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# Modeling Approaches and Robust Stability Conditions for Networked Controlled Systems with Uncertain Delays ⋆

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**Abstract:** In this article two modeling approaches for Networked Controlled Systems (NCS) with different types of uncertainly varying bounded transmission delays and static discrete–time control laws are presented. Different models are offered for each case, all linked to the objective of designing robust discrete-time controllers. It is analytically shown how the careful mixing of asynchronous (event–driven) and synchronized (clocked) signals can lead to discrete time uncertain (possibly switched) systems, where results form robust control analysis and synthesis can be applied. After showing the implications of these modelling results for control synthesis purposes, sufficient conditions for the robust stability are given for each approach and a comparison of the conservatism of results is discussed.

Keywords: Networked Control Systems, Uncertain dynamic systems, Robust Stability, Control over networks.

## 1. INTRODUCTION

It is well known that one of the primary effects and major control challenges in Networked Controlled System (NCS) is the presence of uncertain network-induced delays stemming from the very fact of utilizing a common communication channel for closing the loop Baillieul and Antsaklis [2007], Hespanha et al. [2007].

Network-induced delays in NCS commonly appear in the information flow between the sensor and the controller (delay $\tau_{sc}(k)$), as also between the controller and the actuator (delay $\tau_{ca}(k)$), where '$k$' denotes the dependence on the $k^{th}$ sampling period. As has been shown, when a static feedback law is employed, it is allowed to lump $\tau_{sc}(k)$ and $\tau_{ca}(k)$ into one delay $\tau^k \triangleq \tau_{sc}(k) + \tau_{ca}(k)$ (Nilsson et al. [1998], Zhang et al. [2001], Tzes et al. [2005]).

The type and the characteristics of the underlying delays are varying and in most of the cases depend on the utilized network protocol, the scheduling methods, the communication overhead (collisions/retrasmissions), the packet losses, and in general to uncertain factors that can deteriorate the stability and performance of the controlled system, sometimes even driving it to instability Zhang et al. [2001]. Significant effort has hence been invested in developing control methodologies to handle the network delay effect in NCSs. A survey of control methodologies for a closed–loop control system over a data network has been presented in Tipsuwan and Chow [2003], Hespanha et al. [2007].

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For NCS using random access MAC protocols (Ethernet, DeviceNet) the assumption of equidistant sampling and constant network delay may no longer be valid (see Naghshtabrizi and Hespanha [2006], Hespanha et al. [2007] for the variable sampling case). Hence a more cautious treatment of the modeling and discretization procedure is necessary, and even more so for the control synthesis.

The remainder of this work starts with the general setup regarding the modeling of NCS in Section 2. The two proposed modeling schemes (presented in sections 3 and 4), allow the control designer to embed in a combined NCS dynamic model (plant,controller,network, sample and hold devices), network–induced delays smaller that one sampling period,with known bounds, uncertainly varying, or constant and unknown. Moreover the ensuing robust stability conditions do not need the a-priori knowledge of the probability distribution functions of the uncertain delays. Comparison of the two approaches is presented in 5, while the conclusions are drawn in Section 6.

Inhere, the case of SISO systems with "less than one sampling period delay", ($\tau^k < h$) is examined. Recent works concerning Maximum Allowable Transfer Interval ("M.A.T.I.") computations, have revived the interest for this case of systems( Kim et al. [2003]).

## 2. MODELING ISSUES FOR NCS

The dynamics of the NCS under investigation is described by the combination of a continuous–time linear time–invariant plant with a discrete–time controller Zhang et al. [2001]. The sampling period $h$ is assumed to be constant and known, whereas both controller and actuator (including the zero-order-hold ZOH) are event-driven devices in the sense that they update their outputs as soon as they receive a new sample. The control architecture for this case is shown in Figure 1 where a remote controller, non-collocated with the sensor and actuator is employed (see Dritsas et al. [2007b], Dritsas and Tzes [2007] for further analysis and a timing diagram). The dynamics of the plant

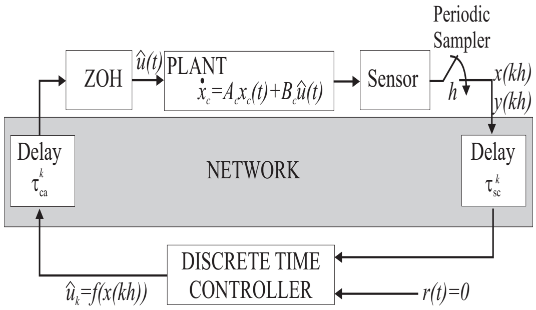


Fig. 1. NCS with uncertain time delays in the measurement and actuation path

in this case can be cast in the following formulation:

$$\begin{aligned}\dot{x}(t) &= A_c x(t) + B_c \hat{u}(t), \quad y(t) = C_c x(t)\\ & \quad t \in \left[kh+\tau^k, kh+h+\tau^{k+1}\right)\\ \hat{u}(t) &= \begin{cases} \hat{u}_{k-1}, \; t \in \left[kh-h+\tau^{k-1}, \; kh+\tau^k\right) \\ \hat{u}_k, \quad\; t \in \left[kh+\tau^k, \; kh+h+\tau^{k+1}\right). \end{cases}\end{aligned} \tag{1}$$

