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Asymptotic tracking of non-sinusoidal periodic references: a switched linear internal model approach

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Abstract: The regulation theory is a very robust way to achieve the asymptotic tracking of input references on Linear Time-Invariant systems generated by an exosystem. Nevertheless, the classical regulation theory cannot be applied to the asymptotic tracking of non-sinusoidal periodic references having an infinite number of harmonics (as triangular waves). In literature many methods have been proposed to deal with this kind of problem. In this paper a novel approach based on a Switched Linear Internal Model is proposed by presenting some preliminary results. This approach seems particularly promising in terms of robustness to disturbances not "captured" by the internal model and this is the main motivation of starting this study.

Keywords: Switching control; Regulation; Tracking;

1. INTRODUCTION

In many applications the control objective is to track periodic references even in presence of uncertainties on system parameters (consider for instance automotive, industrial robotics and power electronics applications). This kind of problem is often tackled using the so-called Internal Model Principle (IMP) (Isidori et al. (2003)), to prevent the use of high-gain/large-bandwidth or adaptive model predictive methods. Nevertheless, a "simple" solution usually is not straightforward for "real world" applications. As a matter of fact, even restricting to the case of Linear Time-Invariant (LTI) systems, if the reference has an infinite number of harmonics (as often happens in practice), the well-known linear regulation theory based on linear and finite-dimensional internal model cannot guarantee the asymptotic tracking. This is the case, for instance, of CNAO Storage Ring Dipole Magnet Power Converter $3000A\,/\,\pm1600V$ (Carrozza et al. (2006)) where the current reference is a set of ramps and constant references connected by 5th order polynomial curves, and the required tracking accuracy is $15mA$.
In the last decades different approaches have been proposed to extend the IMP in order to achieve the asymptotic tracking of periodic infinite-harmonics references.
In *Repetitive Learning Control* (Hara et al. (1988)) the adopted internal model is a closed-loop time-delay system with delay T (in the continuous-time framework) which is able to generate any periodic signal with period T. This allows to deal with infinite-harmonics references, but an infinite dimension internal model is actually required. A different approach relies on the use of *Nonlinear Internal Model Units*. The main difficulty of such approach is to find a structure for the internal model units which can generate the control inputs needed for tracking and, at the same time, is manageable for stabilization. This has usually required to adopt the so-called *immersion assumption*, which limits the applicability of the method to a restricted class of reference/disturbance signals. Recently, this constraint has been removed allowing to cope with every infinite-harmonics or even more complex reference (Byrnes et al. (2005), Marconi and Praly (2006)), but the solution could be very involved even for simple cases and even considering practical regulation. Alternatively, in the *Adaptive Learning Control*, see (Del Vecchio et al. (2003)), a finite number of harmonics of the infinite-harmonics periodic reference/disturbance are compensated and an explicit bound on the residual error can be fixed.
One of the main critical issues with the IMP-based methods is the robustness to external disturbances which are not "canceled" by the internal model unit. Considering the linear SISO case with an internal model realized by a bank of oscillators, it is easy to realize that, owing to the Bode's Integral constraints on the sensitivity function, a relevant sensitivity to disturbances not "captured" by the internal model will result. A similar behavior is expected also for more sophisticated methodologies as the abovementioned ones. Potentially some adaptation mechanism could be added to deal with "strange" disturbances, but at the cost of a relevant complexity increasing.
Motivated by the issues on robustness and considering that in many practical applications, the references to be tracked are given by a sequence of curves which can be generated by different LTI systems, the main purpose of this paper is to propose a novel approach based on a *Switched Linear Internal Model* (SLIM). Focusing on LTI system framework (or, by simple extension, on feedback-linearizable systems), the basic idea is to define as exosystem a free linear system with periodic piecewise constant coefficients (*periodic switched system*) which can generate

