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Hopf bifurcations in normal forms of third order nonlinear affine control systems ⋆

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Abstract: The paper investigates Hopf bifurcations in a class of *simple* nonlinear systems, i.e., third order affine control systems described in terms of "quadratic plus cubic" normal forms and subject to linear state feedback control laws. By employing Harmonic Balance (HB) tools, the set of system parameters corresponding to supercritical and subcritical bifurcations is analytically determined. Also, a second order harmonic approximation of the bifurcated periodic solution is provided. Such analytical results can be exploited as starting points to investigate complex behaviours of the considered class of simple nonlinear systems.

Keywords: Hopf bifurcation, harmonic balance, normal forms.

1. INTRODUCTION

Recent years have witnessed a strong interest in discovering third order nonlinear systems with very few nonlinear terms which are able to display rich dynamical behaviours (see, e.g., Eichhorn et al. [1998], Sprott and Linz [2000], Sprott [2003], Zhou and Chen [2006], Yang et al. [2006]). It is believed that these *simple* systems play a key role for a deeper understanding of the elementary mechanisms which are at the basis of the birth of complex behaviours. The standard approach to single out simple systems consists in selecting a parameterized family of systems with a fixed structure and employing numerical tools for parameter bifurcation analysis. Unfortunately, it is well known that these tools are not well suited for the analysis of nonlinear systems when many free parameters are involved. To this respect, the knowledge of suitable starting points in the parameter space appears of fundamental importance. Obviously, these starting points should correspond to behaviours which are close to complex ones and, most importantly, should be determined analytically.
A first effort to locate starting points has been provided in Innocenti et al. [2006, 2008], where families of third order nonlinear systems, described by differential equations involving only a quadratic term in addition to the linear ones, are considered. For these systems an analytical characterization of supercritical Hopf bifurcations has been provided in the system parameter space via Harmonic Balance (HB) techniques (see, e.g., Mees [1981], Moiola and Chen [1996]). Also, approximations of the bifurcated periodic solutions have been singled out, to be used as starting points for a more complete bifurcation analysis.

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The aim of the present paper is to extend the analysis in Innocenti et al. [2006] to a more general setting. Specifically, we consider the "quadratic plus cubic" normal form of third order nonlinear affine control systems (Kang and Krener [1992], Kang [1994]) subject to linear state feedback control laws. The resulting state space model involves a quadratic term and three cubic terms in addition to the linear ones and depends on seven free parameters. It is first shown that these systems admit a feedback representation composed by a linear block in the forward path and an explicit nonlinearity in the feedback path. Subsequently, employing HB techniques, a general procedure to analytically determine the system parameters which activate supercritical Hopf bifurcations in the seven-dimensional parameter space is derived, together with second order harmonic approximations of the bifurcated periodic solutions.
The paper is organized as follows. The quadratic plus cubic normal form of third order nonlinear affine control systems is introduced in Section 2. Section 3 contains the equivalent feedback block representation and some background on the Hopf theorem in the HB setting. Section 4 provides the complete characterization of supercritical/subcritical Hopf bifurcations via HB. An illustrative example is discussed in Section 5, while some brief comments are drawn in Section 6.

Notation
$\mathbb{R}$: real space;
$\mathbb{C}$: complex space;
$\jmath$: imaginary unit;
$\Re[x]$: real part of $x \in \mathbb{C}$;
$\Im[x]$: imaginary part of $x \in \mathbb{C}$;
D: $\mathrm{d}/\mathrm{d}t$ operator;
$^\top$: the transpose operator.

2. PROBLEM FORMULATION AND PRELIMINARIES RESULTS

Consider the third order nonlinear systems described by the state space equations

$$\begin{cases} \dot{x}_1 = x_2 + a x_3^2 + (b_1 x_1 + b_2 x_2 + b_3 x_3) x_3^2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{cases} \tag{1}$$

where $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ is the state vector, $u \in \mathbb{R}$ is the control input, and $a \in \mathbb{R}$, $b_i \in \mathbb{R}$, $i = 1, 2, 3$, are scalar parameters. For the sake of simplicity, we assume hereafter $a \neq 0$ and $\prod_{1=1}^{3} b_i \neq 0$, to avoid the pure quadratic or cubic special cases, which due to space limitation are not dealt with in this paper.

We are interested in characterizing Hopf bifurcations of the equilibrium point at $x = 0_3$ once (1) is subject to a (negative) linear state feedback control law, i.e.

$$u = -c_1 x_1 - c_2 x_2 - c_3 x_3 , \quad c_i \in \mathbb{R}, \, i = 1, 2, 3 . \tag{2}$$

System (1) is exactly the quadratic plus cubic normal form of third order nonlinear control systems which are affine in the control input, i.e.

$$\dot{x} = f^{[1]}(x) + f^{[2]}(x) + \ldots + u \left(g^{[0]}(x) + g^{[1]}(x) + \ldots \right) ,$$

where $f^{[i]}$ and $g^{[j]}$ are homogeneous polynomials in $x \in \mathbb{R}^3$, respectively of orders i and j (Kang and Krener [1992], Kang [1994]). Thus, Hopf analysis of system (1)-(2) can provide bifurcation conditions for quite general classes of nonlinear systems subject to linear state feedback control laws. In the following we will provide analytical tools to study the properties of the limit cycles activated by means of (2).

