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Analysis and Improvements of a Systematic Componentwise Ultimate-bound Computation Method

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Abstract: We perform in-depth analysis and provide improvements of a systematic componentwise ultimate-bound computation method recently introduced in the literature. This method was shown to have many advantages over traditional ultimate-bound computation methods based on the use of quadratic Lyapunov functions. The analysis performed enhances our understanding of the componentwise methodology, and simplifies the search for improvements. The improvements provided aim at reducing the conservatism of the componentwise ultimate-bound computation methods even further, hence leading to tighter bounds. These improvements do not alter the systematic nature of the method.

Keywords: Ultimate bounds, componentwise analysis, systematic method, Lyapunov methods.

1. INTRODUCTION

The design of any realistic control system must necessarily take account of the effect of perturbations on the performance of the closed-loop system. Typically, the exact value of a perturbation variable is unknown but assumed to be bounded. In the presence of bounded perturbations that do not vanish as the state approaches an equilibrium point, asymptotic stability is in general not possible. However, under certain conditions, the *ultimate boundedness* of the system's trajectories can be guaranteed (Khalil, 2002). A guaranteed ultimate bound on the system's trajectories can be associated with good "attenuation" of the effect of perturbations. Estimation of an ultimate bound is of interest, e.g., in systems involving quantization (Williamson, 1991), unknown disturbance signals (Rapaport and Astolfi, 2002), and controller design via approximate discrete-time models (Nešić and Teel, 2004).

A standard approach for the computation of ultimate bounds is the use of level sets of suitable Lyapunov functions (see, e.g., Section 9.2 of Khalil, 2002). This Lyapunov approach can be applied to very general classes of systems, including nonlinear, and it is thus very powerful. However, this approach may result in conservative bounds in the linear case due to the loss of the structure of the system and also possibly of the perturbation, whose norm typically needs to be bounded for the analysis. Kofman (2005), Kofman et al. (2007a) and Haimovich (2006) presented a new method for ultimate bound computation based on componentwise analysis of the system in modal coordinates. This method allows direct derivation of componentwise ultimate bounds, exploiting the system geometry as well as the perturbation structure and requiring neither the computation of a Lyapunov function for the system nor bounding the norm of the perturbation vector. The examples in the latter references show that this componentwise approach may in some cases provide bounds that are much tighter than those obtained via standard Lyapunov analysis. In addition, this componentwise ultimate-bound computation approach has been successfully applied to the analysis of sampled-data systems involving quantization (Haimovich et al., 2007) and to the development of novel controller design methods (Kofman et al., 2007b, 2008).

The current work analyzes and improves the componentwise ultimate-bound computation method of Kofman et al. (2007a) for continuous-time systems in the case of constant perturbation bounds. We provide three main contributions. First, we modify Theorem 1 of Kofman et al. (2007a) so that a tighter ultimate-bound can be obtained. Second, we show exactly in what sense the ultimate bounds computed with our method are invariant under different choices of a Jordan decomposition required. The importance of this result lies in that it indicates whether and how our method can be further improved. Third, we outline a method for obtaining the tightest ultimate bound possible via our componentwise approach, point out the difficulties involved, and suggest a possible solution. An additional contribution of the current paper is the illustration of the shape of the implicit ultimate bounds given, which is not evident at first sight.

Notation. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $|M|$ and $\mathbb{R}\mathrm{e}(M)$ denote the *elementwise* magnitude and real part, respectively, of a matrix or vector M. If $x(t)$ is a vector-valued function, then $\limsup_{t\to\infty} x(t)$ denotes the vector obtained by taking $\limsup_{t\to\infty}$ of each component of $x(t)$, and similarly for

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'max'. If $x, y \in \mathbb{R}^n$, the expression '$x \preceq y$' denotes the set of componentwise inequalities $x_i \leq y_i$, $i = 1, \ldots, n$, between the components of x and y. $\mathbb{R}_+$ and $\mathbb{R}_{+,0}$ denote the positive and nonnegative real numbers. I_r denotes the $r \times r$ identity matrix, and j the imaginary unit ($j^2 = -1$).

2. PROBLEM STATEMENT AND PREVIOUS RESULTS

We consider a system defined by

$$\dot{x}(t) = Ax(t) + u(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, $u(t) \in \mathbb{R}^n$ a perturbation input and $A \in \mathbb{R}^{n\times n}$ is Hurwitz. Although the precise evolution of the perturbation input u is unknown, it is assumed that u is *componentwise* bounded as follows:

$$|u(t)| \preceq \mathbf{u}, \quad \text{for all } t \geq 0, \tag{2}$$

where $\mathbf{u} \in \mathbb{R}^n_{+,0}$. We emphasize that (2) represents a bound for *each component* of the perturbation input u. The issue that we address in this paper is the computation of an ultimate bound for the state x of system (1) when the perturbation u satisfies the componentwise constant bound (2). We will analyze and improve on the following result of Kofman et al. (2007a).

