**Proceedings of the 17th World Congress**
**The International Federation of Automatic Control**
**Seoul, Korea, July 6-11, 2008**

# Dynamic Non-rational Anti-windup for Time-delay Systems with Saturating Inputs

**I. Ghiggi, A. Bender, J.M. Gomes da Silva Jr.**

*UFRGS - Department of Electrical Engineering,*
*Av. Osvaldo Aranha 103, 90035-190 Porto Alegre-RS, Brazil.*
*E-mail: {jmgomes, ilca}@ece.ufrgs.br, fernando.bender@ufrgs.br*

**Abstract:** This paper addresses the design of dynamic anti-windup compensators for time-delay systems under amplitude control constraints. Considering that the system is subject to the action of $\mathcal{L}_2$ bounded disturbances, a method for computing a non-rational dynamic anti-windup compensator in order to guarantee both that the trajectories of the system are bounded and a certain $\mathcal{L}_2$ performance level is achieved by the regulated outputs, is proposed. Based on Lyapunov-Krazovskii functionals, the use of a modified sector condition, and a classical change of variables, sufficient LMI conditions, both in local as well as global contexts, are derived to ensure the input-to-state and the internal stability of the closed-loop system. From these conditions, LMI-based optimization problems are proposed in order to minimize the $\mathcal{L}_2$ gain, or in order to maximize the bound on the admissible disturbances for which the trajectories are bounded. The results apply to both stable and unstable open-loop systems and, in particular, for systems presenting delayed states.

Keywords: time-delay, saturation, anti-windup, LMI.

## 1. INTRODUCTION

It is well-known that the presence of delays in control systems can lead to poor time-domain performances or even to the instability of the closed-loop system (see for instance Niculescu [2001], Richard [2003] and references therein). The difficulty in controlling time-delay systems becomes even greater if the control is forced to be bounded. Unfortunately, this is a practical constraint, which comes from the physical fact that actuator cannot deliver unlimited signals to the controlled plants. As a consequence, input saturation may occur and it can be source of performance degradation and nonlinear behaviors such that limit cycles, multiple equilibria and instability.

Motivated by such problems, many works addressing the problem of stability analysis and stabilization of time-delay systems presenting bounded controls can be found in the literature. We can cite for instance: Oucheriah [1996], Chen et al. [1988], Niculescu et al. [1996], Tissir and Hmamed [1992], Tarbouriech and Gomes da Silva Jr. [2000], Cao et al. [2002] and Fridman et al. [2003]. In those papers, global and local stability (with the characterization of a set of admissible initial conditions) results considering delay-dependent and independent approaches are derived. However, it should be pointed out that all these works deal with state feedback control laws.

On the other hand, the anti-windup approach deals with the actuator saturation problem in a more practical perspective. Considering a pre-computed dynamic output feedback controller, whose design neglected the possibility of input saturation, the idea in this case consists of feeding the controller with the difference between the actuator input and its output, through a static or dynamic compensator. The aim of the anti-windup compensation is to correct the controller state in order to recover, as much as possible, the nominal performance of the system under saturation, i.e. when there effectively exists a difference between the input and the output of the actuator (see for instance Kapoor et al. [1998], Kothare and Morari [1999], Grimm et al. [2003], Turner and Postlethwaite [2004] and references therein). Considering works dealing with the anti-windup problems for time-delay systems, we can cite for instance Park et al. [2000], Tarbouriech et al. [2004] Zaccarian et al. [2005] and Gomes da Silva Jr. et al. [2006].

In Park et al. [2000] it is considered a plant subject to input delay and saturation. An anti-windup compensator is synthesized for minimizing a cost function, given as the absolute value of the difference between the states of the controller considering a saturation-free actuator, and the controller states when the plant is subject to input saturation and an anti-windup augmentation. It should however to be pointed out that the results apply only to stable open-loop systems and that the approach does not consider systems presenting state delays. In Zaccarian et al. [2005], the optimal controller obtained in Park et al. [2000] is extended in order to cope with disturbances and robustness issues, and output delays.

In Tarbouriech et al. [2004] and Gomes da Silva Jr. et al. [2006], an LMI approach to synthesize stabilizing *static* anti-windup gains have been proposed. Differently from the classical objective of recovering performance, in those works the anti-windup compensation have been used to enlarge the region of attraction of the closed-loop system. In particular, the presence of additive disturbances and closed-loop performance issues were not considered.

* This work was supported in part by CNPq and CAPES, Brazil.

In the present work, we focus on the synthesis of *dynamic* anti-windup compensators for systems presenting *delayed states*. Considering that the system is subject to the action of $\mathcal{L}_2$ bounded disturbances, a method for computing the non-rational dynamic anti-windup compensator in order to guarantee both that the trajectories of the system are bounded and a certain $\mathcal{L}_2$ performance level is achieved by the regulated outputs, is proposed. The approach we follow is based on the use of a Lyapunov-Krasovskii functional (see for instance Niculescu [2001]), a modified sector condition, as proposed in Tarbouriech et al. [2004], and the use of non-rational controllers, as proposed in de Oliveira and Geromel [2004]. From these ingredients, LMI conditions to ensure $\mathcal{L}_2$ input-to-state stability as well as internal asymptotic stability of the closed-loop system, are derived both in the local and global contexts. The proposed results apply therefore to both stable and unstable open-loop systems. Considering these conditions, convex optimization problems are proposed in order to address two synthesis objectives: the maximization of the $\mathcal{L}_2$ upper bound on the admissible disturbances, for which the trajectories are bounded (i.e. disturbance tolerance maximization); and, considering a given upper bound on the $\mathcal{L}_2$ norm of the admissible disturbances, the minimization of the $\mathcal{L}_2$ gain between a regulated output and the disturbance (i.e. disturbance rejection maximization).