as output the required reference [1]. Subsequently, as in the classic regulation theory, the exosystem model (i.e. the internal model unit) is embedded in the controller and the asymptotic tracking is achieved by stabilizing the overall system.
Despite the proposed SLIM approach cannot be applied to all the possible periodic references, it covers a quite interesting class of references and it looks promising in terms of robustness to unmodeled disturbances and reduced complexity, since the switching nature of the internal model allows to use low-order systems.
In this paper some preliminary results for the SLIM approach are presented considering a SISO LTI system to be controlled. The definition of an exosystem with a suitable structure is proposed with particular attention to constraints in the switching instants. In addition a preliminary solution for asymptotic tracking is proposed exploiting the approach proposed in (Devasia et al. (1997)).
The paper is organized as follows. In Section 2 the problem of the synthesis of free switched-linear system with given periodic output is considered: a parallel solution is presented and the problem of the minimal solution is briefly discussed. The parallel structure is crucial since it is adopted for exosystem model. In Section 3 a preliminary result on asymptotic regulation is presented. It relies on an exosystem state observer, reported in Subsection 3.1, and on the use of the Differential Sylvester Equation, according to (Devasia et al. (1997)) as recalled in Subsection 3.2. Finally, Section 4 presents the application to a simple system of these preliminary results. Remarks and future work directions for this kind of approach are summarized in the Conclusions.

2. SYNTHESIS OF A FREE SWITCHED-LINEAR SYSTEM WITH GIVEN PERIODIC OUTPUT

The reference signals considered in this work are given by the smooth concatenation of LTI systems outputs. In particular, the smoothness requirement obviously arises from the reproducibility constraints imposed by the plant to be controlled.
The focus of this Section is on the synthesis of a free switched-linear dynamical system, without jumps in the state trajectories, able to generate a given signal with the above-mentioned characteristics. In Subsection 2.1 the synthesis problem is formulated and discussed. In Subsection 2.2 a solution is proposed and in Subsection 2.3 some considerations on minimization of the generating system are presented. This Section ends with Subsection 2.4 where the class of signals which can be generated with the proposed solutions is considered; particular attention is paid to the effects of the initial conditions of the generating systems.

2.1 The problem formulation

Let $y_d(t) : \mathbb{R}_0^+ \to \mathbb{R}$ be a T–periodic function defined over the positive time axis (i.e. $y_d(t_0) = y_d(t_0 + T)$, $\forall t_0 \in \mathbb{R}_0^+$), with the following properties.

- *Smoothness*: $y_d(t) \in C^r(\mathbb{R}_0^+)$, with r greater than or equal to the *relative degree* of the plant to be controlled.
- *LTI-Sys Concatenation*: $y_d(t)$ is piecewise output of different LTI systems or equivalently:

$$y_d(t) = \begin{cases} y_{d1}(t) & t_0 = 0 \leqslant t < t_1 \\ y_{d2}(t) & t_1 \leqslant t < t_2 \\ & \vdots \\ y_{dN}(t) & t_{N-1} \leqslant t < t_N = T \end{cases} \tag{1}$$
$$y_{di}(t) = \sum_{k=0}^{m_i} A_{ik} t^{a_{ik}} e^{b_{ik}t} \sin(\omega_{ik}t + \phi_{ik})$$

The problem to be solved is to find N matrix pairs

$$(S_1, q_1), (S_2, q_2), \ldots, (S_N, q_N),$$

a T-periodic piecewise constant function $\sigma(t)$

$$\sigma(t) : \mathbb{R}_0^+ \to \{1, 2, \ldots, N\}, \qquad \sigma(t + T) = \sigma(t),$$

and a vector w_0, such that:

$$\Sigma : \begin{cases} \dot{w}(t) = S_{\sigma(t)} w(t) & w(0) = w_0 \\ y_d(t) = q_{\sigma(t)} w(t) \end{cases} \tag{2}$$

Remark 1. According to (2), the class of possible system generating y_d is restricted to linear systems with switching parameters, but *continuous state solution*. This choice is not restrictive and appears profitable to simplify the stabilization problem.

In the following a quite straightforward solution based on parallel structure is presented, proving the solvability of the problem. Afterwards, a different solution reducing the system order is briefly discussed.

2.2 Parallel synthesis

The main idea of the "parallel synthesis" is to use N parallel-connected subsystems, each one representing a linear piece according to (1). By changing the dynamical matrix of the overall system, each subsystem is "turned on", during suitable time-intervals, "rewound" and "frozen" during the rest of time, while it is not observable.