Theorem 1. (Theorem 1(ii) of Kofman et al. (2007a)). Consider system (1) where $x(t), u(t) \in \mathbb{R}^n$, and where $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix with (complex) Jordan canonical form $\Lambda = V^{-1}AV$. Suppose that $u(t)$ satisfies (2) and define

$$\beta(\Lambda, V, \mathbf{u}) \triangleq \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| \cdot |V^{-1}| \cdot \mathbf{u}, \tag{3}$$

$$\gamma(\Lambda, V, \mathbf{u}) \triangleq |V|\beta(\Lambda, V, \mathbf{u}). \tag{4}$$

Then, given any initial condition $x(0) \in \mathbb{R}^n$ and positive vector $\epsilon \in \mathbb{R}^n_+$, a finite time $t_f = t_f(\epsilon, x(0))$ exists so that for all $t \geq t_f$,

a') $|V^{-1}x(t)| \preceq \beta(\Lambda, V, \mathbf{u}) + \epsilon$,
b') $|x(t)| \preceq \gamma(\Lambda, V, \mathbf{u}) + |V|\,\epsilon$,

or, equivalently, for any initial condition $x(0) \in \mathbb{R}^n$,

a) $\limsup_{t\to\infty} |V^{-1}x(t)| \preceq \beta(\Lambda, V, \mathbf{u})$,
b) $\limsup_{t\to\infty} |x(t)| \preceq \gamma(\Lambda, V, \mathbf{u})$.

Note that the quantities β and γ defined in (3)–(4) are vectors with nonnegative components, and that a) and b) above express bounds for each component of $V^{-1}x(t)$ or $x(t)$ (see also the Notation section above). We refer to the bound given by a) above as *implicit* and to that given by b) as *explicit*.

The main aim of the current paper is to investigate the properties of and to derive improvements on the bounds given by Theorem 1.

3. BOUND PROPERTIES AND IMPROVEMENTS

In this section, we present the main contributions of the paper. In Section 3.1, we modify the result of Theorem 1 so that tighter ultimate bounds may be obtained in some cases. In Section 3.2, we derive useful properties of both the bounds given by Theorem 1 and their improved counterparts derived in Section 3.1. In Section 3.3, we outline a method to compute the tightest ultimate bounds that can be obtained by means of the componentwise methods presented.

3.1 Tighter Bound

The following theorem is only a slight modification of Theorem 1. This modification gives a tighter ultimate bound in some cases.

Theorem 2. Consider system (1) where $x(t), u(t) \in \mathbb{R}^n$, and where $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix with (complex) Jordan canonical form $\Lambda = V^{-1}AV$. Suppose that $u(t)$ satisfies (2) and define

$$\psi(\Lambda, V, \mathbf{u}) \triangleq \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| \max_{|u|\preceq\mathbf{u}} |V^{-1}u|, \tag{5}$$

$$\phi(\Lambda, V, \mathbf{u}) \triangleq |V|\psi(\Lambda, V, \mathbf{u}), \tag{6}$$

where the maximum in (5) is taken componentwise. Then, for any initial condition $x(0) \in \mathbb{R}^n$,

a) $\limsup_{t\to\infty} |V^{-1}x(t)| \preceq \psi(\Lambda, V, \mathbf{u})$,
b) $\limsup_{t\to\infty} |x(t)| \preceq \phi(\Lambda, V, \mathbf{u})$.

In addition, $\psi(\Lambda, V, \mathbf{u}) \preceq \beta(\Lambda, V, \mathbf{u})$ and $\phi(\Lambda, V, \mathbf{u}) \preceq \gamma(\Lambda, V, \mathbf{u})$, with β and γ as defined in (3)–(4).

Proof. The proof of Theorem 1 of Kofman et al. (2007a) is based on the coordinate transformation $x = Vz$, whereby the system equation (1) reads

$$\dot{z}(t) = \Lambda z(t) + v(t),$$

with $v(t) \triangleq V^{-1}u(t)$. Any subsequent derivation is based on a bound for each component of $|v(t)|$. We have

$$|v(t)| = |V^{-1}u(t)| \preceq \max_{|u|\preceq\mathbf{u}} |V^{-1}u| \preceq |V^{-1}|\mathbf{u}. \tag{7}$$

This establishes that $\max_{|u|\preceq\mathbf{u}} |V^{-1}u|$ is a suitable bound for $|v(t)|$, which may be tighter than the bound $|V^{-1}|\mathbf{u}$ employed in Kofman et al. (2007a). Items a) and b) above are then established by following exactly the same steps as in the proof of Theorem 1 of Kofman et al. (2007a). The fact that $\psi(\Lambda, V, \mathbf{u}) \preceq \beta(\Lambda, V, \mathbf{u})$ and $\phi(\Lambda, V, \mathbf{u}) \preceq \gamma(\Lambda, V, \mathbf{u})$ follows straightforwardly from (7). $\square$

The computation of (5) requires the solution to the n optimization problems $\max_{|u|\preceq\mathbf{u}} |V^{-1}u|$. We next show that the solution to these problems can be easily obtained. Express the matrix $V^{-1} \in \mathbb{C}^{n\times n}$ as

$$W \triangleq V^{-1} = X + jY, \tag{8}$$

where $X = \mathbb{R}\mathrm{e}(V^{-1})$ and $Y = \mathbb{I}\mathrm{m}(V^{-1})$. Let W_i^T, X_i^T, and Y_i^T denote the i-th rows of W, X, and Y, respectively, and note then that $|V^{-1}u| = \mathrm{col}(|W_1^T u|, \ldots, |W_n^T u|)$. We have

$$\begin{aligned} |W_i^T u| &= \sqrt{(X_i^T u)^2 + (Y_i^T u)^2} \\ &= \sqrt{u^T(X_i X_i^T + Y_i Y_i^T)u}. \end{aligned} \tag{9}$$

Therefore, $\max_{|u|\preceq\mathbf{u}} |V^{-1}u|$ is a set of n constrained convex maximization problems, each over the convex and bounded polyhedral constraint set $C \triangleq \{u \in \mathbb{R}^n : |u| \preceq \mathbf{u}\}$. Therefore, its solution can be found by evaluating each component of $|V^{-1}u|$ on the extreme points of C, $E(C)$ (see, e.g., Borwein and Lewis, 2000):

$$\max_{|u|\preceq\mathbf{u}} |V^{-1}u| = \max_{u\in C} |V^{-1}u| = \max_{u\in E(C)} |V^{-1}u|. \tag{10}$$

We show how this is computed in Example 1 below.