**Notations.** For two symmetric matrices, $A$ and $B$, $A > B$ means that $A - B$ is positive definite. $A'$ denotes the transpose of $A$. $A_{(i)}$ denotes the $i^{th}$ row of matrix $A$. $\star$ stands for symmetric blocks; $\bullet$ stands for an element that has no influence on the development. $I_m$ denotes an identity matrix of order $m$. $\lambda_{max}(P)$ denotes the maximal eigenvalue of matrix $P$. $\mathcal{C}_\tau = \mathcal{C}([-\tau, 0], \Re^n)$ is the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\Re^n$ with the norm $\| \phi \|_c = \sup_{-\tau \le t \le 0} \| \phi(t) \|$. $\| \cdot \|$ refers to the Euclidean vector norm. $\mathcal{C}_\tau^v$ is the set defined by $\mathcal{C}_\tau^v = \{\phi \in \mathcal{C}_\tau \ ; \ \| \phi \|_c < v, \ v > 0\}$ For $v \in \Re^m$, $\text{sat}(v) : \Re^m \to \Re^m$ denotes a classical vector-valued saturation function defined as $(\text{sat}(v))_{(i)} = \text{sat}(v_{(i)}) = \text{sign}(v_{(i)})\min\{u_{0(i)}, |v_{(i)}|\}$, for all $i = 1, \ldots, m$, where $u_{0(i)} > 0$ denotes the $i$-th amplitude bound.

## 2. PROBLEM STATEMENT

Consider the linear continuous-time delay system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bu(t) + B_w w(t) \\ y(t) &= C_y x(t) + C_{y,d} x(t-\tau) + D_{y,w} w(t) \\ z(t) &= C_z x(t) + C_{z,d} x(t-\tau) + D_z u(t) + D_{z,w} w(t) \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $w(t) \in \Re^q$ are the state, the input and the disturbance vectors, respectively. $y(t) \in \Re^p$ corresponds to the measured output and $z(t) \in \Re^r$ is the regulated output. $A$, $A_d$, $B$, $B_w$, $C_y$, $C_{y,d}$, $C_z$, $C_{z,d}$, $D_{y,w}$, $D_z$ and $D_{z,w}$ are real constant matrices of appropriate dimensions. The scalar $\tau$ is a constant time delay. The disturbance vector $w$ is assumed to be limited in energy, that is, $w(t) \in \mathcal{L}_2$ and for some scalar $\delta$, $0 \le \frac{1}{\delta} < \infty$, it follows that:

$$\|w\|_2^2 = \int_0^\infty w'(t)w(t)dt \le \frac{1}{\delta} \tag{2}$$

The inputs are supposed to be bounded as follows:

$$-u_{0(i)} \le u_{(i)} \le u_{0(i)}, i = 1, \ldots, m \tag{3}$$

For the plant given in (1) a dynamic output feedback controller, possibly non-rational (de Oliveira and Geromel [2004]), is supposed to have been designed, disregarding the saturation, as follows:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + A_{c,d} x_c(t-\tau) + B_c u_c(t) \\ y_c(t) &= C_c x_c(t) + C_{c,d} x_c(t-\tau) + D_c u_c(t) \end{aligned} \tag{4}$$

where $x_c(t) \in \Re^{n_c}$ is the controller state, $u_c(t) = y(t)$ is the controller input and $y_c(t)$ is the controller output, Matrices $A_c$, $A_{c,d}$, $B_c$, $C_c$, $C_{c,d}$, $D_c$ are of appropriated dimensions.

As the plant input saturates in amplitude, we have:

$$u(t) = sat(y_c(t)) \tag{5}$$

In order to mitigate the possible undesirable effects of saturation, the following non-rational anti-windup dynamic compensator is proposed:

$$\begin{aligned} \dot{x}_a(t) &= A_a x_a(t) + A_{a,d} x_a(t-\tau) + B_a \psi(y_c(t)) \\ y_a(t) &= C_a x_a(t) + C_{a,d} x_a(t-\tau) + D_a \psi(y_c(t)) \end{aligned} \tag{6}$$

where $x_a(t) \in \Re^{n+n_c}$ is the compensator state vector, $y_a(t)$ is the compensator output, $\psi(y_c(t))$ is the dead-zone function defined as $\psi(y_c(t)) = y_c(t) - sat(y_c(t))$. Matrices $A_a$, $A_{a,d}$, $B_a$, $C_a$, $C_{a,d}$ and $D_a$ have appropriated dimensions.

The compensation effect is applied to the system by injecting the signal $y_a$ in the controller (4). Thus, the final controller structure becomes:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + A_{c,d} x_c(t-\tau) + B_c u_c(t) + y_a(t) \\ y_c(t) &= C_c x_c(t) + C_{c,d} x_c(t-\tau) + D_c u_c(t) \end{aligned} \tag{7}$$

In this work we address the problem of synthesizing the anti-windup compensator with the structure given in (6) in order to ensure that the trajectories of the system are bounded for any disturbance satisfying (2) and, in addition, to ensure an upper bound for the $\mathcal{L}_2$ gain from the disturbance to the regulated output.

## 3. PRELIMINARIES

Define the following matrices:

$$\mathcal{A} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} A + BD_cC_y & BC_c \\ B_cC_y & A_c \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ I_{n_c} \end{bmatrix}, \ \mathcal{B}_1 = \begin{bmatrix} 0 & \mathbf{B}_1 \\ I_a & 0 \end{bmatrix}, \ \mathbf{C} = [\, 0 \ I_a \,]$$

$$\mathcal{A}_d = \begin{bmatrix} \mathbf{A}_d & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{A}_d = \begin{bmatrix} A_d + BD_cC_{y,d} & BC_{c,d} \\ B_cC_{y,d} & A_{c,d} \end{bmatrix}$$

$$\mathbf{K}_1 = \begin{bmatrix} A_a \\ C_a \end{bmatrix}, \ \mathbf{K}_{1,d} = \begin{bmatrix} A_{a,d} \\ C_{a,d} \end{bmatrix}, \ \mathbf{K}_2 = \begin{bmatrix} B_a \\ D_a \end{bmatrix}$$