As a matter of fact, for each of $y_{d1}(t), \ldots, y_{dN}(t)$, defined according to (1), an observable realization (Ω_i, Θ_i) of dimension $n_i \geqslant m_i$, suitably initialized with state w_{0i}, can be found in order to obtain:

$$\Sigma_i : \begin{cases} \dot{w}_i(t) = \Omega_i w_i(t) & w_i(t_{i-1}) = w_{0i} \\ y_{di}(t) = \Theta_i w_i(t) & \forall t \in [t_{i-1}, t_i[\end{cases} \tag{3}$$

Hence a solution for the problem defined in Subsection 2.1, can be obtained by a suitable parallel connection of the N subsystems Σ_i as reported in the following.
The matrix pairs (S_i, q_i) have the form:

$$S_1 = \begin{bmatrix} \Omega_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -\alpha_N \Omega_N \end{bmatrix} \tag{4}$$
$$q_1 = [\Theta_i \; 0 \ldots 0 \; 0 \; 0 \ldots 0]$$

[1] The switching instants are predetermined and constant within a period: a further extension of the SLIM, not considered in this work, could be an exosystem with a state-driven switching law. This would involve very different tools for synthesis and stabilization.

$$S_i = \begin{bmatrix} 0 & \dots\dots & \dots & \dots\dots\dots & 0 \\ \vdots & \ddots \dots & \dots & \dots\dots\dots & \vdots \\ \vdots & \dots\ 0 & 0 & 0\ \ 0\ \dots & \vdots \\ \vdots & \dots\ 0 & -\alpha_{i-1}\Omega_{i-1} & 0\ \ 0\ \dots & \vdots \\ \vdots & \dots\ 0 & 0 & \Omega_i\ \ 0\ \dots & \vdots \\ \vdots & \dots\ 0 & 0 & 0\ \ 0\ \dots & \vdots \\ \vdots & \dots\dots & \dots & \dots\dots\ \ddots & \vdots \\ 0 & \dots\dots & \dots & \dots\dots\dots & 0 \end{bmatrix} \tag{5}$$

$$q_i = [0\ \dots\ 0\ 0\ \Theta_i\ 0\ \dots\ 0] \quad i = 2 \dots N.$$

where the scaling factors α_i are:

$$\alpha_i = \frac{t_i - t_{i-1}}{t_{i+1} - t_i} \tag{6}$$

The switching function $\sigma(t)$ has the simple form:

$$\sigma(t) = j \qquad \text{for } t_{j-1} \leqslant t < t_j \tag{7}$$

The initial condition w_0 is defined as follows:

$$w(0) = w_0 = \begin{bmatrix} w_{01} \\ w_{02} \\ \vdots \\ w_{0N-1} \\ w_{fN} \end{bmatrix} \tag{8}$$

where w_{0i}, $i = 1 \dots N$ denote the initial conditions of the subsystems Σ_i according to (3) and w_{fi} the corresponding "*final conditions*", i.e. $w_{fi} = e^{\Omega_i(t_i - t_{i-1})} w_{0i}$.

According to (4)-(5), in a generic time interval $[t_{i-1}, t_i)$ only the subsystems Σ_{i-1} and Σ_i are "turned on", while other subsystems are "frozen". In particular, the state of Σ_i is the only observable part from the output y_d and it effectively determines the y_d trajectory, evolving from w_{0i} to w_{fi}. Differently, Σ_{i-1} is "rewinding" the state trajectory from $w_{f(i-1)}$ (reached at the end of interval $[t_{i-2}, t_{i-1})$) to w_{0i-1}, since a dynamic matrix $-\alpha_{i-1}\Omega_{i-1}$ is adopted instead of Ω_{i-1} (note that α_{i-1} defined in (6) is used for time-interval normalization only).

Remark 2. The "rewinding" stage for the different subsystems could be replaced by an instantaneous reset with a suitable mapping. Anyway, the "continuous rewinding" is preferred to avoid jumps in the state trajectories.

2.3 Minimal synthesis

The solution presented in previous Subsection is characterized by a very large system dimension ($\sum n_i$, with n_i defined just before (3)). Hence the main purpose of this part is to present some considerations toward minimization of the system generating y_d.