Proposition 3. Consider the statements of Theorems 1 and 2, and suppose that all the entries of V are real. Then, $\beta(\Lambda, V, \mathbf{u}) = \psi(\Lambda, V, \mathbf{u})$ and $\gamma(\Lambda, V, \mathbf{u}) = \phi(\Lambda, V, \mathbf{u})$, for all possible Λ, V and $\mathbf{u}$.

Proof. Since all the entries of V are real, then those of V^{-1} also are real. Let W_i^T denote the i-th row of V^{-1}. Since $W_i \in \mathbb{R}^n$, then $\max_{|u|\preceq\mathbf{u}} |W_i^T u| = |W_i^T|\mathbf{u}$. Therefore, $\max_{|u|\preceq\mathbf{u}} |V^{-1}u| = |V^{-1}|\mathbf{u}$, whence $\beta(\Lambda, V, \mathbf{u}) = \psi(\Lambda, V, \mathbf{u})$. In turn, it follows from this fact, (4) and (6) that $\gamma(\Lambda, V, \mathbf{u}) = \phi(\Lambda, V, \mathbf{u})$. □

Proposition 3 shows that the bounds given by Theorems 1 and 2 coincide if the entries of V are all real. However, if this is not the case, then the bounds given by Theorem 2 may indeed be tighter than those of Theorem 1, as the following example shows.

Example 1. Consider system (1) where

$$A = -\begin{bmatrix} 4.4 & 9.2 & 4.1 \\ 6.2 & 7.4 & 9.4 \\ 7.9 & 1.8 & 9.2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

A Jordan canonical form of A above is $\Lambda = V^{-1}AV$, with

$$\Lambda = \begin{bmatrix} -19.7 & 0 & 0 \\ 0 & -0.64 + j3 & 0 \\ 0 & 0 & -0.64 - j3 \end{bmatrix},$$

$$V = \begin{bmatrix} 0.54 & 0.71 & 0.71 \\ 0.67 & -0.045 - j0.35 & -0.045 + j0.35 \\ 0.52 & -0.55 + j0.27 & -0.55 - j0.27 \end{bmatrix}.$$

We directly compute, from (3) and (4) in Theorem 1,

$$\beta(\Lambda, V, \mathbf{u}) = \begin{bmatrix} 0.17 \\ 7.23 \\ 7.23 \end{bmatrix}, \quad \gamma(\Lambda, V, \mathbf{u}) = \begin{bmatrix} 10.32 \\ 5.26 \\ 8.91 \end{bmatrix}. \tag{11}$$

To compute ψ from (5), we first compute the extreme points of the polyhedral set $C \triangleq \{u \in \mathbb{R}^n : |u| \preceq \mathbf{u}\}$, namely $E(C)$. This gives $E(C) = \{u_i : i = 1, \ldots, 8\}$ with $u_1 = [3, 2, 1]^T$, $u_2 = [-3, 2, 1]^T$, $u_3 = [3, -2, 1]^T$, $u_4 = [-3, -2, 1]^T$, and $u_i = -u_{i-4}$ for $i = 5, \ldots, 8$. Since $|V^{-1}u| = |V^{-1}(-u)|$, we need to evaluate $|V^{-1}u|$ on only half of the extreme points to obtain the required maximum. We have

$$|V^{-1}u_1| = \begin{bmatrix} 3.4 \\ 0.87 \\ 0.87 \end{bmatrix}, \quad |V^{-1}u_2| = \begin{bmatrix} 0.04 \\ 3.3 \\ 3.3 \end{bmatrix},$$

$$|V^{-1}u_3| = \begin{bmatrix} 1.33 \\ 4.2 \\ 4.2 \end{bmatrix}, \quad |V^{-1}u_4| = \begin{bmatrix} 2.1 \\ 1.7 \\ 1.7 \end{bmatrix}.$$

Therefore, we have

$$\max_{|u|\preceq\mathbf{u}} |V^{-1}u| = \max_{u\in E(C)} |V^{-1}u| = \begin{bmatrix} 3.4 \\ 4.2 \\ 4.2 \end{bmatrix}.$$

We next compute, from (5) and (6) in Theorem 2,

$$\psi(\Lambda, V, \mathbf{u}) = \begin{bmatrix} 0.17 \\ 6.6 \\ 6.6 \end{bmatrix}, \quad \phi(\Lambda, V, \mathbf{u}) = \begin{bmatrix} 9.42 \\ 4.81 \\ 8.14 \end{bmatrix}. \tag{12}$$

Comparison of (11) and (12) shows that the bounds given by Theorem 2 are indeed tighter than their corresponding ones of Theorem 1, i.e., $\psi \preceq \beta$ and $\phi \preceq \gamma$. ∘

3.2 Bound Properties

Theorem 1 and its improved counterpart, Theorem 2, provide ultimate bounds for each component of the system state. These ultimate bounds require the matrices Λ and V from a Jordan decomposition of the system matrix A [see (3)–(6)]. The matrix Λ contains the Jordan blocks corresponding to the eigenvalues of A, and V the corresponding eigenvectors and additional vectors required to yield the Jordan canonical form. Recall that for a given A, these matrices Λ and V are not unique. Therefore, in this section we answer the question of whether different choices of Λ and V may yield different bounds according to the method of Theorem 1 or Theorem 2.

In the sequel, we will employ the following definition.