$$\mathcal{B}_\psi = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\mathcal{B}_w = \begin{bmatrix} \mathbf{B}_w \\ 0 \end{bmatrix}, \ \mathbf{B}_w = \begin{bmatrix} BD_cD_{y,w} + B_w \\ B_cD_{y,w} \end{bmatrix}$$

$$\mathcal{C} = [\, \mathbf{C}_z \ 0 \,], \ \mathbf{C}_z = [\, C_z + D_zD_cC_y \ D_zC_c \,]$$

$$\mathcal{C}_d = [\, \mathbf{C}_{z,d} \ 0 \,], \ \mathbf{C}_{z,d} = [\, C_{z,d} + D_zD_cC_{y,d} \ D_zC_{c,d} \,]$$

$$\mathcal{D}_{z,w} = \mathbf{D}_{z,w} = D_{z,w} + D_zD_cD_{y,w}, \ \mathcal{D}_\psi = \mathbf{D}_\psi = -D_z$$

$$\mathcal{K} = [\, \mathbf{K} \ 0 \,], \ \mathbf{K} = [\, D_cC_y \ C_c \,], \ \mathcal{K}_d = [\, \mathbf{K}_d \ 0 \,]$$

$$\mathbf{K}_d = [\, D_cC_{y,d} \ C_{c,d} \,], \ \mathcal{K}_w = \mathbf{K}_w = D_cD_{y,w}, \ \mathcal{K}_\psi = 0$$

Thus, considering $\xi(t)' = \left[\, x(t)' \; x_c(t)' \; x_a(t)' \,\right]$, the closed-loop system can be written as follows:

$$\begin{aligned} \dot{\xi}(t) &= (\mathcal{A} + \mathcal{B}_1\mathbf{K}_1\mathbf{C})\xi(t) + (\mathcal{A}_d + \mathcal{B}_1\mathbf{K}_{1,d}\mathbf{C})\xi(t-\tau) \\ &\quad -(\mathcal{B}_\psi - \mathcal{B}_1\mathbf{K}_2)\psi(y_c) + \mathcal{B}_w w(t) \\ z(t) &= \mathcal{C}\xi(t) + \mathcal{C}_d\xi(t-\tau) + \mathcal{D}_\psi\psi(y_c) + \mathcal{D}_{z,w}w(t) \end{aligned} \tag{8}$$

also,

$$y_c(t) = \mathcal{K}\xi(t) + \mathcal{K}_d\xi(t-\tau) + \mathcal{K}_w w(t) \tag{9}$$

The closed-loop system has initial conditions expressed as:

$$\begin{aligned} \phi_\xi(\theta) &= \left[\, x(t_0+\theta)' \; x_c(t_0+\theta)' \; x_a(t_0+\theta)' \,\right]' \\ &= \left[\, \phi_x(\theta)' \; \phi_{x_c}(\theta)' \; \phi_{x_a}(\theta)' \,\right]',\ for\ \theta \in [-\tau\,,\,0] \end{aligned} \tag{10}$$

Considering now a matrix $G = [\, G_1 \; G_2 \,] \in \Re^{m\times 2(n+n_c)}$ and defining the polyhedral set

$$\mathcal{S} \triangleq \{\xi \in \Re^{2(n+n_c)};\, |(\mathcal{K}_{(i)} - G_{(i)})\xi| \le u_{0(i)},\, i = 1,...,m\} \tag{11}$$

the following Lemma, regarding the nonlinearity $\psi(y_c)$ can be stated.

*Lemma 1.* If $\xi(t) \in \mathcal{S}$ then the relation

$$\begin{aligned} &\psi(y_c)'T\Big(\psi(y_c) - [\, G \; \mathcal{K}_d \; 0 \; \mathcal{K}_w \,]\cdot \\ &\left[\, \xi(t)' \; \xi(t-\tau)' \; \psi(y_c(t))' \; w(t)' \,\right]'\Big) \le 0 \end{aligned} \tag{12}$$

is verified for any matrix $T \in \Re^{m\times m}$ diagonal and positive definite.

**Proof:** Note that considering $r(t) = G\xi(t) + \mathcal{K}_d\xi(t-\tau) + \mathcal{K}_w w(t)$, it follows from (9) that $y_c(t) - r(t) = (\mathcal{K} - G)\xi(t)$ and therefore, $\forall \xi(t) \in \mathcal{S}$ one has that $|y_{c(i)}(t) - r_{(i)}(t)| \le u_{0(i)}$, $i = 1,...,m$. From here, the proof mimics the one given for Lemma 1 in Tarbouriech et al. [2004], where it is shown that $\psi(y_c(t))'T(\psi(y_c(t)) - r(t)) \le 0$. □

Differently from the classical sector condition, the main advantage provided by (12) is that it will allow the formulation of stability conditions directly in LMI form. Furthermore, since (12) encompasses the classical sector condition, less conservative stability conditions can be obtained (see discussion in Gomes da Silva Jr. and Tarbouriech. [2005]).

## 4. MAIN RESULTS

In the sequel, sufficient LMI conditions are derived for the existence of a dynamic non-rational anti-windup compensator (6) ensuring that, for any disturbance satisfying (2), the closed-loop system trajectories are bounded in an ellipsoidal set (input-to-state stability). Moreover, these conditions ensure that this ellipsoidal set is included in the basin of attraction of the origin (internal asymptotic stability).