In the sequel, for notation simplifications, the following symbols will be used: $\bar{0} \triangleq 0_{1\times n}$ is an $n$-column zero vector, $I_n$ $(0_n)$ is the $n \times n$ identity (zero)matrix, $M > 0$ $(< 0)$ means that $M$ is positive (negative) definite. $\sigma_{\max}(M)$ is the maximum singular value of the matrix M and $\sigma_{\min}(M)$ its minimum singular value. Furthermore, we shall hereafter use the notation $\{x_{k+1}, \; x_k, \ldots\}$ in order to denote the values $\{x(kh+h), x(kh), ...\}$ of the discrete–time signal coming out of the periodic sampler.

The total delay within the $k$th sampling period, is denoted by $\tau^k = \tau^k_{sc} + \tau^k_{ca}$ and is assumed upper bounded as $0 \le \tau_{\min} < \tau^k \le \tau_{\max} = h$. The delay $\tau^k$ is in general a time–varying and uncertain quantity, reflecting the nature of the network involved, the network load, etc. In (1) $\hat{u}(t)$ is the "most recent" control action presented to the event–driven actuator at the time instance $t$ within a sampling period (i.e. within the time interval $[kh, \; kh+h)$), and can take either one of the two values $\hat{u}_{k-1}$ or $\hat{u}_k$.

It must be emphasized that the discrete–time piecewise constant control action $\hat{u}(t)$ experiences a "jump" at the uncertain time instance $kh + \tau^k$, in other words the actuation time instances are not equidistant. Hence (unless $\tau^k$ is constant) it is not in general possible to treat the ensuing NCS in a standard "sampled-data" or "time–delayed" setting. Instead a "hybrid" setup should rather be used. Initial efforts towards this objective have been proposed and successfully used specifically for NCS in Naghshtabrizi et al. [2006], Naghshtabrizi and Hespanha [2006], Hespanha et al. [2007], Dritsas et al. [2007b], Dritsas and Tzes [2007].

Despite the "jump" nature of $\hat{u}(t)$, the discretization of (1) is straightforward and the ensuing discretization is exact in the sense that it correctly describes the evolution of the state vector at the discrete time instances, and is given by Dritsas et al. [2007b], Dritsas and Tzes [2007]:

$$x_{k+1} = \Phi x_k + \Gamma_0(\tau^k)\hat{u}_k + \Gamma_1(\tau^k)\hat{u}_{k-1} \tag{2}$$

where $\Phi = \exp(A_c h)$ and

$$\begin{aligned}\Gamma_0(\tau^k) &= \int_0^{h-\tau^k} \exp(A_c\lambda)B_c d\lambda \;, \\ &= \begin{bmatrix} I_n & \bar{0}^T \end{bmatrix} \exp\left( \begin{bmatrix} A_c & B_c \\ \bar{0} & 0 \end{bmatrix} (h-\tau^k) \right) \begin{bmatrix} \bar{0}^T \\ 1 \end{bmatrix} \\ \Gamma_1(\tau^k) &= \int_{h-\tau^k}^{h} \exp(A_c\lambda)B_c d\lambda = -\Gamma_0(\tau^k) + \int_0^h \exp(A_c\lambda)B_c d\lambda\end{aligned} \tag{3}$$

Equation (2) will be the starting point for two different modeling procedures. Motivated by the arbitrarily varying (but bounded) and uncertain nature of the delay $\tau^k$, and following procedures and arguments similar to the ones described in Tipsuwan and Chow [2003], Wang et al. [1994], the system's nominal model and the corresponding control synthesis is intentionally based on the choice of the average delay $\tau^\circ = (\tau_{\min} + \tau_{\max})/2$ as the nominal value of the uncertain delay. The actual uncertain delay can then be modelled (decomposed) as $\tau^{k'} = \tau^\circ + \delta\tau$ . As analytically presented in Dritsas et al. [2007b, 2006b,a, 2007a] the matrices $\Gamma_0(\tau^k), \; \Gamma_1(\tau^k)$ can then be accordingly decomposed into constant and known nominal parts $\Gamma_0(\tau^\circ), \quad \Gamma_1(\tau^\circ)$ and uncertain (though bounded) parts which are related as $\Delta\Gamma_0(\tau^k, \tau^\circ) = -\Delta\Gamma_1(\tau^k, \tau^\circ)$.