In order to investigate Hopf bifurcations, we first write system (1)-(2) in the following parametrized form

$$\dot{x} = \mathcal{A}(\mu) x + a(\mu) f^{[2]}(x) + b^\top(\mu) x f^{[2]}(x) \tag{3}$$

where:

$$\mathcal{A}(\mu) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_1(\mu) & -c_2(\mu) & -c_3(\mu) \end{bmatrix} , \tag{4}$$

$$b(\mu) = \begin{bmatrix} b_1(\mu) \\ b_2(\mu) \\ b_3(\mu) \end{bmatrix} , \quad f^{[2]}(x) = \begin{bmatrix} x_3^2 \\ 0 \\ 0 \end{bmatrix} ,$$

being $\mu \in \mathbb{R}$ the so-called scalar bifurcation parameter.

Without loss of generality, let us suppose the bifurcation happens at $\mu = 0$ and assume that the system parameters admit the Taylor series expansion, i.e.

$$\begin{cases} a(\mu) = a_0 + a_1 \mu + O(\mu^2) \\ b_i(\mu) = b_{i0} + b_{i1} \mu + O(\mu^2) & i = 1, 2, 3 \\ c_j(\mu) = c_{j0} + c_{j1} \mu + O(\mu^2) & j = 1, 2, 3. \end{cases} \tag{5}$$

Consider the eigenvalues of matrix $\mathcal{A}(\mu)$ which clearly define the local stability properties of the equilibrium point at $x = 0_3$. According to the Hopf Theorem and its so-called "transversality condition" (Farkas [1994], Marsden and McCracken [1976]), a couple of complex conjugate eigenvalues of the equilibrium must cross the imaginary axis as μ crosses 0, while the third eigenvalue must remain negative real during such a crossing. In other words, the equilibrium point must be asymptotically stable on one side of the range of the parameter μ. Without loss of generality, we can suppose that the equilibrium point at $x = 0_3$ of (3) is asymptotically stable for $\mu < 0$.

It is not difficult to verify that the above stability assumption and the transversality condition amount to impose the following constraints on the coefficients in (5)

$$\begin{cases} c_{10} > 0 , \quad c_{30} > 0 \\ c_{20} c_{30} - c_{10} = 0 \\ c_{11} - c_{20} c_{31} - c_{30} c_{21} > 0 . \end{cases} \tag{6}$$

In particular, it turns out that the eigenvalues of $\mathcal{A}(\mu)$ at the bifurcation point are

$$-c_{30} , \quad \jmath \sqrt{c_{20}} , \quad -\jmath \sqrt{c_{20}} ,$$

and, furthermore, that:

$$\exists \, \hat{\mu} > 0 \, : \, c_1(\mu) \neq 0 \quad \forall \, \mu \in (-\hat{\mu}, \hat{\mu}) . \tag{7}$$

Clearly, the eigenvalues analysis is not sufficient to study the supercritical or subcritical nature of the bifurcation. A possible way to assess the stability of the bifurcation is to compute the bifurcation stability coefficient via the algorithm proposed in Howard [1979] (see also Hassard et al. [1981], Fu and Abed [1993], Kuznetsov [1995]). On the contrary, we will follow a different approach based on the frequency version of the Hopf Theorem which is related to the HB techniques (see Mees [1981], Allwright [1977], Moiola and Chen [1996]). Besides the well-known power of the HB approach for classical control of feedback systems, our choice is motivated by the observation that such approach can be easily exploited to provide a local approximation of the related bifurcated periodic solution. This approximation may be particularly useful as a starting point for a more complete bifurcation analysis via standard numerical tools.

3. FEEDBACK BLOCK REPRESENTATION AND HARMONIC BALANCE

Our approach requires first the transformation of the system from the state space model (3) to a feedback block representation form, which is instrumental for applying the HB method.

To this purpose, we first transform system (3) into a differential equation form.