*Theorem 1.* If there exist symmetric positive definite matrices $X_0$, $Y_0$, $X_{11}$, $X_{22} \in \Re^{(n+n_c)\times(n+n_c)}$, a positive definite diagonal matrix $S \in \Re^{m\times m}$, matrices $X_{12}$, $\hat{A}_a$, $\hat{A}_{a,d} \in \Re^{(n+n_c)\times(n+n_c)}$, $\hat{C}_a$, $\hat{C}_{a,d} \in \Re^{(n_c)\times(n+n_c)}$, $Z_1$, $Z_2$, $Q_1$, $Q_2 \in \Re^{m\times(n+n_c)}$, and positive scalars $\gamma$ and $\delta$ such that LMIs (13) and (14) are verified,

$$\begin{bmatrix} X_0 & \star & \star \\ I_{n+n_n} & Y_0 & \star \\ \mathbf{K}_{(i)}X_0 - Z_{1(i)} & \mathbf{K}_{(i)} - Z_{2(i)} & \delta u_{0(i)}^2 \end{bmatrix} \ge 0 \quad i = 1,...,m \tag{14}$$

then, considering that $\phi_\xi(\theta) = 0$, $\forall\ \theta\ \in [-\tau, 0]$ and that $\|w\|_2^2 \le \delta^{-1}$, the dynamic anti-windup compensator (6) with matrices

$$\begin{aligned} A_a &= V_0^{-1}[\hat{A}_a - (Y_0\mathbf{A}X_0 + Y_0\hat{C}_a)](U_0')^{-1} \\ A_{a,d} &= V_0^{-1}[\hat{A}_{a,d} - (Y_0\mathbf{A}_dX_0 + Y_0\hat{C}_{a,d}](U_0')^{-1}\mathbf{B}_1 \\ B_a &= V_0^{-1}(Q_2 + Y_0Q_1)S^{-1} \\ C_a &= \hat{C}_a(U_0')^{-1} \\ C_{a,d} &= \hat{C}_{a,d}(U_0')^{-1} \\ D_a &= (Q_1 + \mathbf{B}S)S^{-1} \end{aligned} \tag{15}$$

where matrices $U_0$ and $V_0$ verify $V_0U_0' = I - Y_0X_0$, is such that:

(1) when $w \ne 0$
   a. the closed-loop trajectories remain bounded in the set $\mathcal{E}(P_0, \delta^{-1}) \triangleq \{\xi \in \Re^{2(n+n_c)}; \xi'P_0\xi \le \delta^{-1}\}$, with
   $$P_0 = \begin{bmatrix} Y_0 & V_0 \\ V_0' & \bullet \end{bmatrix} \text{ and } P_0^{-1} = \begin{bmatrix} X_0 & U_0 \\ U_0' & \bullet \end{bmatrix}$$
   b. $\|z\|_2^2 < \gamma\|w\|_2^2$
(2) if $w(t) = 0, \forall t > t_1 \ge 0$, $\xi(t)$ converges asymptotically to the origin.

**Proof:** Consider the following Lyapunov-Krasovskii candidate functional

$$V(t) = \xi'(t)P_0\xi(t) + \int_{t-\tau}^{t} \xi'(\theta)P_1\xi(\theta)d\theta. \tag{16}$$

Since $\phi_\xi(\theta) = 0$, it follows that $V(0) = 0$. Define now $\mathcal{J} = \dot{V}(t) - w'(t)w(t) + \frac{1}{\gamma}z'(t)z(t)$. If $\mathcal{J} < 0$, one obtains that $\int_0^T \mathcal{J}dt = V(T) - V(0) - \int_0^T w'(t)w(t)dt + \frac{1}{\gamma}\int_0^T z'(t)z(t)dt < 0$, $\forall\ T$. Hence, it follows that:

- $\xi(T)'P_0\xi(T) \le V(T) < V(0) + \|w\|_2^2 \le \delta^{-1}$, $\forall T > 0$, i.e. the trajectories of the system do not leave the set $\mathcal{E}(P_0, \delta^{-1})$ for $w(t)$ satisfying (2)
- for $T \to \infty$, $\|z\|_2^2 < \gamma\|w\|_2^2$
- if $w(t) = 0, \forall t > t_1 \ge 0$, then $\dot{V}(t) < -\frac{1}{\gamma}z'(t)z(t) < 0$, which ensures that $\xi(t) \to 0$ as $t \to \infty$.

Let us now evaluate $\mathcal{J}$ along the system trajectories: $\mathcal{J} = 2\xi'(t)(\mathcal{A}+\mathcal{B}_1K_1\mathbf{C})'P_0\xi(t) + 2\xi'(t-\tau)(\mathcal{A}_d+\mathcal{B}_1K_{1,d}\mathbf{C})'\cdot P_0\xi(t) - 2\psi'(y_c)(t)(\mathcal{B}_\psi - \mathcal{B}_1\mathbf{K}_2)'P_0\xi(t) + 2w'(t)\mathcal{B}_w'P_0\xi(t) + \xi'(t)P_1\xi(t) - \xi'(t-\tau)P_1\xi(t-\tau) - w'(t)w(t) + \frac{1}{\gamma}z'(t)z(t)$.

From Lemma 1, provided that $\xi(t) \in \mathcal{S}$ it follows that:

$$\begin{aligned} &\mathcal{J} \le \mathcal{J} - 2\psi(y_c)'T\big(\psi(y_c) - [\, G \; \mathcal{K}_d \; 0 \; \mathcal{K}_w \,]\cdot \\ &\left[\, \xi'(t) \; \xi'(t-\tau) \; \psi'(y_c(t)) \; w'(t) \,\right]'\big) \end{aligned} \tag{17}$$

Since $z(t) = \mathcal{C}\xi(t) + \mathcal{C}_d\xi(t-\tau) + \mathcal{D}_\psi\psi(y_c) + \mathcal{D}_{z,w}w(t)$, the right hand side of (17) can be written as $\mu(t)'\Xi\mu(t)$ with $\mu(t) = \left[\, \xi(t)' \; \xi(t-\tau)' \; \psi(y_c(t))' \; w(t)' \,\right]'$ and $\Xi$ given by (18). Then, if $\Xi < 0$, one obtains $\mathcal{J} < 0$ . Applying now Schur complement, note that $\Xi < 0$ is equivalent to (19).