## 3. NCS-MODELING RELYING ON THE AUGMENTED CLOSED–LOOP STATE VECTOR

Suppose that a discrete–time state feedback law (for regulation purposes) with static gain $K_{sf}$ is employed in (2), i.e. $\hat{u}_k = -K_{sf}x_k, \quad \hat{u}_{k-1} = -K_{sf}x_{k-1}$. Substituting into (2) the state space description of the time–varying closed–loop system is:

$$x_{k+1} = \left[\Phi - \Gamma_0(\tau^k)K_{sf}\right] x_k + \left[-\Gamma_1(\tau^k)K_{sf}\right] x_{k-1} \; . \tag{4}$$

Noting that in (4) above, only periodically–sampled state vector values $\{x_{k+1}, \; x_k, \; x_{k-1}\}$ are present, it is allowable to express it in terms of an augmented periodic vector $\xi_k$ defined as

$$\xi_k^T = \xi(kh)^T \triangleq \left[x_k^T, \; x_{k-1}^T\right] \;, \tag{5}$$

yielding the following uncertain time–varying discrete–time description for the closed–loop dynamics

$$\xi_{k+1} = \begin{bmatrix} \Phi - \Gamma_0(\tau^k)K_{sf} & -\Gamma_1(\tau^k)K_{sf} \\ I_n & 0_n \end{bmatrix} \xi_k \triangleq A_{sf}\xi_k \tag{6}$$

The closed–loop system matrix $A_{sf}(\tau^k, K_{sf}) \in R^{2n\times 2n}$ in (6) can be accordingly decomposed into a nominal time invariant part $A^\circ_{sf}(\tau^\circ, K_{sf})$ and an uncertain part $\Delta A_{sf}(\tau^k, \tau^\circ, \tau^k, \tau^\circ)$

$$\begin{aligned}A_{sf} &= \begin{bmatrix} \Phi - \Gamma_0(\tau^\circ)K_{sf} & -\Gamma_1(\tau^\circ)K_{sf} \\ I_n & 0_n \end{bmatrix} + \\ &\quad \begin{bmatrix} -\Delta\Gamma_0(\tau^k, \tau^\circ)K_{sf} & -\Delta\Gamma_1(\tau^k, \tau^\circ)K_{sf} \\ 0_n & 0_n \end{bmatrix} \\ &\triangleq A^\circ_{sf} + \Delta A_{sf}(\tau^k, \tau^\circ)\end{aligned} \tag{7}$$

The corresponding expressions for the output feedback case ($\hat{u}_k = -K_{of}y(kh) = -K_{of}C_c x_k$), are easily derived, formally using $K_{sf} = K_{of}C_c$ in the previous state feedback results. Using $\Delta\Gamma_1(\tau^k) = -\Delta\Gamma_0(\tau^k)$, the uncertain "perturbation" matrix $\Delta A_{sf}$ can be expressed as

$$\Delta A_{sf}(\tau^k, \tau^\circ) = \begin{bmatrix} I_n \\ 0_n \end{bmatrix} \Delta\Gamma_1(\tau^k, \tau^\circ) K_{sf} \left[\, I_n \; -I_n \,\right]$$
$$\triangleq S_3 \Delta\Gamma_1(\tau^k, \tau^\circ) K_{sf} S_4 \ . \tag{8}$$

where $\sigma_{\max}(S_3) = 1$ and $\sigma_{\max}(S_4) = \sqrt{2}$ independently of the dimension of the state vector. In the sequel, for notation brevity, we omit the dependence of $\Delta A_{sf}(\tau^k, \tau^\circ), \Delta A_{of}(\tau^k, \tau^\circ)$ and $\Delta\Gamma_1(\tau^k, \tau^\circ)$ on $\tau^k$ and $\tau^\circ$.

Inhere, the following issue is addressed: Given a stabilizing controller $\hat{u}_k = \left[-K_{sf}\ ,\ 0\right]^T \xi_k$ such that the nominally delayed closed–loop system is stable, or $\left|\mathrm{eig}(A^\circ_{sf})\right| < 1$, what is it to be expected (mainly in terms of stability) when the same control law is used for the uncertain discrete–time system in (7)? Moreover, since the uncertain varying network delay is reflected into the uncertain matrices $\Delta A_{sf}$ and $\Delta\Gamma_1$, is it possible to quantify the answer in terms of a "delay–range" for which asymptotic stability of the closed–loop system is guaranteed?