Proposition 1. Suppose constraints (6) hold and let $\hat{\mu}$ be defined as in (7). Then, for each $\mu \in (-\hat{\mu}, \hat{\mu})$, system (3) admits the differential equation form:

$$\begin{aligned} \dddot{y} + c_3(\mu) \ddot{y} + c_2(\mu) \dot{y} + c_1(\mu) y &= -a(\mu) c_1(\mu) \dot{y}^2 + \\ &- (b_2(\mu) c_1(\mu) - b_1(\mu) c_2(\mu)) y \dot{y}^2 + \\ &- (b_3(\mu) c_1(\mu) - b_1(\mu) c_3(\mu)) \dot{y}^3 + b_1(\mu) \ddot{y} \dot{y}^2 . \end{aligned} \tag{8}$$

Proof. Since $\det \mathcal{A}(\mu) = -c_1(\mu) \neq 0$, it turns out that for each $\mu \in (-\hat{\mu}, \hat{\mu})$ $\mathcal{A}(\mu)$ is invertible. Under the coordinate change imposed by the linear transformation $x = \mathcal{A}^{-1}(\mu) z$, system (3) boils down to:

$$\begin{aligned} \dot{z} &= \mathcal{A}(\mu) z + \left(a + b^\top \mathcal{A}^{-1}(\mu) z \right) \mathcal{A}(\mu) f^{[2]} \left(\mathcal{A}^{-1}(\mu) z \right) = \\ &= \mathcal{A}(\mu) z + h(z) \end{aligned} \tag{9}$$

where

$$h(z) = \begin{bmatrix} 0 \\ 0 \\ -a(\mu) c_1(\mu) z_2^2 + \\ -(b_2(\mu) c_1(\mu) - b_1(\mu) c_2(\mu)) z_1 z_2^2 + \\ -(b_3(\mu) c_1(\mu) - b_1(\mu) c_3(\mu)) z_2^3 + b_1(\mu) z_3 z_2^2 \end{bmatrix} . \tag{10}$$

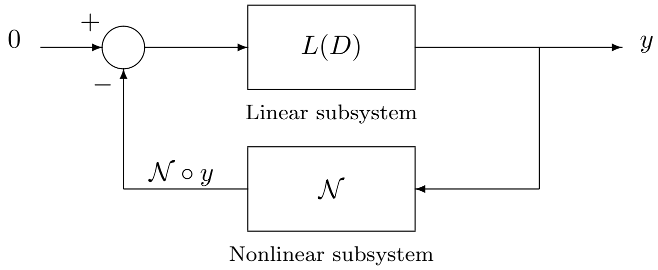

Fig. 1. Block diagram representation of the differential equation system (15).

Since the explicit state space form of (9) is given by

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = -c_1(\mu)z_1 - c_2(\mu)z_2 - c_3(\mu)z_3 - a(\mu)c_1(\mu)z_2^2 + \\ \quad -(b_2(\mu)c_1(\mu) - b_1(\mu)c_2(\mu))z_1 z_2^2 + \\ \quad -(b_3(\mu)c_1(\mu) - b_1(\mu)c_3(\mu))z_2^3 + b_1(\mu)z_3 z_2^2 \, , \end{cases} \tag{11}$$

it is straightforward to derive (8) from (11) once y is chosen as $y = z_1$, and thus $y = x_2$ in the original variables. □

It is well known that differential equations of the form (8) can be represented in the feedback block diagram of Fig. 1, where the linear subsystem is described by a linear time-invariant operator $L(D)$ and the nonlinear subsystem by a scalar time-invariant nonlinear operator $\mathcal{N}$. Hence, the next result pertains to system (3).

Proposition 2. Suppose constraints (6) hold and let $\hat{\mu}$ be defined as in (7). Then, for each $\mu \in (-\hat{\mu}, +\hat{\mu})$, system (3) admits the feedback block diagram of Fig. 1, once

$$L(D) = \frac{1}{D^3 + c_3(\mu)D^2 + c_2(\mu)D + c_1(\mu)} \doteq L_\mu(D) \tag{12}$$

and the nonlinear operator $\mathcal{N}$ is such that:

$$\begin{aligned} \mathcal{N} \circ y &= a(\mu)c_1(\mu)\,(Dy)^2 + \\ &+ (b_2(\mu)c_1(\mu) - b_1(\mu)c_2(\mu))\,(Dy)^2\, y + \\ &+ (b_3(\mu)c_1(\mu) - b_1(\mu)c_3(\mu))\,(Dy)^3 + \\ &- b_1(\mu)\,(Dy)^2\,(D^2 y) \doteq n_\mu(y, Dy, D^2 y) \, . \end{aligned} \tag{13}$$

Proof. Consider the differential equation form (8). Introducing the derivative operator D, it can be rewritten into the form:

$$\begin{aligned} y = &-\frac{1}{(D^3 + c_3(\mu)D^2 + c_2(\mu)D + c_1(\mu))} \cdot \\ &\cdot \Big(a(\mu)c_1(\mu)\,(Dy)^2 + \\ &+ (b_2(\mu)c_1(\mu) - b_1(\mu)c_2(\mu))\,(Dy)^2\, y + \\ &+ (b_3(\mu)c_1(\mu) - b_1(\mu)c_3(\mu))\,(Dy)^3 + \\ &- b_1(\mu)\,(Dy)^2\,(D^2 y) \Big) \, . \end{aligned} \tag{14}$$

Then, exploiting the linear operator (12) and the nonlinear one in (13), we finally obtain:

$$y = -L_\mu(D) \cdot n_\mu(y, Dy, D^2 y) = -L(D) \cdot (\mathcal{N} \circ y) \tag{15}$$

which exactly describes the feedback block interconnection of Fig. 1. □

The remaining part of this section briefly summarizes the HB approach to Hopf bifurcations in systems which admit the block diagram representation of Fig. 1.