$$\begin{bmatrix} \text{sym}\{\mathbf{A}X_0 + B_1\hat{C}_a\} + X_{11} & \mathbf{A} + \hat{A}_a' + X_{12} & \mathbf{A}_d X_0 + B_1 \hat{C}_{a,d} & \mathbf{A}_d & -Q_1 + Z_1' & \mathbf{B}_w' & X_0\mathbf{C}_z' \\ \star & Y_0\mathbf{A} + \mathbf{A}'Y_0 + X_{22} & \hat{A}_{a,d} & Y_0\mathbf{A}_d & -Q_2 + Z_2' & Y_0\mathbf{B}_w' & \mathbf{C}_z' \\ \star & \star & -X_{11} & -X_{12} & X_0'\mathbf{K}_d & 0 & X_0\mathbf{C}_{z,d}' \\ \star & \star & \star & -X_{22} & \mathbf{K}_d & 0 & \mathbf{C}_{z,d}' \\ \star & \star & \star & \star & -2S & \mathbf{K}_w & -S\mathbf{D}_\psi' \\ \star & \star & \star & \star & \star & -I & \star \\ \star & \star & \star & \star & \star & \mathbf{D}_{z,w} & -\gamma I \end{bmatrix} < 0 \qquad (13)$$

$$\Xi = \begin{bmatrix} \begin{pmatrix} (\mathcal{A}+\mathcal{B}_1\mathbf{K}_1\mathbf{C})'P_0 + \\ P_0(\mathcal{A}+\mathcal{B}_1\mathbf{K}_1\mathbf{C}) + \\ P_1 + \frac{1}{\gamma}\mathcal{C}'\mathcal{C} \end{pmatrix} & \begin{pmatrix} P_0(\mathcal{A}_d + \mathcal{B}_1\mathbf{K}_{1,d}\mathbf{C}) + \\ \frac{1}{\gamma}\mathcal{C}'\mathcal{C}_d \end{pmatrix} & \begin{pmatrix} -P_0(\mathcal{B}_\psi - \mathcal{B}_1\mathbf{K}_2) + \\ G'T + \frac{1}{\gamma}\mathcal{C}'\mathcal{D}_\psi \end{pmatrix} & P_0\mathcal{B}_w + \frac{1}{\gamma}\mathcal{C}'\mathcal{D}_{z,w} \\ \star & -P_1 + \frac{1}{\gamma}\mathcal{C}_d'\mathcal{C}_d & \mathcal{K}_d'T + \frac{1}{\gamma}\mathcal{C}_d'\mathcal{D}_\psi & \frac{1}{\gamma}\mathcal{C}_d'\mathcal{D}_{z,w} \\ \star & \star & -2T + \frac{1}{\gamma}\mathcal{D}_\psi'\mathcal{D}_\psi & T\mathcal{K}_w + \frac{1}{\gamma}\mathcal{D}_\psi'\mathcal{D}_{z,w} \\ \star & \star & \star & \frac{1}{\gamma}\mathcal{D}_{z,w}'\mathcal{D}_{z,w} - I \end{bmatrix} \qquad (18)$$

$$\begin{bmatrix} (\mathcal{A}+\mathcal{B}_1\mathbf{K}_1\mathbf{C})'P_0 + P_0(\mathcal{A}+\mathcal{B}_1\mathbf{K}_1\mathbf{C}) + P_1 & \star & \star & P_0\mathcal{B}_w & \mathcal{C}' \\ (\mathcal{A}_d + \mathcal{B}_1\mathbf{K}_{1,d}\mathbf{C})'P_0 & -P_1 & \mathcal{K}_d'T & 0 & \mathcal{C}_d' \\ -(\mathcal{B}_\psi + \mathcal{B}_1\mathbf{K}_2)'P_0 + TG & \star & -2T & T\mathcal{K}_w & \mathcal{D}_\psi' \\ \star & \star & \star & -I & \mathcal{D}_{z,w}' \\ \star & \star & \star & \star & -\gamma I \end{bmatrix} < 0 \qquad (19)$$

Define now a matrix $\Pi = \begin{bmatrix} X_0 & I \\ U_0' & 0 \end{bmatrix}$, (see Scherer et al. [1997] and de Oliveira and Geromel [2004]). Note that, from condition (14), it follows that $I - Y_0X_0$ is nonsingular. This implies that it is always possible to compute nonsingular matrices $V_0$ and $U_0$ verifying the equation $V_0U_0' = I - Y_0X_0$, which ensures $\Pi$ nonsingular.

Pre and post-multiplying (19) respectively by the block diagonal matrix $Diag\begin{pmatrix}\Pi' & \Pi' & S & I & I\end{pmatrix}$ and its transpose, with $S = T^{-1}$, one gets (20).

Defining $\Pi'P_1\Pi = X = \begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix}$ and by considering $\hat{A}_a = Y_0\mathbf{A}X_0 + Y_0\mathbf{B}_1C_aU_0' + V_0A_aU_0'$, $\hat{A}_{a,d} = Y_0\mathbf{A}_dX_0 + Y_0\mathbf{B}_1C_{a,d}U_0' + V_0A_{a,d}U_0'$, $\hat{C}_a = C_aU_0'$, $\hat{C}_{a,d} = C_{a,d}U_0'$, $Z_2 = G_1$, $Z_1 = G_1X_0 + G_2U_0$, $Q_1 = \mathbf{B}S - \mathbf{B}_1D_aS$, $Q_2 = Y_0Q_1 - V_0B_aS$, it follows that:

$$\begin{aligned} &\Pi'P_0\Pi = \begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix}, \; \Pi'P_0(\mathcal{B}_\psi + \mathcal{B}_1\mathbf{K}_2)S = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\ &\Pi'\mathcal{K}_d' = \begin{bmatrix} X_0\mathbf{K}_d' \\ \mathbf{K}_d' \end{bmatrix}, \; \Pi'P_0\mathcal{B}_w = \begin{bmatrix} \mathbf{B}_w \\ Y_0\mathbf{B}_w \end{bmatrix} \\ &\Pi'P_0(\mathcal{A}+\mathcal{B}_1\mathbf{K}_1\mathbf{C})\Pi = \begin{bmatrix} \mathbf{A}X_0 + \mathbf{B}_1\hat{C}_a & \mathbf{A} \\ \hat{A}_a & Y_0\mathbf{A} \end{bmatrix} \\ &\Pi'P_0(\mathcal{A}_d+\mathcal{B}_1\mathbf{K}_{1,d}\mathbf{C})\Pi = \begin{bmatrix} \mathbf{A}_dX_0 + \mathbf{B}_1\hat{C}_{a,d} & \mathbf{A}_d \\ \hat{A}_{a,d} & Y_0\mathbf{A}_d \end{bmatrix} \\ &\Pi'G' = \begin{bmatrix} Z_1' \\ Z_2' \end{bmatrix}, \; \Pi'\mathcal{C}' = \begin{bmatrix} X_0\mathbf{C}_z' \\ \mathbf{C}_z' \end{bmatrix}, \; \Pi'\mathcal{C}_d' = \begin{bmatrix} X_0\mathbf{C}_{z,d}' \\ \mathbf{C}_{z,d}' \end{bmatrix} \end{aligned} \qquad (21)$$

Hence, since $\Pi$ and $S$ are nonsingular, it follows that if (13) is verified, i.e. $\Xi < 0$ and then $\mathcal{J} < 0$ holds with the matrices $A_a$, $A_{a,d}$, $B_a$, $C_a$, $C_{a,d}$, and $D_a$ defined as in (15), provided $\xi(t) \in \mathcal{S}$. Relation (14) ensures that $\mathcal{E}(P_0, \delta^{-1}) \subset \mathcal{S}$. Hence, provided (13) and (14) holds it follows that $\xi(t)$ never leaves $\mathcal{E}(P_0, \delta^{-1})$; therefore we have effectively $\mathcal{J} < 0$, which concludes the proof of item 1.

In case $w(t) = 0$, for $t > t_1 \geq 0$, since $\xi(t) \in \mathcal{E}(P_0, \delta^{-1}) \subset \mathcal{S}$, the satisfaction (13) ensures that $\dot{V}(t) < 0, \forall\, t > t_1 \geq 0$, and the trajectories converge asymptotically to the origin. This completes the proof of item 2. □

The following corollary provides a sufficient condition to ensure stability in a global context. It can be particularly applied when the open-loop system is asymptotically stable. In this case, it can be ensured that the trajectories of the closed-loop system are bounded for any $w(t) \in \mathcal{L}_2$. Moreover, the origin of the system is ensured to be globally asymptotically stable.

*Corollary 1.* If there exist symmetric positive definite matrices $X_0$, $Y_0$, $X_{11}$, $X_{22} \in \Re^{(n+n_c)\times(n+n_c)}$, a positive definite diagonal matrix $S \in \Re^{m\times m}$, matrices $X_{12}$, $\hat{A}_a$, $\hat{A}_{a,d} \in \Re^{(n+n_c)\times(n+n_c)}$, $\hat{C}_a$, $\hat{C}_{a,d} \in \Re^{(n_c)\times(n+n_c)}$, $Q_1$, $Q_2 \in \Re^{m\times(n+n_c)}$, and positive scalars $\gamma$ and $\delta$ such that (22) is verified, then the dynamic controller (7) with matrices defined as in (15), where matrices $U_0$ and $V_0$ verify $V_0U_0' = I - Y_0X_0$, is such that

(1) when $w \neq 0$
  a. the closed-loop trajectories remain bounded $\forall w(t) \in \mathcal{L}_2$ and $\forall \phi_\xi(\theta) \in \mathcal{C}_\tau^v$.
  b. $\|z\|_2^2 < \gamma\|w\|_2^2 + \gamma V(0)$.
(2) if $w(t) = 0, \forall t > t_1 \geq 0$, $\xi(t)$ converges asymptotically to the origin.

**Proof:** Consider $G = \mathcal{K}$. It follows that the sector condition (12) is verified for all $\xi(t) \in \Re^{2(n+n_c)}$. □

*Remark 1.* Without the non-rational terms, given by $A_{a,d}x_a(t-\tau)$ and $C_{a,d}x_a(t-\tau)$, in the anti-windup compensator structure, it would not be possible to obtain the

$$\begin{bmatrix} \begin{pmatrix} \Pi'(\mathcal{A}+\mathcal{B}_1\mathbf{K}_1\mathbf{C})'P_0\Pi+ \\ \Pi'P_0(\mathcal{A}+\mathcal{B}_1\mathbf{K}_1\mathbf{C})\Pi+ \\ \Pi'P_1\Pi \end{pmatrix} & \Pi'P_0(\mathcal{A}_d+\mathcal{B}_1\mathbf{K}_{1,d}\mathbf{C})\Pi & \begin{pmatrix} -\Pi'P_0(\mathcal{B}_\psi+\mathcal{B}_1\mathbf{K}_2)S+ \\ \Pi'G' \end{pmatrix} & \Pi'P_0\mathcal{B}_w & \Pi'\mathcal{C}' \\ \star & -\Pi'P_1\Pi & \Pi'\mathcal{K}'_d & 0 & \Pi'\mathcal{C}'_d \\ \star & \star & -2S & \mathcal{K}_w & S\mathcal{D}'_\psi \\ \star & \star & \star & -I & \mathcal{D}'_{z,w} \\ \star & \star & \star & \star & -\gamma I \end{bmatrix} < 0 \quad (20)$$

$$\begin{bmatrix} \text{sym}\{\mathbf{A}X_0+B_1\hat{C}_a\}+X_{11} & \mathbf{A}+\hat{A}'_a+X_{12} & \mathbf{A}_dX_0+B_1\hat{C}_{a,d} & \mathbf{A}_d & -Q_1+\mathbf{K}'X_0 & \mathbf{B}'_w & X_0\mathbf{C}'_z \\ \star & Y_0\mathbf{A}+\mathbf{A}'Y_0+X_{22} & \hat{A}_{a,d} & Y_0\mathbf{A}_d & -Q_2+\mathbf{K} & Y_0\mathbf{B}'_w & \mathbf{C}'_z \\ \star & \star & -X_{11} & -X_{12} & X'_0\mathbf{K}_d & 0 & X_0\mathbf{C}'_{z,d} \\ \star & \star & \star & -X_{22} & \mathbf{K}_d & 0 & \mathbf{C}'_{z,d} \\ \star & \star & \star & \star & -2S & \mathbf{K}_w & -S\mathbf{D}'_\psi \\ \star & \star & \star & \star & \star & -I & \mathbf{D}_{z,w} \\ \star & \star & \star & \star & \star & \star & -\gamma I \end{bmatrix} < 0 \quad (22)$$

synthesis conditions in a LMI form using the proposed approach. Furthermore, these terms introduce more degrees of freedom in the synthesis, which are clearly useful in order to reduce conditions conservatism.