We are now ready to state a main contribution of our work (inspired by ideas presented in Konstantopoulos and Antsaklis [1996], Dritsas and Tzes [2007]), which concerns sufficient robust stability conditions relating the network delay with the feedback controller gains for the perturbed closed-loop system with output and state feedback.

*Theorem 1.* Consider the closed-loop linear discrete–time system with the structure presented in (7),(8) and the feedback controller $\hat{u}_k = -K_{sf}x_k$ designed such that the nominal part $A^\circ_{sf}$ in (7) is asymptotically stable (a Schur-Matrix) satisfying the Lyapunov equation

$$L^\circ = (A^\circ_{sf})^T P A^\circ_{sf} - P = -Q < 0 \tag{9}$$

with $P = P^T > 0$, $Q = Q^T > 0$ both of dimension $R^{2n\times 2n}$. Then the perturbed closed-loop system remains asymptotically stable if

$$\sigma^2_{\max}(\Delta\Gamma_1) < \frac{\sigma_{\min}(Q) - \frac{1}{\alpha}\sigma^2_{\max}(A^\circ_{sf})\sigma^2_{\max}(P)}{2\ \sigma_{\max}(\alpha I_{2n} + P)\sigma^2_{\max}(K_{sf})} \tag{10}$$

with $\alpha$ any positive number satisfying

$$\alpha > \frac{\sigma^2_{\max}(A^\circ_{sf})\sigma^2_{\max}(P)}{\sigma_{\min}(Q)} \ . \tag{11}$$

**Proof** The following standard properties of the singular values of matrices $X$ and $Y$ with compatible dimensions will be used

$$\sigma_{\max}(X+Y) \leq \sigma_{\max}(X) + \sigma_{\max}(Y) \tag{12}$$

$$\sigma_{\max}(XY) \leq \sigma_{\max}(X)\sigma_{\max}(Y) \tag{13}$$

If, furthermore, $X$ and $Y$ are symmetric positive definite matrices the following relation is valid

$$\sigma_{\max}(X) < \sigma_{\min}(Y) \Rightarrow X < Y \ . \tag{14}$$

Let the following "perturbed" Lyapunov equation (inspired by (9)) be defined for the resulting closed–loop system

$$\left(A^\circ_{sf} + (\Delta A_{sf})\right)^T P \left(A^\circ_{sf} + (\Delta A_{sf})\right) - P$$
$$= -Q + (\Delta A_{sf})^T P (\Delta A_{sf}) + (\Delta A_{sf})^T P A^\circ_{sf} + A^{\circ T}_{sf} P (\Delta A_{sf})$$
$$\triangleq -Q + M_\Delta \tag{15}$$

The investigation for sufficient conditions for asymptotic stability of the closed–loop perturbed system, is transformed into a search of conditions for

$$-Q + M_\Delta < 0 \ . \tag{16}$$

The quantity $M_\Delta$ is bounded by

$$M_\Delta = \Delta A^T_{sf} P \Delta A_{sf} + \alpha \Delta A^T_{sf}\Delta A_{sf} + \frac{1}{\alpha} A^{\circ T}_{sf} P^T P A^\circ_{sf} -$$
$$-\alpha \left[(\Delta A_{sf}) - \frac{1}{\alpha} P A^\circ_{sf}\right]^T \left[(\Delta A_{sf}) - \frac{1}{\alpha} P A^\circ_{sf}\right]$$
$$\leq \Delta A^T_{sf} \left[\alpha I_{2n} + P\right] \Delta A_{sf} + \frac{1}{\alpha} A^{\circ T}_{sf} P^T P A^\circ_{sf} \ , \tag{17}$$

where $\alpha > 0$.

Henceforth, (16) is satisfied if

$$Q - \Delta A^T_{sf}\left[\alpha I_{2n} + P\right]\Delta A_{sf} - \frac{1}{\alpha}\left[A^{\circ T}_{sf} P P A^\circ_{sf}\right] > 0. \tag{18}$$

Since both $Q$ (by selection) and $\Delta A^T_{sf}[\alpha I_{2n} + P]\Delta A_{sf} + \frac{1}{\alpha}[A^{\circ T}_{sf} P P A^\circ_{sf}]$ are symmetric (and positive definite) matrices, (18) is satisfied if

$$\sigma_{\max}\left(\Delta A^T_{sf}\left[\alpha I_{2n} + P\right]\Delta A_{sf} + \frac{1}{\alpha}\left[A^{\circ T}_{sf} P P A^\circ_{sf}\right]\right)$$
$$\leq \sigma^2_{\max}\left(\Delta A_{sf}\right)\sigma_{\max}\left(\alpha I_{2n} + P\right) + \frac{1}{\alpha}\sigma^2_{\max}\left(A^\circ_{sf}\right)\sigma^2_{\max}\left(P\right)$$
$$\leq \sigma_{\min}(Q) \tag{19}$$

where the first inequality stems from properties (12),(13) and the latter stems from (14).