Such an approach relates the solution of the so-called second order HB problem with the supercritical or subcritical nature of the bifurcation. More specifically, consider the following prototype periodic solution of period $2\pi/\omega$:

$$\begin{aligned} y_p(t) &= A + B\cos(\omega t) + P\cos(2\omega t) + Q\sin(2\omega t) = \\ &= \Re\left[A + Be^{\jmath\omega t} + (P - \jmath Q)\, e^{\jmath 2\omega t}\right] \end{aligned} \tag{16}$$

with $B \neq 0$, $P^2 + Q^2 \neq 0$.

For any μ the output of the nonlinear subsystem corresponding to (16) is given by $n_\mu(y_p(t), Dy_p(t), D^2 y_p(t))$, whose Fourier development can be written by using polar notation as

$$\begin{aligned} &n_\mu(y_p(t), Dy_p(t), D^2 y_p(t)) = \\ &= \Re\left[N_0 A + N_1 B e^{\jmath\omega t} + N_2\,(P - \jmath Q)\, e^{\jmath 2\omega t}\right] + \Delta_\mu y(t) \, , \end{aligned} \tag{17}$$

where $\Delta_\mu y(t)$ contains the higher harmonics and

$$\begin{aligned} N_0 &= N_0(A, B, P, Q, \omega; \mu) \in \mathbb{R} \\ N_1 &= N_1(A, B, P, Q, \omega; \mu) \in \mathbb{C} \\ N_2 &= N_2(A, B, P, Q, \omega; \mu) \in \mathbb{C} \end{aligned}$$

can be computed analytically.

The periodic output of the linear subsystem driven by the signal in (17) is given by

$$\begin{aligned} &-\Re\left[L_\mu(0)N_0 A + L_\mu(j\omega)N_1 B e^{\jmath\omega t} + \right. \\ &\left. + L_\mu(j2\omega)N_2\,(P - \jmath Q)\, e^{\jmath 2\omega t}\right] + \hat{\Delta}_\mu y(t), \end{aligned} \tag{18}$$

where $L_\mu(D)$ is as in (12) and $\hat{\Delta}_\mu y(t)$ contains the higher harmonics.

Then, balancing the terms up to the second harmonic between (16) and (18), thus neglecting $\hat{\Delta}_\mu y(t)$, we arrive at the second order HB problem:

$$\begin{cases} A = -L_0 N_0 A & (19) \\ B = -L_1 N_1 B & (20) \\ (P - \jmath Q) = -L_2 N_2 (P - \jmath Q), & (21) \end{cases}$$

where $L_0 \doteq L_\mu(0) \in \mathbb{R}$, $L_1 \doteq L_\mu(j\omega) \in \mathbb{C}$, $L_2 \doteq L_\mu(j2\omega) \in \mathbb{C}$.

Note that (19) is a real equation related to the continuous component of (16), while (20) and (21) are complex and are due to the first and second order harmonics, respectively. Hence, it follows that the second order HB problem consists of five scalar equations in the five unknowns (A, B, P, Q, ω), for each value of the bifurcation parameter μ.

Next, we summarize known results on the relation between Hopf bifurcation and the solution of the second order HB problem (19)-(21). Such rigorous results are based on the intuitive idea that the limit cycle arising from the bifurcation initially has a small distortion, so that it can be approximated by $y_p(t)$ for small values of μ (see for details Mees [1981], Allwright [1977]).

Proposition 3. Suppose constraints (6) hold and let $\hat{\mu}$ be defined as in (7). The equilibrium at $x = 0_3$ of (3) undergoes a Hopf bifurcation for $\mu = 0$ if and only if there exists $\bar{\mu} \in \mathbb{R} \, : \, 0 < \bar{\mu} \leq \hat{\mu}$ such that the solution of second order HB problem (19)-(21) exists and it is defined either $\forall \mu \in (0, \bar{\mu})$ (supercritical case) or $\forall \mu \in (-\bar{\mu}, 0)$ (subcritical case).