Definition 1. We say that (Λ, V) is a *Jordan decomposition* of a matrix $A \in \mathbb{R}^{n\times n}$ if $\Lambda = V^{-1}AV$ and Λ is in (complex) Jordan canonical form. ∘

We begin by finding all possible Jordan decompositions of a given matrix. We will then analyze whether it is possible to optimize the bounds given by Theorems 1 or 2 by selecting different Jordan decompositions.

Given the system matrix $A \in \mathbb{R}^{n\times n}$, let $(\bar{\Lambda}, \bar{V})$ be a Jordan decomposition of A. We must have

$$\bar{\Lambda} = \mathrm{diag}(J_1, \ldots, J_{\mathrm{K}}), \tag{13}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 0 & \ldots & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{C}^{\mathrm{R}_i\times \mathrm{R}_i} \tag{14}$$

where for $i = 1, \ldots, \mathrm{K}$, J_i is a Jordan block of order R_i with eigenvalue λ_i. The orders R_i satisfy $\sum_{i=1}^{\mathrm{K}} \mathrm{R}_i = n$. Let the columns of $\bar{V}$ be grouped as follows:

$$\bar{V} = \left[\bar{V}_1 | \ldots | \bar{V}_{\mathrm{K}}\right], \quad \bar{V}_i \in \mathbb{C}^{n\times \mathrm{R}_i}, \tag{15}$$

where for $i = 1, \ldots, \mathrm{K}$, $\bar{V}_i$ is a group of R_i columns of $\bar{V}$, corresponding to the Jordan block J_i. From $\bar{\Lambda}$ and $\bar{V}$ we may obtain all other Jordan decompositions of A as follows. First, we may multiply $\bar{V}$ by the following matrix:

$$M \triangleq \mathrm{diag}(M_1, \ldots, M_{\mathrm{K}}), \tag{16}$$

where each $M_i \in \mathbb{C}^{\mathrm{R}_i\times \mathrm{R}_i}$ is a Toeplitz [2] matrix of the form

$$M_i = \begin{bmatrix} 1 & q_{i,1} & \cdots & q_{i,\mathrm{R}_i-1} \\ 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & q_{i,1} \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{17}$$

Second, we may scale the columns of $\bar{V}$ in such a way that the same scaling factor α_i be applied to each group of columns $\bar{V}_i$:

$$\left[\alpha_1 \bar{V}_1 | \ldots | \alpha_{\mathrm{K}} \bar{V}_{\mathrm{K}}\right] = \bar{V} D,$$

$$D \triangleq \mathrm{diag}(\alpha_1 \mathrm{I}_{\mathrm{R}_1}, \ldots, \alpha_{\mathrm{K}} \mathrm{I}_{\mathrm{R}_{\mathrm{K}}}), \quad \alpha_i \in \mathbb{C} \setminus \{0\}. \tag{18}$$

Third, we may also reorder the Jordan blocks J_i of $\bar{\Lambda}$ and the corresponding groups of columns of $\bar{V}$, $\bar{V}_i$. In matrix notation, this is achieved via multiplication by a permutation matrix $P \in \{0, 1\}^{n\times n}$, as follows:

$$V = \bar{V} P, \qquad \Lambda = P^{-1} \bar{\Lambda} P. \tag{19}$$

Combining the three operations mentioned above, we may thus obtain other Jordan decompositions (Λ, V) of A from $(\bar{\Lambda}, \bar{V})$ as follows:

$$V = \bar{V} M D P, \qquad \Lambda = P^{-1} \bar{\Lambda} P. \tag{20}$$

We have the following result.

[2] Toeplitz matrices are constant along diagonals.

Lemma 4. Let $(\bar{\Lambda}, \bar{V})$ be a Jordan decomposition of A. Then all other Jordan decompositions (Λ, V) of A are given by (20), where M satisfies (16)–(17) for some $q_{i,\ell} \in \mathbb{C}$, $i = 1, \dots, \mathrm{K}$, $\ell = 1, \dots, \mathrm{R}_i - 1$, K is the number of Jordan blocks of $\bar{\Lambda}$, R_i is the order of the i-th Jordan block, D satisfies (18), and $P \in \{0,1\}^{n \times n}$ is a permutation matrix such that Λ is just a reordering of the Jordan blocks of $\bar{\Lambda}$ and each Jordan block remains unchanged.

Proof. We shall establish that (Λ, V) is a Jordan decomposition of A. The proof that these are all possible Jordan decompositions is left to the reader. Since $\bar{\Lambda}$ is in Jordan canonical form and Λ is just a reordering of the Jordan blocks of $\bar{\Lambda}$, then Λ also is in Jordan canonical form. Next, we have:

$$\begin{aligned} V\Lambda V^{-1} &= \bar{V}MDP\,P^{-1}\bar{\Lambda}P\,P^{-1}D^{-1}M^{-1}\bar{V}^{-1} \\ &= \bar{V}MD\bar{\Lambda}D^{-1}M^{-1}\bar{V}^{-1}. \end{aligned}$$

From (13) and (18), it follows that $D\bar{\Lambda} = \bar{\Lambda}D$. Also, from (13)–(14) and (16)–(17), it can easily be shown that $M\bar{\Lambda} = \bar{\Lambda}M$. Using these facts in the equation above yields

$$\begin{aligned} V\Lambda V^{-1} &= \bar{V}MD\bar{\Lambda}D^{-1}M^{-1}\bar{V}^{-1} \\ &= \bar{V}\bar{\Lambda}MDD^{-1}M^{-1}\bar{V}^{-1} = \bar{V}\bar{\Lambda}\bar{V}^{-1}, \end{aligned}$$

which establishes that (Λ, V) is indeed a Jordan decomposition of A. □

We are now ready to state the main result of this section.