It is now clear that if condition

$$\sigma^2_{\max}(\Delta A_{sf}) < \frac{\sigma_{\min}(Q) - \frac{1}{\alpha}\sigma^2_{\max}(A^\circ_{sf})\sigma^2_{\max}(P)}{\sigma_{\max}\left(\alpha I_{2n} + P\right)}, \tag{20}$$

is satisfied, then our "wish" for robust stability (expressed via (18)) is guaranteed with $\alpha$ any positive number satisfying $\alpha > \frac{\sigma^2_{\max}(A^\circ_{sf})\sigma^2_{\max}(P)}{\sigma_{\min}(Q)}$, since the numerator of (20) must be a positive scalar. The perturbation matrix $\Delta A_{sf}$ in (8) can be bounded as

$$\sigma_{\max}(\Delta A_{sf}) \leq \sqrt{2}\sigma_{\max}(\Delta\Gamma_1)\ \sigma_{\max}(K_{sf}) \tag{21}$$

which combined with (20) gives

$$\sigma^2_{\max}(\Delta\Gamma_1) < \frac{\sigma_{\min}(Q) - \frac{1}{\alpha}\sigma^2_{\max}(A^\circ_{sf})\sigma^2_{\max}(P)}{2\ \sigma_{\max}(\alpha I_{2n} + P)\sigma^2_{\max}(K_{sf})} \ \text{Q.E.D.}\square$$

Expression (10) along with the constraint (11), can now be used either as an "a posteriori" robust stability check for a given (range of) stabilizing gains $K_{sf}$s, or for the synthesis of a $K_{sf}$ that maximizes the uncertainty bounds that can be tolerated without losing stability. The latter approach was followed in Konstantopoulos and Antsaklis [1996], where both stabilization and performance objective functions were considered and the Broyden family of nonlinear minimization algorithms was used for an iterative search on the optimum $K_{sf}$. Note also that although the "$\alpha$–constraint" in (11) reads for "any positive number", it is advised to select an $\alpha$ that maximizes the right hand side (RHS) of the expression (10).

From a robust control point of view, given $Q, \tau^\circ, h$ and a set of stabilizing gains $K_{sf}$, the objective is to maximize over all feasible values of the positive scalar $\alpha$, the quantity

$$F(\alpha_{\rm opt}) = \max_{\alpha}\Big|_{Q,\tau^\circ,K_{sf},h} \frac{\sigma_{\min}(Q) - \frac{1}{\alpha}\sigma^2_{\max}(A^\circ_{sf})\sigma^2_{\max}(P)}{2\ \sigma_{\max}(\alpha I_{2n}+P)\sigma^2_{\max}(K_{sf})} .$$

subject to

$$\alpha > \frac{\sigma^2_{\max}(A^\circ_{sf})\sigma^2_{\max}(P)}{\sigma_{\min}(Q)} \tag{22}$$

After the computation of $F$, the maximum allowable total delay can be computed, using a nonlinear optimization algorithm for the computation of the values of $\tau^{k,\max}$, $\tau^{k,\min}$ such that

$$\tau^{k,\max} = \max\left(\tau^k\right) \text{ subject to } \sigma_{\min}\left(\Delta\Gamma_1(\tau^k)\right) \le \sqrt{F}$$
$$\tau^{k,\min} = \min\left(\tau^k\right) \text{ subject to } \sigma_{\min}\left(\Delta\Gamma_1(\tau^k)\right) \le \sqrt{F} .$$

Consider the following open–loop stable continuous–time system $G(s) = \frac{2}{s^2+3s+2}$, with state space description:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t)$$
$$y(t) = [\,1\ \ 0\,]\, x(t) \tag{23}$$

The system is sampled with $h = 1.3333$ seconds, while the uncertain delay varies between the bounds $\tau_{\min} = 0$ and $\tau_{\max} = h$. The nominal delay upon which the nominally delayed discrete–time system is obtained is the average value of $\tau^\circ = 0.6667$ seconds. The output–feedback version of the Theorem 1 will be used, for which the sufficient conditions are formally expressed (by setting $K_{sf} \to K_{of}C_c$) as

$$\sigma^2_{\max}(\Delta\Gamma_1) < \frac{\sigma_{\min}(Q) - \frac{1}{\alpha}\sigma^2_{\max}(A^\circ_{of})\sigma^2_{\max}(P)}{2\ \sigma_{\max}(\alpha I_{2n}+P)\sigma^2_{\max}(K_{of}C_c)} \tag{24}$$

with $\alpha$ any positive number satisfying $\alpha > \frac{\sigma^2_{\max}(A^\circ_{of})\sigma^2_{\max}(P)}{\sigma_{\min}(Q)}$ .