Clearly, the dimension of a system Σ solving the problem stated in Subsection 2.1 cannot be lower than $\max n_i$. In the following, the realization of a system generating y_d and having dimension $\max n_i$ is considered.

Starting from the N subsystems Σ_i, defined in (3) for the parallel solution, it is always possible to regularize the order of each subsystem to $\max n_i$ by extending the matrices Ω_i and vectors Θ_i and w_{0i} without modifying the output behavior, i.e. adding an unobservable dynamics to the system. This operation can be simply done, for instance, by appending zeros to the above mentioned matrices and vectors. Hence, denoting with $\bar{\Omega}_i$, $\bar{\Theta}_i$ and $\bar{w}_{0i}$ the "extended elements" and introducing N non-singular matrices T_i with dimension $\max n_i$, it can be easily proved that

$$\begin{aligned} S_i &= T_i \bar{\Omega}_i T_i^{-1},\ q_i = \bar{\Theta}_i T_i^{-1},\ i = 1 \dots N \\ \sigma(t) &= j \quad \text{for } t_{j-1} \leqslant t < t_j \\ w(0) &= T_1 \bar{w}_{01} \end{aligned} \tag{9}$$

is a solution with dimension $\max n_i$ for the problem defined in Subsection 2.1 iff the following condition is satisfied

$$\begin{cases} T_1 w_1(t_1^-) = T_2 w_2(t_1^+) \\ T_2 w_2(t_2^-) = T_3 w_3(t_2^+) \\ \qquad \vdots \\ T_N w_N(t_N^-) = T_1 w_1(t_0^+) \end{cases} . \tag{10}$$

The general idea underlying the proposed minimal solution is to substitute the parallelization of system Σ_i of (3) with a "true" commutation, but preserving continuity of the state evolution by means of suitable coordinate transformations (if possible). Condition (10) represents this requirement.

From a operative viewpoint, solving the equation (10) with respect to T_i under the constraint of non-singularity, will give a direct way to build a minimal solution from a given parallel one.

Remark 3. Obviously, (10) cannot be solved for any given parallel solution, i.e. a minimal solution with the proposed structure need not to exist. On the other hand, it is worth noting that solvability of (10) could depend also on the way the elements Ω_i, Θ_i, w_{0i} are extended to $\bar{\Omega}_i$, $\bar{\Theta}_i$, $\bar{w}_{0i}$.

2.4 Class of output signals generated by the proposed systems

After considering the synthesis problem for an assigned y_d, it is quite natural wondering what class of functions can be generated with the proposed systems structures by simply varying the initial condition.

Clearly, for the parallel solution of Subsection 2.2 the initial condition is crucial to guarantee the smoothness of the solution. It is easy to prove that, using a generic initial state, the generated output is T–periodic, complies the *LTI–Sys Concatenation* condition, but, in general, does not satisfy the *Smoothness* property. An important class of references that satisfies the smoothness property is that generated by systems initialized at $w(0) = kw_0, k \in \mathbb{R}$.

Differently, for the minimal solution proposed in Subsection 2.3, setting a generic initial state, also T–periodicity property is not guaranteed, beside the *Smoothness* property violation. A T–periodic behavior can be imposed for every $w(0)$ if the following additional condition on the non-singular matrices $T_1, \dots, T_N$ is satisfied:

$$\begin{aligned} &e^{(t_{N-1} - t_{N-2}) T_N^{-1} \Omega_N T_N} \cdot e^{(t_{N-1} - t_{N-2}) T_{N-1}^{-1} \Omega_{N-1} T_{N-1}} \cdot \dots \\ &\qquad \dots \cdot e^{(t_2 - t_1) T_2^{-1} \Omega_2 T_2} e^{(t_1 - t_0) T_1^{-1} \Omega_1 T_1} = I \end{aligned} \tag{11}$$

This condition comes directly from composition of the evolutions of the different LTI systems.