Remark 4. The solution of the second order HB problem provides the prototype periodic solution

$$\begin{aligned} y_p(t) &= A(\mu) + B(\mu)\cos(\omega(\mu)t) + \\ &+ P(\mu)\cos(2\omega(\mu)t) + Q(\mu)\sin(2\omega(\mu)t), \end{aligned} \tag{22}$$

which is, for any μ sufficiently small, a strict approximation of the real limit cycle arising from the Hopf bifurcation. Moreover, when $\mu \to 0$ such a solution collapses to the equilibrium point at $x = 0_3$, according to the following conditions:

$$\begin{cases} A(\mu) \to 0 \\ B(\mu) \to 0 \\ P(\mu) \to 0 \\ Q(\mu) \to 0 \\ \omega(\mu) \to \sqrt{c_{20}} \quad . \end{cases} \tag{23}$$

4. MAIN RESULTS

According to Proposition 3, Hopf bifurcations of the equilibrium point at $x = 0_3$ of (3) are completely characterized once the second order HB problem (19)-(21) is solved. Hence, let us consider the real equation (19) which via some straightforward computations boils down to:

$$\begin{aligned} &- a(\mu)c_1(\mu)\left(\frac{1}{2}B^2 + 2P^2 + 2Q^2\right)\omega^2 + \\ &- (b_2(\mu)c_1(\mu) - b_1(\mu)c_2(\mu)) \cdot \\ &\cdot \left(\frac{1}{2}AB^2 + \frac{3}{4}B^2P + 2AP^2 + 2AQ^2\right)\omega^2 + \\ &+ \frac{3}{2}\left(b_3(\mu)c_1(\mu) - b_1(\mu)c_3(\mu)\right)B^2Q\omega^3 - c_1(\mu)A = 0 \ . \end{aligned} \tag{24}$$

Equation (20) instead is a complex one. Then, it can be equivalently written as two scalar equations, which assume the following explicit forms:

$$\begin{cases} -2a(\mu)c_1(\mu)P\omega^2 - (b_2(\mu)c_1(\mu) - b_1(\mu)c_2(\mu)) \cdot \\ \quad \cdot \left(\frac{1}{4}B^2 + 2AP + 2P^2 + 2Q^2\right)\omega^2 + \\ \quad -b_1(\mu)\left(\frac{1}{4}B^2 + 2P^2 + 2Q^2\right)\omega^4 + \\ \quad +c_3(\mu)\omega^2 - c_1(\mu) = 0 \\ -2a(\mu)c_1(\mu)Q\omega^2 + \\ \quad -2\left(b_2(\mu)c_1(\mu) - b_1(\mu)c_2(\mu)\right)AQ\omega^2 + \\ \quad + \left(b_3(\mu)c_1(\mu) - b_1(\mu)c_3(\mu)\right) \cdot \\ \quad \cdot \left(\frac{3}{4}B^2 + 6P^2 + 6Q^2\right)\omega^3 + \\ \quad -\omega^3 + c_2(\mu)\omega = 0 \end{cases} \tag{25}$$

where B has been simplified according to the hypothesis $B \neq 0$. In the same way, the complex equation (21) boils down to the following two scalar equations:

$$\begin{cases} \frac{1}{2}a(\mu)c_1(\mu)B^2\omega^2 - (b_2(\mu)c_1(\mu) - b_1(\mu)c_2(\mu)) \cdot \\ \quad \left(-\frac{1}{2}AB^2 + \frac{1}{2}B^2P + P^3 + PQ^2\right)\omega^2 + \\ \quad - \left(b_3(\mu)c_1(\mu) - b_1(\mu)c_3(\mu)\right) \cdot \\ \quad \left(3B^2Q + 6P^2Q + 6Q^3\right)\omega^3 + \\ \quad +b_1(\mu)\left(2B^2P - 4P^3 - 4PQ^2\right)\omega^4 + \\ \quad +8\omega^3Q + 4c_3(\mu)\omega^2P - 2c_2(\mu)\omega Q - c_1(\mu)P = 0 \\ - \left(b_2(\mu)c_1(\mu) - b_1(\mu)c_2(\mu)\right) \cdot \\ \quad \left(\frac{1}{2}B^2Q + P^2Q + Q^3\right)\omega^2 + \\ \quad + \left(b_3(\mu)c_1(\mu) - b_1(\mu)c_3(\mu)\right) \cdot \\ \quad \left(3B^2P + 6P^3 + 6PQ^2\right)\omega^3 + \\ \quad +b_1(\mu)\left(2B^2Q - 4P^2Q - 4Q^3\right)\omega^4 + \\ \quad -8\omega^3P + 4c_3(\mu)\omega^2Q + 2c_2(\mu)\omega P - c_1(\mu)Q = 0 \ . \end{cases} \tag{26}$$

Observe that in the second order HB equations (24)-(26) the amplitude of the first harmonic appears only through B^2. Then, the sought solution can be represented in the following vector form:

$$S(\mu) = [\, A(\mu)\ \mathcal{B}(\mu)\ P(\mu)\ Q(\mu)\ \omega(\mu)\,]^\top \in \mathbb{R}^5 \tag{27}$$

where $\mathcal{B}(\mu) \doteq B^2(\mu)$. Clearly, such a solution exists only for the values of $\mu > 0$ such that $\mathcal{B}(\mu) > 0$.
To compute (27), we find it convenient to express $S(\mu)$ via its Taylor series expansion:

$$S(\mu) = S_0 + S_1 \cdot \mu + O(\mu^2) \tag{28}$$

where

$$S_0 \doteq \begin{bmatrix} A_0 \\ \mathcal{B}_0 \\ P_0 \\ Q_0 \\ \omega_0 \end{bmatrix} \in \mathbb{R}^5 , \quad S_1 \doteq \begin{bmatrix} A_1 \\ \mathcal{B}_1 \\ P_1 \\ Q_1 \\ \omega_1 \end{bmatrix} \in \mathbb{R}^5 \quad . \tag{29}$$

