substituting (4),(13),(14) into it, and using Lemma 1, we have

$$\begin{aligned} \dot{V}(t) &= s^T\dot{s} = s^T[\dot{d(t)} + \beta\gamma diag(|\dot{d}(t)|^{\gamma-1})\ddot{d}(t)] \\ &= s^T[\dot{d}(t) + \beta\gamma diag(|\dot{d}(t)|^{\gamma-1})M^{-1}(U(t)+\omega(t) \\ &+B_\omega\omega_1 - C\dot{d}(t) - Kd(t))] \\ &= s^T[\dot{d}(t) + \beta\gamma diag(|\dot{d}(t)|^{\gamma-1})M^{-1}(U_2+\omega(t) \end{aligned}$$

$$
\begin{aligned}
&+B_\omega\omega_1 - \hat{\omega}(t) - M\beta^{-1}\gamma^{-1}sig(\dot{d}(t))^{2-\gamma})] \\
&= s^T[\dot{d}(t) + \beta\gamma diag(|\dot{d}(t)|^{\gamma-1})M^{-1}(U_2 + \omega(t) \\
&+B_\omega\omega_1 - \hat{\omega}(t)) - diag(|\dot{d}(t)|^{\gamma-1})sig(\dot{d}(t))^{2-\gamma}] \\
&= s^T[\dot{d}(t) + \beta\gamma diag(|\dot{d}(t)|^{\gamma-1})M^{-1}(U_2 + \omega(t) \\
&+B_\omega\omega_1 - \hat{\omega}(t)) - |\dot{d}(t)|sign(\dot{d}(t))] \\
&= s^T\beta\gamma diag(|\dot{d}(t)|^{\gamma-1})M^{-1}(U_2 + \omega(t) \\
&+B_\omega\omega_1 - \hat{\omega}(t)) \\
&= -s^T\beta\gamma diag(|\dot{d}(t)|^{\gamma-1})[K_1 s + K_2 sig(s)^\rho \\
&+M^{-1}(\omega(t) + B_\omega\omega_1 - \hat{\omega}(t))] \qquad (20)
\end{aligned}
$$

which can be further changed into the following two forms:

$$
\begin{aligned}
&\dot{V}(t) = -s^T\beta\gamma diag(|\dot{d}(t)|^{\gamma-1})[(K_1 - diag(M^{-1}(\omega(t) \\
&-\hat{\omega}(t) + B_\omega\omega_1))diag^{-1}(s))s + K_2 sig(s)^\rho] \qquad (21) \\
&\dot{V}(t) = -s^T\beta\gamma diag(|\dot{d}(t)|^{\gamma-1})[K_1 s + (K_2 - diag(M^{-1} \\
&(\omega(t) - \hat{\omega}(t) + B_\omega\omega_1))diag^{-1}(sig(s)^\rho))sig(s)^\rho] \qquad (22)
\end{aligned}
$$

For Eq. (21), if we can keep the matrix $K_1 - diag(M^{-1}(\omega(t) - \hat{\omega}(t) + B_\omega\omega_1))diag^{-1}(s)$ positive definite,then have

$$\dot{V} \le -s^T\underline{K}_1 s - s^T\underline{K}_2 sig(s)^\rho \qquad (23)$$

Where $\underline{K}_1$, $\underline{K}_2$ are denoted in Theorem 2. Obviously $\underline{K}_1, \underline{K}_2$are positive definite diagonal matrices if any $\dot{d}(t) \neq 0$. From Lemma 1, we have

$$\dot{V} \le -2\underline{k}_1 V - 2^{(\rho+1)/2}\underline{k}_2 V^{(\rho+1)/2} \qquad (24)$$

Where $\underline{k}_1$, $\underline{k}_2$ are also denoted in Theorem 2. According to the finite-time stability criterion(See Feng[2002]), TSM (12) will be reached in the finite time. The setting time is given

$$T \le \frac{1}{\underline{k}_1(1-\rho)} \ln((\underline{k}_1 V^{(1-\rho)/2} + 2^{(1-\rho)/2}\underline{k}_2)/2^{(1-\rho)/2}\underline{k}_2)$$

In addition, if keep the matrix $K_1 - diag(M^{-1}(\omega(t) - \hat{\omega}(t)))diag^{-1}(s)$ positive definite, then yields

$$|s_i| \le \frac{|[M^{-1}(\omega(t) - \hat{\omega}(t) + B_\omega\omega_1)]_i|}{k_1}$$

and

$$
\begin{aligned}
||s|| &\le \frac{||M^{-1}(\omega(t) - \hat{\omega}(t) + B_\omega\omega_1)||}{k_1} \\
&\le \frac{||M^{-1}(\omega(t) - \hat{\omega}(t)|| + ||M^{-1}B_\omega\omega_1)||}{k_1} \\
&\le \frac{(||\lambda M^{-1}V|| + ||M^{-1}B_\omega||)||\omega_1||}{k_1} = \Delta_1 \qquad (25)
\end{aligned}
$$

For Eq. (22), by similar analysis for (21) and Lemma 1, we can have that the region

$$|s_i|^\rho \le \frac{|[M^{-1}(\omega(t) - \hat{\omega}(t) + B_\omega\omega_1)]_i|}{k_2}$$

and

$$
\begin{aligned}
||s|| &\le (\frac{||M^{-1}(\omega(t) - \hat{\omega}(t) + B_\omega\omega_1)||}{k_2})^{1/\rho} \\
&\le (\frac{||M^{-1}(\omega(t) - \hat{\omega}(t)|| + ||M^{-1}B_\omega\omega_1)||}{k_2})^{1/\rho} \\
&\le (\frac{(||\lambda M^{-1}V|| + ||M^{-1}B_\omega||)||\omega_1||}{k_2})^{1/\rho} = \Delta_2 \qquad (26)
\end{aligned}
$$

Therefore, based on (25),(26),we have $||s|| < \Delta = min(\Delta_1, \Delta_2)$.
Moreover, because$||s|| \le \Delta$ means $|s_i| \le \Delta, i = 1, ...n$, then

$$d_i + \beta_i|\dot{d}_i|^{\gamma_i} sign(\dot{d}_i) = \varphi_i, \quad |\varphi_i| \le \Delta \qquad (27)$$

Eq. (27) can be rewritten as

$$d_i + (\beta_i - \frac{\varphi_i}{|\dot{d}_i|^{\gamma_i} sign(\dot{d}_i)})|\dot{d}_i|^{\gamma_i} sign(\dot{d}_i) = 0. \qquad (28)$$

Then when $\beta_i - \frac{\varphi_i}{|\dot{d}_i|^{\gamma_i} sign(\dot{d}_i)} > 0$, Eq. (28) is still kept in the form of TSM, which also means that the $\dot{d}(t)$ will converge to the region

$$\dot{d}_i \le (\frac{\Delta}{\beta_i})^{1/\gamma_i} = \Delta_{\dot{d}_i}$$

in finite time. Furthermore, with the TSM dynamics (27), the $d(t)$ will converge to the region

$$|d_i| \le \beta_i|\dot{d}_i|^{\gamma_i} + |\varphi_i| \le \Delta + \Delta = 2\Delta = \Delta_d.$$

in finite time. This completes the proof.