Rather than examining the bounds $\tau^{k,\min}$ and $\tau^{k,\max}$ for one stabilizing gain $K_{of}$, in the sequel the stability region is presented in the domain $\left[K, \tau^k \in \left[\tau^{k,\min}, \tau^{k,\max}\right]\right]$ within which the closed–loop system remains stable.

In Figure 2, one can compare the stability margins for two values of $\alpha$ and $Q = 0.5 \times I_4$ for the Lyapunov equation: a) the blue-colored one (denoted as $\alpha_{opt}$) corresponds to the bounds using the optimum-$\alpha$, while b) the red-colored one corresponds to the bounds obtained using as $\alpha$ the next largest integer (denoted as $\alpha_{ceil}$) that satisfies the "alpha" inequality.

The continuous system's response in combination with the asynchronous command input for a set of output feedback gains $K_{of} \in \{-1, \ldots, 1\}$ (stabilizing the nominal system) and a randomly varying delay $\tau^k \in \left\{\tau^{k,\min}, \tau^{k,\max}\right\}$ is presented in Figure 3. As expected, the closed–loop system is stable, while the speed of the output's convergence depends heavily on the selected gain. For small values of $K_{of}(\simeq -1)$ the system responds slower since the real part of one of its poles approaches one.

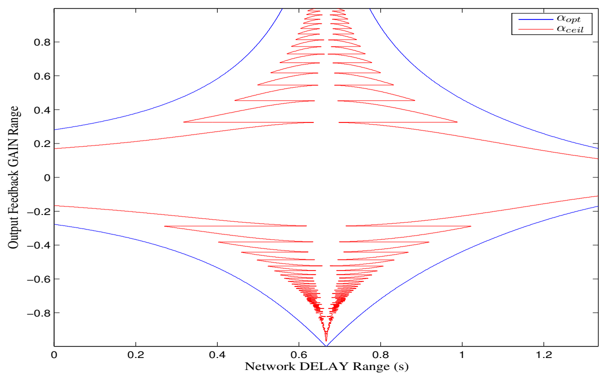


Fig. 2. Stability Contours for $K_{of} \in \{-1, \ldots, 1\}$, $h = 1.333$s, $Q = 0.5 \times I_4$, and two admissible choices for $\alpha$.

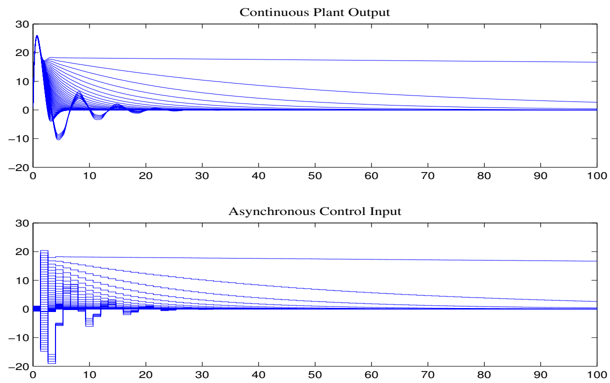


Fig. 3. Family of controlled system responses and input commands for $K_{of} \in \{-1, \ldots, 1\}$, $h = 1.333$s, $Q = 0.5 \times I_4$, $x(0) = [1, 100]^T$ and $\alpha_{\rm opt}$.

## 4. NCS-MODELING RELYING ON THE AUGMENTED OPEN–LOOP STATE VECTOR

Starting again from equation (2), a second modeling approach will be presented. The "peculiarity" of this approach is that the proposed augmented vector consists of a mixing of periodic and aperiodic (asynchronous / event–based) discrete–time variables. The ensuing state equations must thus be treated as "iteration maps" rather than classical sampled–data state equations, and the analysis must be carried out in the context of "Asynchronous (Hybrid) Dynamical Systems" as described in Hassibi et al. [1999]. Apart from the arising mathematical subtleties, a clear benefit is that the state equations can be formally treated in a "classical" control formalism. Indeed defining the augmented state vector $z_k$ as

$$z_k^T \triangleq \left[x_k^T,\ \ \hat{u}_{k-1}^T\right] \tag{25}$$

the exact discretization given by equation (2), can be formally expressed as

$$z_{k+1} = \begin{bmatrix} \Phi & \Gamma_1(\tau^k) \\ 0 & 0 \end{bmatrix} z_k + \begin{bmatrix} \Gamma_0(\tau^k) \\ 1 \end{bmatrix} \hat{u}_k$$
$$\triangleq A(\tau^k) z_k + B(\tau^k)\hat{u}_k . \tag{26}$$

Using the delay and matrix decomposition presented before, a decomposition of the open–loop dynamics (26) into a nominal and an uncertain part is possible as follows Dritsas and Tzes [2007], Dritsas et al. [2007b, 2006a,b]:

$$z_{k+1} = \left(A(\tau^\circ) + \Delta A(\tau^k)\right) z_k + \left(B(\tau^\circ) + \Delta B(\tau^k)\right) \hat{u}_k$$
$$\triangleq \left(A^\circ + \Delta A(\tau^k)\right) z_k + \left(B^\circ + \Delta B(\tau^k)\right) \hat{u}_k \quad (27)$$

where $\tau^\circ$ is the assumed nominal value of $\tau$.