We first note that S_0 is indeed already known, since from Remark 4 it follows that $A_0 = 0$, $\mathcal{B}_0 = 0$, $P_0 = 0$, $Q_0 = 0$, and $\omega_0 = \sqrt{c_{20}}$. Hence, we need to compute S_1 only.
Exploiting (5) and (28) and taking into account the known expression of S_0, it is not difficult to verify that system (24)-(26) can be written as

$$(M_1 \cdot S_1 - V_1)\mu + O(\mu^2) = 0_5, \tag{30}$$

for suitable $M_1 \in \mathbb{R}^{5\times5}$ and $V_1 \in \mathbb{R}^{5\times1}$. Indeed, let us consider (24) along with the μ power developments of all its terms. Since S_0 solve the HB problem at $\mu = 0$, all the constant terms cancel each other. Moreover, it is straightforward to check that the first order μ powers contain at most one element of S_1. Analogous results hold for each equation in (25) and (26), leading to the form (30). Hence, the sought S_1 can be computed as

$$S_1 = M_1^{-1} \cdot V_1 \, . \tag{31}$$

We can now prove the main result of the paper.

Theorem 5. Suppose constraints (6) hold and $\det M_1 \neq 0$. Then, the equilibrium at $x = 0_3$ of (3) undergoes a Hopf bifurcation for $\mu = 0$ which is supercritical if $\mathcal{B}_1 > 0$ and subcritical if $\mathcal{B}_1 < 0$.

Proof. According to Proposition 3, the nature of the Hopf bifurcation can be assessed by evaluating the existence range of the solution $S(\mu)$ in (28)-(29) of the second order HB equations (24)-(26). Since $\det M_1 \neq 0$, S_1 can be computed according to (31). From Remark 4 it follows that $\mathcal{B}_0 = 0$ and thus we have $\mathcal{B}(\mu) > 0$ if and only if $\mathcal{B}_1\mu > 0$. Hence, the proof follows directly from Proposition 3. □

Remark 6. Once S_1 has been computed according to (31), an approximation of the bifurcated periodic solution, similar to that in Remark 4, can be computed analytically as follows:

$$\begin{aligned} \hat{y}(t;\mu) &= A_1\mu + \sqrt{\mathcal{B}_1\mu}\cos(\sqrt{c_{20}}t + \omega_1\mu t) + \\ &+ P_1\mu\cos\left(2\sqrt{c_{20}}t + 2\omega_1\mu t\right) + Q_1\mu\sin\left(2\sqrt{c_{20}}t + 2\omega_1\mu t\right) . \end{aligned} \tag{32}$$

According to Proposition 1, it turns out that $\hat{y}(t;\mu)$ provides an approximation of the component x_2 of the real bifurcated solution of (3).

5. DISCUSSION AND NUMERICAL EXAMPLE

To show how Theorem 5 can be efficiently exploited to characterize Hopf bifurcations in system (3), let us

Table 1.

$$M_1 = \begin{bmatrix} c_{20}c_{30} & \frac{1}{2}a_0c_{20}^2c_{30} & 0 & 0 & 0 \\ 0 & \frac{1}{4}b_{20}c_{20}^2c_{30} & 2a_0c_{20}^2c_{30} & 0 & -2c_{30}\sqrt{c_{20}} \\ 0 & -\frac{3}{4}(b_{30}c_{20}c_{30}-b_{10}c_{30})c_{20}\sqrt{c_{20}} & 0 & 2a_0c_{20}^2c_{30} & 2c_{20} \\ 0 & -\frac{1}{2}a_0c_{20}^2c_{30} & -3c_{20}c_{30} & -6c_{20}\sqrt{c_{20}} & 0 \\ 0 & 0 & 6c_{20}\sqrt{c_{20}} & -3c_{20}c_{30} & 0 \end{bmatrix} \qquad V_1 = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{c_{20}} \\ 0 \\ 0 \end{bmatrix}$$