Theorem 5. Let $(\bar{\Lambda}, \bar{V})$ be a Jordan decomposition of $A \in \mathbb{R}^{n\times n}$ and let Λ and V satisfy (20), where M satisfies (16)–(17), D satisfies (18), and the permutation matrix $P \in \{0,1\}^{n\times n}$ is such that Λ is just a reordering of the Jordan blocks of $\bar{\Lambda}$. Then, for all $\mathbf{u} \in \mathbb{R}^n_{+,0}$

$$\gamma(\Lambda, V, \mathbf{u}) = \gamma(\bar{\Lambda}, \bar{V}M, \mathbf{u}), \tag{21}$$

$$\phi(\Lambda, V, \mathbf{u}) = \phi(\bar{\Lambda}, \bar{V}M, \mathbf{u}). \tag{22}$$

Before proving Theorem 5, the following comments are in order. First, note that Theorem 5 establishes that the explicit bounds given by either Theorem 1 or Theorem 2 are invariant under reordering of the Jordan blocks of $\bar{\Lambda}$. Second, Theorem 5 shows that these bounds are, in addition, invariant under scaling of the groups of columns of $\bar{V}$ corresponding to each Jordan block of $\bar{\Lambda}$. Consequently, the only operation on $\bar{V}$ that may alter the explicit bounds provided by Theorems 1 or 2 is multiplication by M, as defined in (16)–(17).

The proof of Theorem 5 requires the following result, whose proof is immediate.

Lemma 6. Let $T \in \mathbb{C}^{n\times n}$ be an arbitrary invertible matrix, $D \in \mathbb{C}^{n\times n}$ be diagonal and nonsingular, and $P \in \{0,1\}^{n\times n}$ be a permutation matrix. Then,

i) $|D^{-1}| = |D|^{-1}$.
ii) $|TD| = |T||D|$.
iii) $|D^{-1}T| = |D^{-1}||T|$.
iv) $|TP| = |T|P$.
v) $|P^{-1}T| = P^{-1}|T|$.

Proof of Theorem 5. Since $\mathbb{R}\mathrm{e}(\Lambda)$ constitutes a componentwise operation on the elements of Λ, and since $\Lambda = P^{-1}\bar{\Lambda}P$ is just a reordering of the elements of $\bar{\Lambda}$, then $\mathbb{R}\mathrm{e}(\Lambda) = P^{-1}\mathbb{R}\mathrm{e}(\bar{\Lambda})P$, whence $[\mathbb{R}\mathrm{e}(\Lambda)]^{-1} = P^{-1}[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}P$. Applying Lemma 6 iv) and v), then

$$\left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| = P^{-1}\left|[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}\right|P. \tag{23}$$

From (20), we have

$$|V| = |\bar{V}MDP| \overset{\text{iv)}}{=} |\bar{V}MD|P \overset{\text{ii)}}{=} |\bar{V}M||D|P \tag{24}$$

$$\begin{aligned} |V^{-1}| &= |P^{-1}D^{-1}M^{-1}\bar{V}^{-1}| \overset{\text{v)}}{=} P^{-1}|D^{-1}M^{-1}\bar{V}^{-1}| \\ &\overset{\text{iii)}}{=} P^{-1}|D^{-1}||M^{-1}\bar{V}^{-1}|, \end{aligned} \tag{25}$$

where the text over the equal signs above shows which part of Lemma 6 is being applied. From (13), it follows that

$$\left|[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}\right| = \mathrm{diag}(|\mathbb{R}\mathrm{e}(J_1)^{-1}|, \dots, |\mathbb{R}\mathrm{e}(J_{\mathrm{K}})^{-1}|),$$

and from (18)

$$|D| = \mathrm{diag}(|\alpha_1|\mathrm{I}_{r_1}, \dots, |\alpha_{\mathrm{K}}|\mathrm{I}_{\mathrm{R}_{\mathrm{K}}}).$$

It thus follows that

$$|D|\left|[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}\right| = \left|[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}\right||D|. \tag{26}$$

Combining (23)–(25) yields

$$\begin{aligned} &|V|\left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right||V^{-1}| \\ &= |\bar{V}M||D|PP^{-1}\left|[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}\right|PP^{-1}|D^{-1}||M^{-1}\bar{V}^{-1}| \\ &\overset{(26)}{=} |\bar{V}M|\left|[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}\right||D||D^{-1}||M^{-1}\bar{V}^{-1}| \\ &\overset{\text{i)}}{=} |\bar{V}M|\left|[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}\right||M^{-1}\bar{V}^{-1}|. \end{aligned} \tag{27}$$

Therefore, (21) is established from (4), (3), and (27). Next, we have

$$\begin{aligned} \max_{|u|\preceq\mathbf{u}} |V^{-1}u| &= \max_{|u|\preceq\mathbf{u}} P^{-1}|D^{-1}||M^{-1}\bar{V}^{-1}u| \\ &= P^{-1}|D^{-1}|\max_{|u|\preceq\mathbf{u}} |M^{-1}\bar{V}^{-1}u|. \end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned} &|V|\left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right|\max_{|u|\preceq\mathbf{u}}|V^{-1}u| = \\ &\qquad |\bar{V}M|\left|[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}\right|\max_{|u|\preceq\mathbf{u}}|M^{-1}\bar{V}^{-1}u|. \end{aligned} \tag{29}$$

Then, (22) is established from (6), (5), and (29). This concludes the proof of Theorem 5. □