The relation $\Delta\Gamma_1(\tau^k) = -\Delta\Gamma_0(\tau^k)$ allows to decompose the uncertain matrices $\Delta A(\tau^k), \Delta B(\tau^k)$ as:

$$\Delta A(\tau^k) = \begin{bmatrix} I_n \\ \bar{0} \end{bmatrix} \Delta\Gamma_1(\tau^k) \begin{bmatrix} \bar{0} & 1 \end{bmatrix}$$
$$\triangleq S_1 \Delta\Gamma_1(\tau^k) S_2 \quad (28)$$

$$\Delta B(\tau^k) = -\begin{bmatrix} I_n \\ \bar{0} \end{bmatrix} \Delta\Gamma_1(\tau^k) \triangleq -S_1 \Delta\Gamma_1(\tau^k) \ , \quad (29)$$

with $\sigma_{\max}(S_1) = \sigma_{\max}(S_2) = 1$.

Closing the loop with a discrete–time state feedback $\hat{u}_k = -K_{sf} x_k = [-K_{sf} \quad 0]\, z_k \triangleq -K^a_{sf} z_k$ results in the following state equation for the closed–loop system:

$$z_{k+1} = \begin{bmatrix} \Phi - \Gamma_0(\tau^k) K_{sf} & \Gamma_1(\tau^k) \\ -K_{sf} & 0 \end{bmatrix} z_k \triangleq A_{sf} z_k. \quad (30)$$

The decomposition of the closed–loop system matrix in (30), into a nominal part and an uncertain part is

$$A_{sf} = \begin{bmatrix} \Phi - \Gamma_0(\tau^\circ) K_{sf} & \Gamma_1(\tau^\circ) \\ -K_{sf} & 0 \end{bmatrix} + \begin{bmatrix} -\Delta\Gamma_0(\tau^k) K_{sf} & \Delta\Gamma_1(\tau^k) \\ \bar{0} & 0 \end{bmatrix}$$
$$\triangleq A^\circ_{sf} + \Delta A_{sf}(\tau^k, K_{sf}). \quad (31)$$

Analogous results can be easily derived for the static output–feedback case, by formally using $K_{sf} \to K_{of} C_c$ in the previous results. Note that the symbols $A_{sf}, A_{of}$ in this section have a different notion and different dimension compared to the same symbols used in the previous section. The following relation between the open–loop system matrices $A^\circ, B^\circ, \Delta A, \Delta B$ and the closed–loop system matrices $A^\circ_{sf}$, $\Delta A_{sf}$can be easily derived.

$$A^\circ_{sf} = A^\circ - B^\circ K^a_{sf}$$
$$\Delta A_{sf}(\tau^k) = \Delta A(\tau^k) - \Delta B(\tau^k) K^a_{sf} \ . \quad (32)$$

The Theorem below is derived in a manner analogous to the previous Theorem 1, while the output–feedback version can be found in Dritsas and Tzes [2007].

*Theorem 2.* Consider the linear discrete–time system in (30) with the uncertain matrices $\Delta A, \Delta B$ obeying the structure presented in (28) and (29) respectively, and the feedback controller $\hat{u}_k = -K_{sf} x_k = -K^a_{sf} z_k$ designed such that $A^\circ_{sf}$ is asymptotically stable, (a Schur-Matrix) satisfying the Lyapunov equation

$$\left(A^\circ_{sf}\right)^T P A^\circ_{sf} - P = -Q, \quad (33)$$

with $P = P^T > 0$, $Q = Q^T > 0$. Then the perturbed closed-loop system remains asymptotically stable if

$$\sigma^2_{\max}(\Delta\Gamma_1) < \frac{\sigma_{\min}(Q) - \frac{1}{\alpha}\sigma^2_{\max}(A^\circ_{sf})\sigma^2_{\max}(P)}{\sigma_{\max}(\alpha I_{n+1} + P)(1 + \sigma_{\max}(K^a_{sf}))^2} \quad (34)$$

with $\alpha$ any positive number satisfying $\alpha > \frac{\sigma^2_{\max}(A^\circ_{sf})\sigma^2_{\max}(P)}{\sigma_{\min}(Q)}$.