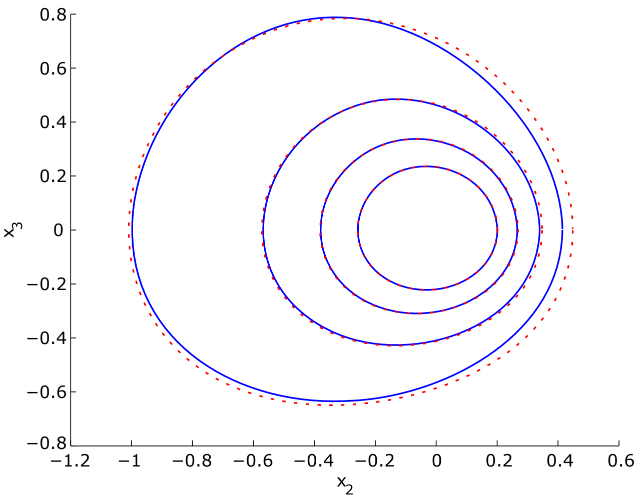

Fig. 2. Comparison between the real limit cycles (solid line) and the approximated ones (dotted line) on the plane x_2-x_3 (i.e. y-$\dot{y}$) for $\mu = 0.05$, $\mu = 0.10$, $\mu = 0.20$ and $\mu = 0.50$.

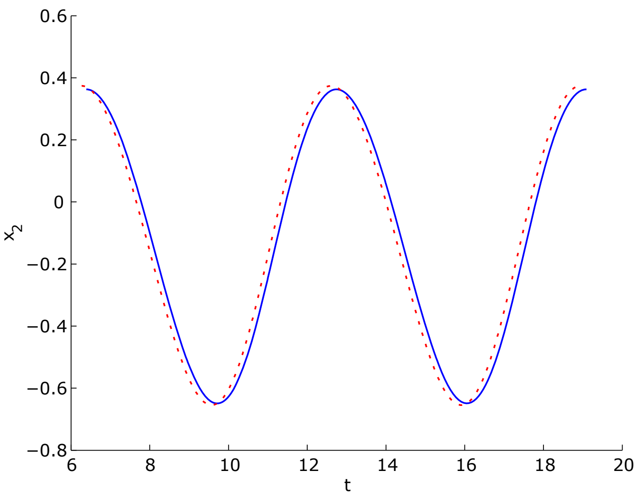

Fig. 3. Time behaviour of the real periodic solution $x_2(t;\mu)$ (solid line) and its approximation $\hat{y}(t;\mu)$ (dotted line) for $\mu = 0.25$.

consider the case in which the system parameters in (5) are given by:

$$\begin{cases} a = a_0 \\ b_i = b_{i0}\,, \quad i = 1,2,3 \\ c_1 = c_{20}c_{30} \\ c_2 = c_{20} - \mu\,, \quad c_{20} > 0 \\ c_3 = c_{30} > 0 \end{cases}$$

It is straightforward to check that the constraints in (6) are then satisfied and that $\hat{\mu} = +\infty$. Hence, the expansion of the second order HB problem (24)-(26) as a power series of μ leads to the following explicit form of (30):

$$\begin{cases} \left(c_{20}c_{30}A_1 + \frac{1}{2}a_0c_{20}^2c_{30}\mathcal{B}_1\right)\mu + O(\mu^2) = 0 \\ \left(-2c_{30}\sqrt{c_{20}}\omega_1 + 2a_0c_{20}^2c_{30}P_1 + \frac{1}{4}b_{20}c_{20}^2c_{30}\mathcal{B}_1\right)\mu + \\ \quad +O(\mu^2) = 0 \\ \left(\sqrt{c_{20}} + 2c_{20}\omega_1 + 2a_0c_{20}^2c_{30}Q_1 + \right. \\ \quad \left. -\frac{3}{4}(b_{30}c_{20}c_{30} - b_{10}c_{30})c_{20}\sqrt{c_{20}}\mathcal{B}_1\right)\mu + O(\mu^2) = 0 \\ \left(-6c_{20}\sqrt{c_{20}}Q_1 - 3c_{20}c_{30}P_1 - \frac{1}{2}a_0c_{20}^2c_{30}\mathcal{B}_1\right)\mu + \\ \quad +O(\mu^2) = 0 \\ (6c_{20}\sqrt{c_{20}}P_1 - 3c_{20}c_{30}Q_1)\,\mu + O(\mu^2) = 0\,. \end{cases}$$

Thus, the related matrix M_1 and vector V_1 in (30) are given according to Table 1. It turns out that Theorem 5 applies if

$$\det M_1 = 9c_{20}^5c_{30}^2\left[\left(2c_{20} + \frac{1}{2}c_{30}^2\right)\cdot \right. \\ \left. \cdot\,(b_{20}c_{20} + 3b_{10}c_{30} - 3b_{30}c_{20}c_{30}) - 2a_0^2c_{20}^2c_{30}^2\right] \neq 0\,. \quad (33)$$

In this case, (31) yields the following expression for S_1:

$$\begin{bmatrix} A_1 \\ \mathcal{B}_1 \\ P_1 \\ Q_1 \\ \omega_1 \end{bmatrix} = \frac{3c_{20}^4c_{30}^2}{\det M_1}\begin{bmatrix} 3a_0c_{20}\left(4c_{20}+c_{30}^2\right) \\ -6\left(4c_{20}+c_{30}^2\right) \\ a_0c_{20}c_{30}^2 \\ 2a_0c_{20}c_{30}\sqrt{c_{20}} \\ c_{20}^{\frac{3}{2}}\left(3b_{20}c_{20} - \frac{3}{4}c_{30}^2 + a_0^2c_{20}c_{30}^2\right) \end{bmatrix}. \quad (34)$$