3. ASYMPTOTIC OUTPUT TRACKING USING EXOSYSTEM STATE OBSERVER

The scheme of the proposed controller is reported in Figure 1. An observer of the exosystem state, with the reference

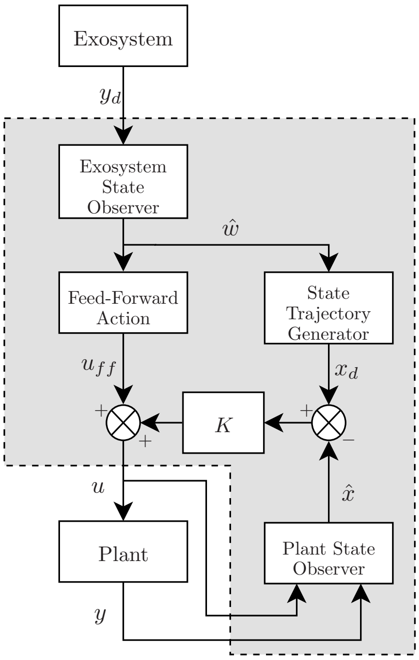

Fig. 1. Preliminary SLIM scheme

y_d as measurement input, is adopted and its estimation $\hat{w}$ is used to generate a state reference trajectory compliant with the desired output. This solution is equivalent to the use of a "pure" internal model, based on the tracking error, when no parameter uncertainty is present, but actually it has not the same intrinsic robustness properties to some kind of parameters uncertainties. Hence the presented scheme is just preliminary in the path toward the realization of SLIM-based controller.

Nevertheless, this preliminary solution is significative since the observer design for the particular exosystem is not straightforward, as reported in Subsection 3.1. In addition, in order to build the state reference, some particular tools must be considered as enlightened in 3.2.

3.1 Exosystem State Observer

A parallel structure according to Subsection 2.2 is adopted for the exosystem model and, accordingly, for the exosystem observer. Taking inspiration form the classic Luenberger approach the following observer structure is considered:

$$\dot{\hat{w}}(t) = S_{\sigma(t)}\hat{w}(t) + G_{\sigma(t)}(y_d - q_{\sigma(t)}\hat{w}) \qquad \hat{w}(0) = \hat{w}_0 \quad (12)$$

where the matrices $G_1, \ldots, G_N$ have the following structure

$$G_i = \left(0 \ \ldots \ 0 \ \Gamma_i^T \ 0 \ \ldots \ 0\right)^T \quad (13)$$

The main problem to be solved with an observer with this structure is that every subsystem (Ω_i, Θ_i) is observable only in the interval $[t_{i-1}, t_i)$. In the "rewind phase" every subsystem Σ_i is completely unobservable and possibly unstable. The proposed solution compensates the possibly unstable behavior of the unobservable phase by means of a suitable eigenvalues assignment during the observable phase. The convergence is then verified using the theorem 4 that exploits the well-known *Multiple Lyapunov Functions* theory developed in the framework of switched systems (Branicky (1994), Branicky (1998), Ye et al. (1996), Ye et al. (1998), Pettersson and Lennartson (1996)).

Theorem 4. Sufficient condition for the asymptotic stability of the observer (12) is that there exist N matrices Γ_i such that the following inequalities are satisfied for all i:

$$\mu_2\left(\Omega_i + \Gamma_i\Theta_i\right) + \mu_2\left(-\Omega_i\right) < 0 \quad (14)$$

where the symbol $\mu_2(A)$ represents the measure of the real matrix A w.r.t. euclidean norm, i.e. $\mu_2(A) = \frac{1}{2}\left(\lambda_{max}\{A + A^T\}\right)$.

Proof. For any LTI system $\dot{x}(t) = Ax(t)$ the following relation holds $\forall t \geqslant t_0$ (Desoer and Vidyasagar (1975)):

$$\|x(t)\| \leqslant e^{\mu_2(A)(t-t_0)}\|x(t_0)\| \quad (15)$$

Exploiting the parallel structure of the observer, the stability of the overall observer can be proved by demonstrating the stability of the observers of the N subsystems. Applying the bound of equation 15 to the evolution of the i-th estimation error $\tilde{w}_i(t) = w_i(t) - \hat{w}_i(t)$, it follows that:

$$\|\tilde{w}_i(t_0 + T)\| \leqslant m_{2i}m_{1i}\|\tilde{w}_i(t_0)\| \quad (16)$$

where the scalars m_{1i} and m_{2i} are defined as:

$$m_{1i} = e^{\mu_2(\Omega_i + \Gamma_i\Theta_i)(t_i - t_{i-1})}$$
$$m_{2i} = e^{\alpha_i\mu_2(-\Omega_i)(t_{i+1} - t_i)} = e^{\mu_2(-\Omega_i)(t_i - t_{i-1})}$$