Corollary 7. Suppose that the system matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable and let $(\bar{\Lambda}, \bar{V})$, (Λ, V) be any two Jordan decompositions of A. Then, for all $\mathbf{u} \in \mathbb{R}^n_{+,0}$

$$\gamma(\bar{\Lambda}, \bar{V}, \mathbf{u}) = \gamma(\Lambda, V, \mathbf{u}), \tag{30}$$

$$\phi(\bar{\Lambda}, \bar{V}, \mathbf{u}) = \phi(\Lambda, V, \mathbf{u}). \tag{31}$$

Proof. Since A is diagonalizable, then any Jordan decomposition of A must necessarily comprise n Jordan blocks, each of order 1. Consequently, given an arbitrary Jordan decomposition $(\bar{\Lambda}, \bar{V})$ any other Jordan decomposition (Λ, V) must satisfy (20), with $M = \mathrm{I}_n$ from (16)–(17) because $\mathrm{K} = n$ and $\mathrm{R}_i = 1$ for $i = 1, \dots, \mathrm{K}$. The result is then established by application of Theorem 5. □

The main conclusion of this subsection is thus that the explicit bounds provided by either Theorem 1 or Theorem 2 might be improved by selecting a suitable Jordan decomposition only in the case when the system matrix A is not diagonalizable, and that this may be achieved only via multiplying the eigenvector matrix V by a matrix M satisfying (16)–(17).

3.3 Optimized Bound

We have already shown that the explicit bounds given by Theorems 1 and 2 are invariant under reordering of the Jordan blocks of the system matrix and under scaling of columns of V. In this section, we outline a method to optimize the bounds given by Theorems 1 or 2.

Given a Jordan decomposition $(\bar{\Lambda}, \bar{V})$ of the system matrix A, according to Theorem 5 we know that the only other Jordan decompositions of A that may yield different explicit componentwise ultimate bounds are given by $(\bar{\Lambda}, \bar{V}M)$, where M satisfies (16)–(17). We next seek the matrix M that yields the smallest explicit componentwise ultimate bounds γ or ϕ, as given by Theorems 1 and 2. We thus pose the following optimization problems.

Problem 8. Given a Jordan decomposition $(\bar{\Lambda}, \bar{V})$ of the system matrix $A \in \mathbb{R}^{n\times n}$, such that $\bar{\Lambda}$ satisfies (13)–(14), where K is the number of Jordan blocks of $\bar{\Lambda}$ and R_i is the order of the i-th Jordan block, for $i = 1, \ldots, \mathrm{K}$, solve

$$\tilde{\gamma}_s \triangleq \min_M \gamma_s(\bar{\Lambda}, \bar{V}M, \mathbf{u}), \text{ or } \tilde{\phi}_s \triangleq \min_M \phi_s(\bar{\Lambda}, \bar{V}M, \mathbf{u})$$

for $s = 1, \ldots, n$, where γ_s and ϕ_s denote the s-th components of the bounds γ and ϕ given by Theorems 1 and 2, respectively, subject to

- M satisfies (16)–(17), $q_{i,\ell} \in \mathbb{C}$ for $i = 1, \ldots, \mathrm{K}$ and $\ell = 1, \ldots, \mathrm{R}_i - 1$. ∘

The solution to Problem 8 gives the tightest explicit ultimate bounds that can be obtained by application of Theorems 1 or 2. However, the complexity of these optimization problems increases with the order of the Jordan blocks of $\bar{\Lambda}$, as we next show for γ. A similar analysis can be performed for ϕ.

From (4) and (3), it follows that

$$\gamma(\bar{\Lambda}, \bar{V}M, \mathbf{u}) = |\bar{V}M| \left|[\mathbb{R}\mathrm{e}(\bar{\Lambda})]^{-1}\right| |M^{-1}\bar{V}^{-1}| \mathbf{u}. \tag{32}$$

Let the rows of $\bar{V}^{-1}$ be grouped as follows:

$$\bar{V}^{-1} = \mathrm{col}(Z_1^T, \ldots, Z_{\mathrm{K}}^T), \tag{33}$$

where $Z_i^T \in \mathbb{C}^{\mathrm{R}_i\times n}$, and recall grouping the columns of $\bar{V}$ as in (15). With this notation and recalling (13) and (16), we can write

$$\gamma(\bar{\Lambda}, \bar{V}M, \mathbf{u}) = \sum_{i=1}^{\mathrm{K}} |\bar{V}_i M_i||\mathbb{R}\mathrm{e}(J_i)^{-1}||M_i^{-1} Z_i^T| \mathbf{u}. \tag{34}$$

From (17), it follows that M_i^{-1} is Toeplitz of the form

$$M_i^{-1} = \begin{bmatrix} 1 & p_{i,1} & \ldots & p_{i,\mathrm{R}_i-1} \\ 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & p_{i,1} \\ 0 & \ldots & 0 & 1 \end{bmatrix}, \tag{35}$$

where $p_{i,\ell}$ satisfy the recursion

$$p_{i,\ell} = -\sum_{m=1}^{\ell} q_{i,m} p_{i,\ell-m}, \quad \text{with } p_{i,0} \triangleq 1, \tag{36}$$

for $i = 1, \ldots, \mathrm{K}$ and $\ell = 1, \ldots, \mathrm{R}_i - 1$. Note from (36) that each quantity $p_{i,\ell}$ is a polynomial in the variables $q_{i,1}, \ldots, q_{i,\ell}$, and that the degrees of these polynomials increase with the order R_i of the corresponding Jordan block. As an example, we have, for a Jordan block J_i of order $\mathrm{R}_i = 4$:

$$p_{i,1} = -q_{i,1}, \quad p_{i,2} = -q_{i,2} + q_{i,1}^2, \tag{37}$$

$$p_{i,3} = -q_{i,3} + 2q_{i,2}q_{i,1} - q_{i,1}^3. \tag{38}$$

Therefore, the combination of (32), (34), and (35)–(36) shows that Problem 8 involves optimizing over sums and possibly products of absolute values of polynomials in the quantities $q_{i,\ell}$, where the complexity of such optimization problem increases with the order of the Jordan blocks of $\bar{\Lambda}$. Should the complexity of the optimization problem be prohibitive, a suboptimal solution can easily be obtained by optimizing over only some of the $q_{i,\ell}$, leaving the rest fixed at an arbitrary value. For example, for a Jordan block J_i of order $\mathrm{R}_i = 4$, setting $q_{i,1} = 0$ yields linear expressions in (37)–(38). The resulting bounds, though possibly suboptimal, will nonetheless be not worse than those obtained by direct application of Theorem 1 using the given Jordan decomposition $(\bar{\Lambda}, \bar{V})$ of A.

We next provide an example of application of the bound optimization procedure of Problem 8.

Example 2. Consider system (1) where

$$A = \begin{bmatrix} 0.2 & -0.8 \\ 1.8 & -2.2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \tag{39}$$

A Jordan decomposition $(\bar{\Lambda}, \bar{V})$ of A in (39) is

$$\bar{\Lambda} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad \bar{V} = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & -0.2 \end{bmatrix}. \tag{40}$$

Direct application of Theorems 1 or 2 (recall that according to Proposition 3, there is no difference in applying either theorem in this case, since the entries of $\bar{V}$ are all real) yields:

$$\gamma(\bar{\Lambda}, \bar{V}, \mathbf{u}) = \phi(\bar{\Lambda}, \bar{V}, \mathbf{u}) = [9.2 \;\; 12.6]^T. \tag{41}$$

We see that $\bar{\Lambda} = J_1 \in \mathbb{R}^{2\times 2}$, i.e., $\bar{\Lambda}$ consists of only one Jordan block of order 2 ($\mathrm{K} = 1$, $\mathrm{R}_1 = 2$). The matrix M over which the optimization of Problem 8 is performed has the form

$$M = \begin{bmatrix} 1 & q_{1,1} \\ 0 & 1 \end{bmatrix}, \text{ with } M^{-1} = \begin{bmatrix} 1 & p_{1,1} \\ 0 & 1 \end{bmatrix}, \tag{42}$$

where, according to (36), we have $p_{1,1} = -q_{1,1}$. Defining $q \triangleq q_{1,1}$ and operating as outlined above, we obtain

$$\gamma(\bar{\Lambda}, \bar{V}M, \mathbf{u}) = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 0.8\,|1-3q| + 3.6\,|1+2q| + 4.8 \\ 3.6\,|1-3q| + 1.8\,|1+2q| + 7.2 \end{bmatrix} \tag{43}$$

Eq.(43) clearly shows that $\gamma(\bar{\Lambda}, \bar{V}M, \mathbf{u})$ depends on the value of q, and hence depends on the matrix M. The optimized bound is $\tilde{\gamma}_1 = \min_{q\in\mathbb{C}} \gamma_1 = 6.8$, (at $q = -1/2$), $\tilde{\gamma}_2 = \min_{q\in\mathbb{C}} \gamma_2 = 10.2$, (at $q = 1/3$).We thus obtain the optimized bound $[6.8 \;\, 10.2]^T$, tighter than (41). Note that the tightest bounds for each component are obtained for different values of q, and hence correspond to different Jordan decompositions of A. This means that the optimized bound cannot result from application of Theorem 1 using a single Jordan decomposition of A. ∘

4. IMPLICIT BOUND RESULTS

The main aim of this section is to give insight into the shape of the implicit ultimate bounds provided by Theorems 1 and 2. These implicit ultimate bounds are given by Theorems 1a) and 2a). According to these results, if $(\bar{\Lambda}, \bar{V})$ is a Jordan decomposition of the system matrix $A \in \mathbb{R}^{n\times n}$, note that associated with each Jordan block $J_i \in \mathbb{C}^{\mathrm{R}_i\times \mathrm{R}_i}$ of $\bar{\Lambda}$, there is a set of R_i inequalities:

- Theorem 1a):

$$\limsup_{t\to\infty} |Z_i^T x(t)| \preceq |\mathbb{R}\mathrm{e}(J_i)^{-1}||Z_i^T|\mathbf{u}, \tag{44}$$

- Theorem 2a):

$$\limsup_{t\to\infty} |Z_i^T x(t)| \preceq |\mathbb{R}\mathrm{e}(J_i)^{-1}| \max_{u\preceq\mathbf{u}} |Z_i^T u|, \tag{45}$$

for $i = 1, \ldots, \kappa$, where κ is the number of Jordan blocks of $\bar{\Lambda}$ and $Z_i^T \in \mathbb{C}^{\mathrm{R}_i \times n}$ is a group of R_i rows of $\bar{V}^{-1}$ that corresponds to Jordan block J_i, as in (33).

The following result shows that not all inequalities above are necessarily independent.

Lemma 9. Let $(\bar{\Lambda}, \bar{V})$ be a Jordan decomposition of the system matrix $A \in \mathbb{R}^{n \times n}$. Let $\bar{\Lambda}$ satisfy (13)–(14), where κ is the number of Jordan blocks of $\bar{\Lambda}$ and R_i is the order of the i-th Jordan block. Suppose that J_ℓ is a Jordan block of $\bar{\Lambda}$ with complex eigenvalue $\lambda_\ell \notin \mathbb{R}$ and of order R_ℓ. Then, there exists J_s, another Jordan block of $\bar{\Lambda}$, with eigenvalue $\lambda_s = \lambda_\ell^*$, the complex conjugate of λ_ℓ, and of order $\mathrm{R}_s = \mathrm{R}_\ell$. Then, the two sets of $\mathrm{R}_s = \mathrm{R}_\ell$ inequalities given by (44) above for $i = \ell, s$ coincide, and the same happens for the two sets given by (45) for $i = \ell, s$.