The bifurcation is supercritical if $\mathcal{B}_1 > 0$, which turns out to be equivalent to $\det M_1 < 0$. Hence, the bifurcation is supercritical if the following condition holds:

$$b_{20}c_{20} + 3b_{10}c_{30} - 3b_{30}c_{20}c_{30} < \frac{2a_0^2c_{20}^2c_{30}^2}{2c_{20} + \frac{1}{2}c_{30}^2}. \quad (35)$$

Numerical example. To provide an idea of the level of approximation related to (32), let us consider the following numerical values:

$$\begin{cases} a_0 = b_{30} = c_{10} = c_{20} = c_{30} = 1 \\ b_{10} = b_{20} = 0 \\ c_{20} = -1 \\ a_1 = b_{11} = b_{21} = b_{31} = c_{11} = c_{31} = 0\,, \end{cases}$$

which identify the specific system

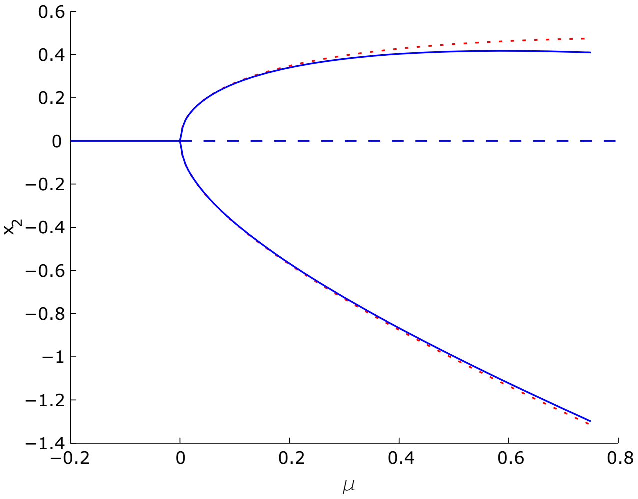

Fig. 4. The bifurcation diagram. Starting from negative values of μ, the stable equilibrium (solid line) undergoes a supercritical Hopf bifurcation at $\mu = 0$. For positive values of μ, it becomes unstable (dashed line) and a new stable solution, the limit cycle (solid line), arises. The behaviour of the latter is compared with the amplitude of the approximated solution $\hat{y}(t;\mu)$ (dotted line).

$$\begin{cases} \dot{x}_1 = x_2 + x_3^2 + x_3^3 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -x_1 - (1-\mu)x_2 - x_3 \end{cases} . \tag{36}$$

In order to define the nature of its Hopf bifurcation for $\mu = 0$ at $x = 0_3$, it is sufficient to compute condition (35). It is straightforward to verify that such a relation is satisfied, thus, the origin undergoes a supercritical Hopf bifurcation. Moreover, according to (34) we obtain

$$\begin{cases} \mathcal{A}_1 = -0.5263 \\ \mathcal{B}_1 = 1.0526 \\ P_1 = -0.0350 \\ Q_1 = -0.0701 \\ \omega_1 = -0.0350 \end{cases}$$

and thus, according to Remark 6, the approximation of the second component of the real bifurcated periodic solution $x(t;\mu)$ of (36) has the following form:

$$\hat{y}(t;\mu) = -0.5263\mu + \sqrt{1.0526\mu}\cos(t - 0.035\mu t) + \\ -0.035\mu\cos(2t - 0.07\mu t) - 0.0701\mu\sin(2t - 0.07\mu t) \quad .$$

The numerical evidence of the level of approximation of $\hat{y}(t;\mu)$ is illustrated by some diagrams.

Fig. 2 shows the comparison between the real limit cycle and the approximated one in the x_2-x_3 plane projection for different values of μ. The time evolution of the second component of the real periodic solution of (36) and its approximation are depicted in Fig. 3 for $\mu = 0.25$. Finally, Fig. 4 reports the real bifurcation diagram of the system in comparison with the behaviour of the amplitude of its approximating quantity $\hat{y}(t;\mu)$.

6. CONCLUSION

An analytic description of the system parameters which activate supercritical and subcritical Hopf bifurcations in the class of "quadratic plus cubic" normal forms of third order affine control systems, under linear state feedback control laws, has been provided via Harmonic Balance (HB) techniques. A distinguished feature of the HB approach, besides its popularity in classical control of feedback systems, is that it provides analytical approximations of the bifurcated periodic solutions, as shown in the numerical example. It is believed that such analytical results can be fruitfully exploited as starting points for a more complete bifurcation analysis of these control system normal forms.

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