If the matrix Γ_i is chosen such that $\mu_2\left(\Omega_i + \Gamma_i\Theta_i\right) + \mu_2\left(-\Omega_i\right) < 0$ it follows that:

$$\|\tilde{w}_i(t_0 + T)\| < \|\tilde{w}_i(t_0)\|$$

The euclidean norm results to be the positive definite function required by the Asymptotic Stability Theorem presented in (Ye et al. (1996)) (see Theorem 3). Therefore, the Exosystem Observer is *asymptotically stable.* □

Remark 5. The problem of finding the matrices Γ_i cannot be solved as a classical *eigenvalues assignment* problem. In general there is not a simple relation between the spectrum of Ω_i and the spectrum $\Omega_i + \Omega_i^T$.

Remark 6. The problem can be remapped into a *Linear Matrix Inequality* introducing the auxiliary matrices M_{1i}. The inequality can be written as:

$$\Omega_i + \Omega_i^T + \gamma_{i1}M_{i1} + \gamma_{i2}M_{i2} + \ldots + \gamma_{in_i}M_{in_i} < -\mu_2(-\Omega_i)I_{n_i} \quad (17)$$

where the matrices M_{1i} are defined as $M_{ij} = (e_j\Theta_i) + (e_j\Theta_i)^T$ and γ_{ij} are the elements of Γ_i.

3.2 Exact Output Tracking of Switched System: the Differential Sylvester Equation

Following the conceptual line depicted in (Devasia et al. (1997)) and according to classical regulation theory, the purpose of this section is to find a trajectory for plant state x that guarantees perfect tracking of input reference. This is done by searching a linear map between exosystem state w (or its estimate $\hat{w}$) and plant state x.

Denoting with r the relative degree of the plant, consider the plant in *Brunovsky canonical form*:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \quad \ldots \\ \dot{\xi}_{r-1} = \xi_r \\ \dot{\xi}_r = a_\xi\xi + b_\eta\eta + b_u u \\ \dot{\eta} = A_\eta\eta + B_\xi\xi \\ y = \xi_1 \end{cases} \quad (18)$$

where the vector $\eta \in \mathbb{R}^{n-r}$ represents the *zero dynamics* of the system.

As a preliminary step, consider a LTI exosystem described by the following equations:

$$\begin{cases} \dot{w}(t) = Sw(t) \qquad w(0) = w_0 \\ y_d(t) = qw(t) \end{cases} \tag{19}$$

In order to obtain the perfect tracking, i.e. impose that $y \equiv y_d, \forall t \geqslant t_0$, it is sufficient that the following conditions are satisfied:

(i) initial conditions on ξ

$$\xi(t_0) = \begin{pmatrix} q \\ qS \\ \dots \\ qS^{r-1} \end{pmatrix} w(t_0) = Qw(t_0);$$

(ii) suitable control input $u(t)$

$$u(t) = b_u^{-1}\left(y_d^{(r)} - a_\xi \xi(t) - b_\eta \eta(t)\right) = $$
$$= b_u^{-1}\left(qS^r w(t) - a_\xi Q w(t) - b_\eta \eta(t)\right)$$

where $\eta(t)$ is a solution of the following differential equation

$$\dot{\eta} = A_\eta \eta + B_\xi \xi \qquad \eta(t_0) = \eta_0$$

The classic regulation theory requires that the *zero dynamics* $\eta(t)$ are a static linear combination of $w(t)$.

$$\eta(t) = \Pi w(t), \qquad \eta(t_0) = \Pi w(t_0) \tag{20}$$

where Π satisfies an *Algebraic Sylvester Equation* (*regulator equation*).