Considering the same system $G(s) = \frac{2}{s^2+3s+2}$ presented in (23) before with sampling period $h = 1.3333$s, the stability contours generated via the output–feedback version of the Theorem 2 are shown in Figure 4. As expected, the bounds generated by the optimum-$\alpha$ ($\alpha_{opt}$) embrace those generated via the $\alpha_{ceil}$. Furthermore, although the bounds computed using the $\alpha_{ceil}$ are continuous, these are not differentiable due to the discontinuous nature of the ceiling function.

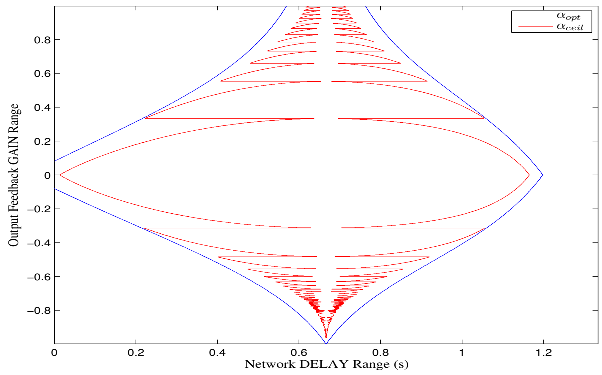


Fig. 4. Stability Contours (for SYS2) with $\{h = 1.333$s, $\tau^\circ = h/2 = 0.666$, $Q = 0.5 \times I_3\}$.

## 5. COMPARISON OF THE TWO APPROACHES IN TERMS OF THE DELAY BOUND

A comparison of the two modeling approaches will now be presented. From a control theoretic point of view the two approaches reflect two completely different "philosophies". The augmented state vector "$\xi$" of the first approach, section 3, appears only after closing the loop with a static state or output feedback law, giving rise to closed–loop analysis and synthesis results in a way closely resembling the "classical" sampled-data theory for systems with norm bounded uncertainties. On the other hand the augmented state vector "$z$" in section 4 is an open loop state vector that mixes clocked with "event–based" signals and thus must be interpreted in the context of hybrid ("Asynchronous Dynamical") systems. The clear benefit of this approach is that it offers itself not only to a wider interpretation (analysis) of the NCS dynamics but also to a broader range of synthesis procedures for the closed–loop control law.

Regarding the conservatism of the robust stability results achieved when following these two approaches, is interesting to compare via simulation the delay bounds achieved for the same range of stabilizing gains. Such a comparison is presented in Figure 5 below where the outer curve is the guaranteed stability bound achieved via Theorem 1 employing the $\xi$ augmented state vector whereas the inner curve corresponds to the result achieved via Theorem 2 employing the $z$ augmented state vector. Simulations concern the same continuous–time system employed for the previous numerical results, a constant sampling period of $h = 1.333$ s and the same range of stabilizing output gains for both cases (value range between $-0.99$ and $0.99$). The horizontal axis is the normalized delay $(\tau/h)$ showing that for low gain range, the first approach covers the whole delay range contrary to the second approach which guarantees robust stabilization for the 90% of the delay range.

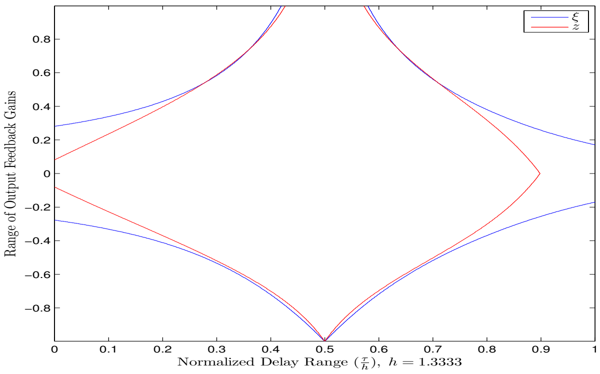


Fig. 5. Comparison of the two approaches in terms of the delay bound ($h = 1.333$)

The experiment was repeated for many different second order systems, values of sampling period and range of stabilizing gains, with the result being always the same: the first approach gave wider delay bounds for robust stability, whereas the second approach is more conservative.

## 6. CONCLUSIONS

Two discrete–time modeling approaches for Networked Controlled Systems (NCS) with uncertainly varying and bounded transmission delays and static discrete–time control laws were presented in this article. A delay decomposition was proposed which results in models well–suited for control synthesis using established robust control techniques, without any requirement for a priori information about the probability distributions of network–induced delays. Sufficient conditions for the robust stability problem were given for each approach and a comparison of the conservatism of results was discussed.

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