## 5. OPTIMIZATION PROBLEMS

The conditions obtained in the previous section can be used to find the anti-windup compensator considering, for instance, the following optimization problems.

*Maximization of the tolerance disturbance.* The idea is maximizing the bound on the disturbance, for which we can ensure that the system trajectories remain bounded. This can be accomplished by the following convex optimization problem.

$$\begin{array}{c} \min \delta \\ \text{subject to } (13),\ (14) \end{array} \quad (23)$$

Note that, in this case, we are not interested in the value of $\gamma$. Indeed, $\gamma$ will assume a finite value to ensure that (13) is verified.

*Minimization of the $\mathcal{L}_2$ gain.* For an *a priori* given bound on the $\mathcal{L}_2$ norm of the admissible disturbances (given by $\frac{1}{\delta}$), the idea is minimizing the upper bound for the $\mathcal{L}_2$-gain of $w(t)$ on $z(t)$. This can be obtained from the solution of the following convex optimization problem:

$$\begin{array}{c} \min \gamma \\ \text{subject to } (13),\ (14)\ (\text{or } (22)) \end{array} \quad (24)$$

## 6. NUMERICAL EXAMPLES

*Example 1.* Consider system (1) given by:

$$\begin{aligned} \dot{x}(t) &= -0.1x(t) + 0.1x(t-\tau) + u(t) + 0.1w(t) \\ y(t) &= x(t) \\ z(t) &= x(t) \end{aligned} \quad (25)$$

i.e., $A = -0.1$, $A_d = 0.1$, $B = 1$, $B_w = 0.1$, $C_y = 1$, $C_{y,d} = 0$, $D_{y,w} = 0$, $C_z = 1$, $C_{z,d} = 0$, $D_z = 0$, $D_{z,w} = 0$.

A PI controller described in the state-space as follows is considered:

$$\begin{aligned} \dot{x}_c(t) &= -0.2y(t) \\ y_c(t) &= x_c(t) - 2y(t) \end{aligned} \quad (26)$$

i.e., $A_c = 0$, $A_{c,d} = 0$, $B_c = -0.2$, $C_c = 1$, $C_{c,d} = 0$, $D_c = -2$.

Consider $u_0 = 1$ and $\tau = 1$. From the resolution of the optimization problem (23) we obtain the following anti-windup compensator:

$$\begin{gathered} D_a = -0.7495; C_a = [\, 0.4966 \;\; 0.6401 \,] \\ C_{a,d} = [\, 0.0074 \;\; 0.0030 \,]\, ; B_a = \begin{bmatrix} 5.1913 \\ 3.6375 \end{bmatrix} \\ A_a = \begin{bmatrix} -2.7810 & 0.6633 \\ -2.8824 & -3.2606 \end{bmatrix}; A_{a,d} = \begin{bmatrix} -3.4736 & -0.3950 \\ -0.6333 & -0.2059 \end{bmatrix} \end{gathered} \quad (27)$$

The resulting optimal value of $\delta$ is $7.7256 \times 10^{-5}$, and the correspondent $\mathcal{L}_2$-bound for $w(t)$ is $1/\sqrt{\delta} = 113.7717$. Note that considering a static anti-windup strategy, i.e. $y_a = D_a\psi(y_c(t))$ (see also Gomes da Silva Jr. et al. [2006]), and solving a similar optimization problem, we find $1/\sqrt{\delta} = 80.3909$, which leads to a smaller bound on the admissible $\mathcal{L}_2$ disturbances.

| $\delta$ | $\sqrt{\gamma}$ |
|---|---|
| $7 \times 10^{-4}$ | 1.2122 |
| $7 \times 10^{-3}$ | 0.0242 |
| $7 \times 10^{-2}$ | 0.0142 |
| $7 \times 10^{-1}$ | 0.0135 |

Table 1. Trade-off between disturbance tolerance ($\delta$) $\times$ rejection ($\gamma$)

Table 1 presents results for different *a priori* values of $\delta$ and the respective values of $\gamma$ optimized through (24). The larger is the disturbance at the plant input (i.e. smaller is $\delta$), the greater is the $\mathcal{L}_2$-gain bound $\sqrt{\gamma}$. As the control is bounded, this relation is nonlinear. Clearly, there is a trade-off between the admissible disturbance bound given by $\delta$ and the achievable $\mathcal{L}_2$ performance given by $\gamma$.

Figure 1 depicts the behavior of the output $z(t) = y(t)$ of the closed-loop system with and without non-rational dynamic anti-windup compensation (27). The disturbance applied had the maximal tolerated $\mathcal{L}_2$-norm obtained from (23). Its amplitude was 11377, and it was applied from instant $t = 0$ until $t = 0.01s$.[1] It is possible to verify that with anti-windup compensation, the output $\mathcal{L}_2$-norm is smaller and the convergence of the trajectory to the origin is faster. Figure 2 illustrates the corresponding inputs $u(t)$. It can be noted that the actuator remains more time saturated in the configuration with no anti-windup compensation.