Proof. That J_s must exist follows from the fact that A has real entries and $\bar{\Lambda}$ is a Jordan canonical form of A. The entries of J_ℓ and Z_ℓ^T necessarily are the complex conjugates of those of J_s and Z_s^T, respectively. Then, $\mathbb{Re}(J_\ell) = \mathbb{Re}(J_s)$, $|Z_\ell^T| = |Z_s^T|$, and $|Z_\ell^T u| = |Z_s^T u|$ and $|Z_\ell^T x| = |Z_s^T x|$, since $x, u \in \mathbb{R}^n$. Therefore,

$$|\mathbb{Re}(J_\ell)^{-1}||Z_\ell^T|\mathbf{u} = |\mathbb{Re}(J_s)^{-1}||Z_s^T|\mathbf{u}$$
$$|\mathbb{Re}(J_\ell)^{-1}| \max_{u \preceq \mathbf{u}} |Z_\ell^T u| = |\mathbb{Re}(J_s)^{-1}| \max_{u \preceq \mathbf{u}} |Z_s^T u|.$$

The result follows from these equalities, jointly with the fact that $|Z_\ell^T x| = |Z_s^T x|$ for all $x \in \mathbb{R}^n$. □

Lemma 9 shows that only half of the inequalities given by Theorems 1a) or 2a) for a pair of complex conjugate eigenvalues need to be considered, since the other half is equivalent. We next provide an illustration of this fact.

Example 3. Consider again A and $\mathbf{u}$, and the Jordan decomposition (Λ, V) given in Example 1. We have $\Lambda = \mathrm{diag}(J_1, J_2, J_3)$ with $J_1 = -19.7$, $J_2 = -0.64 + j3$, $J_3 = -0.64 - j3$, and we can straightforwardly compute

$$V^{-1} = \begin{bmatrix} Z_1^T \\ Z_2^T \\ Z_3^T \end{bmatrix} = \begin{bmatrix} 0.56 & 0.52 & 0.68 \\ 0.49-j0.47 & -0.2+j0.9 & -0.26-j0.68 \\ 0.49+j0.47 & -0.2-j0.9 & -0.26+j0.68 \end{bmatrix}. \quad (46)$$

Note that we can apply Lemma 9 with $\ell = 2$ and $s = 3$. Recalling (11) and (12), it follows from:

- Theorem 1a):
$$\limsup_{t\to\infty} |Z_2^T x(t)| \preceq 7.23, \quad \limsup_{t\to\infty} |Z_3^T x(t)| \preceq 7.23.$$
- Theorem 2a):
$$\limsup_{t\to\infty} |Z_2^T x(t)| \preceq 6.6, \quad \limsup_{t\to\infty} |Z_3^T x(t)| \preceq 6.6.$$

Since the entries of Z_2^T are the complex conjugates of the corresponding ones of Z_3^T [see (46)], then each pair of inequalities above is equivalent. ○

We next analyze the shape of the implicit bounds given by Theorems 1a) and 2a). Again, express V^{-1} as in (8), where $X = \mathbb{Re}(V^{-1})$ and $Y = \mathbb{Im}(V^{-1})$, and let W_i^T, X_i^T, and Y_i^T denote the i-th rows of W, X, and Y, respectively. According to Theorems 1a) and 2a), we have

$$\limsup_{t\to\infty} |W_i^T x(t)| \le \sigma_i, \quad \text{for } i = 1, \ldots, n, \quad (47)$$

and where $\sigma_i \in \mathbb{R}_{+,0}$ is a component of either β or ψ, depending on whether Theorem 1 or 2 is employed. Define

$$U_i \triangleq \{x \in \mathbb{R}^n : |W_i^T x| \le \sigma_i\}. \quad (48)$$

Then, (47)–(48) show that the state $x(t)$ ultimately tends to the ultimate-bound set $U = \bigcap_{i=1}^n U_i$. Note that

$$U_i = \left\{x \in \mathbb{R}^n : x^T(X_i X_i^T + Y_i Y_i^T)x \le \sigma_i^2\right\},$$

which constitutes an ellipsoid. It can be shown that if, for some $s \in \{1, \ldots, n\}$, W_s^T corresponds to a Jordan block of Λ with real eigenvalue, then $\mathrm{rank}(X_s X_s^T + Y_s Y_s^T) = 1$ and hence U_s degenerates into the space contained between two parallel hyperplanes. On the other hand, if W_s^T corresponds to a complex eigenvalue in the same sense, then $\mathrm{rank}(X_s X_s^T + Y_s Y_s^T) = 2$, but according to Lemma 9, there will be some $U_\ell = U_s$ among the $n-1$ remaining sets U_i.

5. CONCLUSION

We have performed in-depth analysis and provided improvements of a systematic componentwise ultimate-bound computation method previously introduced in the literature. As a result of our analysis, we have established the invariance of the componentwise ultimate bounds given by such method under certain choices of Jordan decompositions, and we have illustrated the shape of the implicit componentwise bounds. The improvements provided consist in (a) a modification of the method that allows the obtention of tighter bounds, and (b) the derivation of a set of optimization problems whose solutions yield tighter bounds for each component of the system state. A noteworthy conclusion is that the improved bounds can still be obtained in a systematic way, a key feature of the method analyzed.

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