$$-\Pi S + A_\eta \Pi + B_\xi Q = 0 \tag{21}$$

If the non-resonance conditions on the eigenvalues of A_η and S are satisfied, the Sylvester equation has an unique solution and the initialization of *zero dynamics* η_0 is predetermined. The control input $u(t)$ and the state $x(t)$ become linear combinations of $w(t)$

$$u(t) = b_u^{-1}(qS^r - a_\xi Q - b_\eta \Pi) w(t) \qquad x(t) = \begin{pmatrix} Q \\ \Pi \end{pmatrix} w(t)$$

Unfortunately, these constraints applied to a switched exosystem may lead to discontinuous zero dynamics since for each value of $\sigma(t)$ there is a different $\Pi_{\sigma(t)}$.

In order to cope with this problem, pursuing the idea of (Devasia et al. (1997)), a possible relaxation is to allow a *time-variant* linear dependency between $w(t)$ and $\eta(t)$:

$$\eta(t) = \Pi(t) w(t) \tag{22}$$

The matrix $\Pi(t)$ is the solution of the following *Differential Sylvester Equation*:

$$\dot{\Pi}(t) = -\Pi(t) S + A_\eta \Pi(t) + B_\xi Q \tag{23}$$

with initial condition $\Pi(t_0) = \Pi_0$ such that $\eta_0 = \Pi_0 w_0$.

This extension allows to choose arbitrary initial conditions η_0 for $\eta(t)$ and, above all, allows to obtain a continuous zero dynamics trajectory even in the switching case. In fact, being $S_{\sigma(t)}$ and $Q_{\sigma(t)}$ piecewise constant with finite switches in finite time, the Switching Differential Sylvester Equation satisfies the *Carathéodory Conditions*. Therefore the solution $\Pi(t)$ is *absolutely continuous* and *a.e. differentiable* for every initial condition Π_0 (see Filippov (1988)). Thank to $\Pi(t)$, which can be calculated either online or offline, the exact output tracking can be achieved initializing the plant state x to initial state $x_0 = (Qw(t_0)\ \eta_0)^T$ and feeding the plant with the input u.

$$u(t) = b_u^{-1}(qS^r - a_\xi Q - b_\eta \Pi(t)) w(t)$$

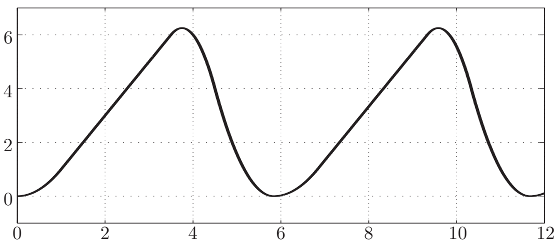

Fig. 2. Reference

The results of section 3.2 can be applied to scheme in Figure 1, provided that the exosystem state is replaced by its estimation (according to Subsection 3.1). Asymptotic convergence can be achieved using the following control input:

$$u(t) = \underbrace{b_u^{-1}\left(qS^r - a_\xi Q - b_\eta \Pi(t)\right)\hat{w}(t)}_{\textit{feed-forward action}} + \underbrace{K\left(x_d(t) - \hat{x}(t)\right)}_{\textit{stabilizing action}} \tag{24}$$

where x_d is the desired state trajectory calculated using $\hat{w}(t)$

$$x_d(t) = \begin{pmatrix} Q_{\sigma(t)} \\ \Pi(t) \end{pmatrix} \hat{w}(t) \tag{25}$$

and $\Pi(t)$ is a solution of (23).

Matrix K can be simply designed to stabilize the plant and the overall stability can be easily verified exploiting the cascade structure of the system.

4. EXAMPLE

The preliminary SLIM scheme described in previous sections has been applied to the simple case of tracking the periodic reference depicted in figure 2 on a stable, minimum phase, 4th order system.