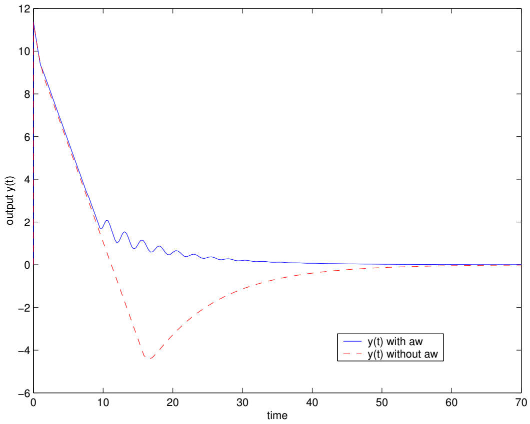


Fig. 1. $y(t)$ with (solid blue) and without (dashed red) anti-windup compensation

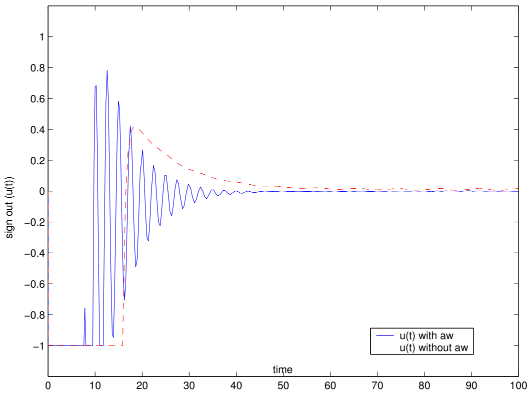


Fig. 2. $u(t)$ with (solid blue) and without (dashed red) anti-windup compensation

## 7. CONCLUDING REMARKS

In this paper, a methodology for computing non-rational dynamical anti-windup compensators has been proposed. The considered system may present delayed states on the dynamics of both plant and controller. Also, the open-loop plant may present unstable modes, since local stability is also addressed. The derived LMI theoretical conditions allow the synthesis of an anti-windup compensator, ensuring $\mathcal{L}_2$ performance enhancement with guaranteed input-to-state and internal stability of the closed-loop system, from the solution of convex optimization problems.

## REFERENCES

Y. Cao, Z. Lin, and T. Hu. Stability analysis of linear time-delay systems subject to input saturation. *IEEE Trans. on Circ. and Syst. I*, 49:233–240, 2002.

B. S. Chen, S. S. Wang, and H. C. Lu. Stabilization of time-delay systems containing saturating actuators. *Int. J. of Contr.*, 47:867–881, 1988.

[1] To obtain the anti-windup compensator, the LMITOOL Box of MATLAB *mincx* function was used, with parameters $\begin{bmatrix} .01 & 400 & 100 & 40 & 0 \end{bmatrix}$.

M.C. de Oliveira and J.C. Geromel. Synthesis of non-rational controllers for linear delay systems. *Automatica*, 40(2):171–188, 2004.

E. Fridman, A. Pila, and U. Shaked. Regional stabilization and $H_\infty$ control of time-delay systems with saturating actuators. *Int. J. of Rob. and Nonlin. Contr.*, 13:885–907, 2003.

J. M. Gomes da Silva Jr. and S. Tarbouriech. Anti-windup design with guaranted regions of stability: an lmi-based approach. *IEEE Trans. on Automatic Control*, 50:106–111, 2005.

J. M. Gomes da Silva Jr., S. Tarbouriech, and G. Garcia. Anti-windup design for time-delay systems subject to input saturation. an lmi-based approach. *European Journal of Control*, 12:622–634, Dec. 2006.

G. Grimm, J. Hatfield, I. Postlethwaite, A. Teel, M. Turner, and L. Zaccarian. Anti-windup for stable systems with input saturation: an LMI-based synthesis. *IEEE Trans. Automat. Contr.*, 48(9):1500–1525, 2003.

N. Kapoor, A. R. Teel, and P. Daoutidis. An anti-windup design for linear systems with input saturation. *Automatica*, 34(5):559–574, 1998.

M. V. Kothare and M. Morari. Multiplier theory for stability analisys of anti-windup control systems. *Automatica*, 35:917–928, 1999.

S-I. Niculescu. *Delay Effects on Stability. A Robust Control Approach.* Springer-Verlag, Berlin, Germany, 2001.

S-I. Niculescu, J. M. Dion, and L. Dugard. Robust stabilization for uncertain time-delay systems containing saturating actuators. *IEEE Trans. Autom. Contr.*, 41 (5):742–747, 1996.

S. Oucheriah. Global stabilization of a class of linear continuous time delay systems with saturating controls. *IEEE Trans. on Circ. and Syst. I*, 43(12):1012–1015, 1996.

J-K. Park, C-H. Choi, and H. Choo. Dynamic anti-windup method for a class of time-delay control systems with input saturation. *Int. J. of Rob. and Nonlin. Contr.*, 10:457–488, 2000.

J.P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39: 1667–1604, 2003.

C. Scherer, P. Gahinet, and M. Chilali. Multiobjective output-feedback control via LMI optimization. *IEEE Trans. on Automat. Contr.*, 42(7):896–911, 1997.

S. Tarbouriech and J. M. Gomes da Silva Jr. Synthesis of controllers for continuous-time delay systems with saturating controls via LMIs. *IEEE Trans. Autom. Contr.*, 45(1):105–111, 2000.

S. Tarbouriech, J. M. Gomes da Silva Jr., and G. Garcia. Delay-dependent anti-windup strategy for linear systems with saturating inputs and delayed outputs. *Int. J. of Rob. and Nonlin. Contr.*, 14:665–682, 2004.

E. Tissir and A. Hmamed. Further results on the stabilization of time delay systems containing saturating actuators. *Int. J. of Syst. Sci.*, 23:615–622, 1992.

M.C. Turner and I. Postlethwaite. A new perspective on static and low-order anti-windup synthesis. *Int. J. Control*, 77(1):27–44, 2004.

L. Zaccarian, D. Nesic, and A.R. Teel. $\mathcal{L}_2$ anti-windup for linear dead-time systems. *Syst. & Contr. Lett.*, 54(12): 1205–1217, 2005.