The reference is a set of ramps and parabolic ramps with period $T = 5.83s$. Its parallel synthesis requires the use of four blocks: three 3rd order systems and one 2nd order system for a overall system of 11th order. The switching instants are $t_1 = 1s$, $t_2 = 3.5s$, $t_3 = 4.5s$, $t_4 = 5.83s$. The matrices (Ω_i, Θ_i) and the initial states w_{0i} of the four systems are:

$$\begin{aligned} &\Omega_1 = \Omega_3 = \Omega_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \Omega_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &\Theta_1 = \Theta_3 = \Theta_4 = (1\ 0\ 0) && \Theta_2 = (1\ 0) \\ &w_{01} = (0\ 0\ 2)^T && w_{02} = (1\ 2)^T \\ &w_{03} = (6\ 2\ {-8})^T && w_{04} = (0\ 0\ 4.5)^T \end{aligned} \tag{26}$$

The plant (A, B, C) is a stable, minimum-phase system with relative degree $r = 2$. In Brunovsky form the plant matrices are:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & -17 & -26 & -6 \\ 1 & -4 & -12 & -1 \\ 0 & 10 & 20 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \tag{27}$$

The matrix $\Pi(t)$ has been online calculated by numerically solving the *Differential Sylvester Equation*.

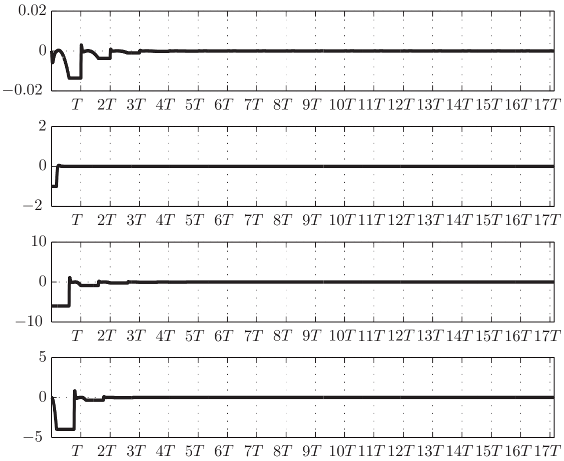

Fig. 3. Convergence of the estimate of exosystem state

The observer gains G_i are reported in eq. (28).

$$\begin{aligned} G_1 &= (\,30\ 275\ 750\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,)^T \\ G_2 &= (\,0\ 0\ 0\ 11\ 30\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,)^T \\ G_3 &= (\,0\ 0\ 0\ 0\ 0\ 30\ 275\ 750\ 0\ 0\ 0\,)^T \\ G_4 &= (\,0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 30\ 275\ 750\,)^T \end{aligned} \tag{28}$$

Although the sufficient condition of Theorem 4 is not satisfied, the exosystem state estimate $\hat{w}(t)$ still converges (see figure 3) and consequently the overall system is asymptotically stable as well (figure 4).

5. CONCLUSIONS

A novel Internal Model based approach, referred as *SLIM - Switched Linear Internal Model*, has been introduced. The main motivation of such method is to cover asymptotic tracking control problems quite common in practice by using a "simple" internal model structure and thus guaranteeing a good robustness to unmodeled disturbances.
In this paper, focusing on trajectory tracking problem, some preliminary results have been presented on the path to exhaustively define and verify this method. Two structures for the exosystem have been proposed and one, referred as *Parallel Structure*, has been exploited to define an observer, which is the basis for the internal model definition. In addition, in order to define a continuous and piecewise differentiable state reference trajectory for the plant to be controlled, the *Differential Silvester Equation* has been exploited.
Some of the issues to be considered in future works are the following.

- Exploit the so-called *Minimal Structure* in the observer design to reduce the observer order.
- Consider the case of unknown switching instants both in exosystem and observer definition. This could be useful to cover a wider class of signals.
- Find a less conservative convergence condition for the exosystem state observer
- Embed the exosystem state observer in the feedback control loop to achieve a "true" Internal Model based scheme.
- Extensively address the robustness properties of the proposed scheme.

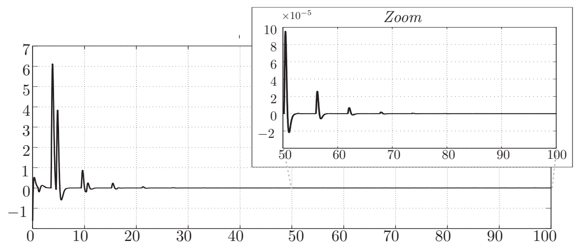

Fig. 4. Tracking Error

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