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LPV Control for Robust Attenuation of Non-stationary Sinusoidal Disturbances with Measurable Frequencies ⋆

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Abstract: Attenuation of sinusoidal disturbances with uncertain and arbitrarily time-varying, yet online measurable frequencies is considered. The disturbances are modeled as the outputs of an autonomous exogenous system, whose system matrix depends on some uncertain parameters and is skew-symmetric for all admissible parameter values. A procedure is then developed for the synthesis of an observer-based controller that uses the online measurements of the uncertain parameters to guarantee a desired level of attenuation at steady-state in the face of all admissible parameter variations. The controller is scheduled by the measurements of the uncertain parameters as well as the matrix-valued outputs of a unit that is also scheduled on the uncertain parameters. The synthesis procedure is based on solving a convex optimization problem in which the variables are subject to a set of parameter-dependent matrix inequality constraints, which can be relaxed into finitely many linear matrix inequalities. The procedure also provides options for improving the transient response.

Keywords: Robust control, regulation, linear parameter-varying systems, robust linear matrix inequalities, convex optimization.

1. INTRODUCTION

Rejection of sinusoidal or periodic disturbances is a common problem in various engineering systems ranging from disk drives, Sacks et al. [1996], to CD players, Lee [1998], Dettori [2001], helicopters Arcara et al. [2000] and steel casting Manayathara et al. [1996]. With the disturbances generated by a known autonomous and unstable exogenous system from unknown initial conditions, it is well-established in the theory of asymptotic regulation (see Saberi et al. [2000], Byrnes et al. [1997]) when and how such a problem can be solved in a stationary setting. In this case, the solution essentially amounts to replicating in the feedback loop the dynamics of the exogenous system as required by the Internal Model Principle of Francis et al. [1974]. The classical asymptotic regulation theory however does not offer an immediate solution when the disturbances have a non-stationary and uncertain nature, i.e. when their frequencies or periods can change in time. Hence, part of the recent interest concerning sinusoidal/periodic disturbance rejection is on reducing the sensitivity of the design against changes in the period/frequency. This can be realized either by robust controller synthesis techniques, Lee and Chung [1998], Tsao et al. [2000], Li and Tsao [2001], Steinbuch [2002], Osburn and Franchek [2004], Kim and Tsao [2004], Steinbuch et al. [2004], Köroğlu and Scherer [2008], or by adaptive methods, Bodson and Douglas [1997], Bodson [2001], Guo and Bodson [2005], Serrani et al. [2001] as specialized to sinusoidal disturbance rejection. When the frequency is measurable or estimable online, as is the case -for instance- in systems with rotational machinery, linear parameter-varying (LPV) controller synthesis techniques can also be applied for robust and adaptive non-stationary sinusoidal disturbance attenuation, Dettori [2001], Du et al. [2003], Hüttner et al. [2005], Kulkarni et al. [2005], Gruenbacher et al. [2007]. Within the LPV control framework, it even becomes possible to systematically handle other performance objectives as well, Köroğlu and Scherer [2007].

It was observed in Köroğlu and Scherer [2007] that exact asymptotic rejection of infinite-energy non-stationary disturbances might not be possible for some plants. Motivated by this observation and inspired by Hu et al. [2005], we formulate in Section 2 the attenuation of non-stationary sinusoidal disturbances based on a generalized notion of asymptotic regulation, in which the steady-state peak of the output need not be zero but is required to be bounded from above. The disturbances are assumed to be generated by a particular type of autonomous exo-system that is dependent on a set of uncertain parameters, which are online measurable and can influence the plant as well. We first investigate in Section 3 the solvability of the problem along parallel lines to the works on exact asymptotic regulation for linear time-varying systems, Zhang and Serrani [2006], Ichikawa and Katayama [2006]. This analysis reveals that the level of generalized asymptotic regulation is determined by the solutions of a system of differential equations, which are required to be bounded. The novel synthesis procedure developed in this paper is based on designing a parameter-dependent feedback between the

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variables involved in these differential equations, as well as suitable initial conditions, and obtaining the solutions online by using the available values of the uncertain parameter. We then describe in Section 4 how to construct an observer-based controller that uses the solutions of the differential regulator equations to guarantee the desired level of attenuation. After a brief example synthesis, the paper is concluded with remarks on possible extensions.

2. PROBLEM FORMULATION

This paper is concerned with the attenuation of multi-sinusoidal disturbances with uncertain and time-varying frequencies that are measurable during online operation. We characterize such disturbances as the outputs of a parameter-dependent autonomous system of the form

$$\dot{v} = A_{\mathrm{e}}(\delta)v;\ A_{\mathrm{e}}(\delta) = -A_{\mathrm{e}}(\delta)^T \in \mathbb{R}^{l\times l}, \tag{1}$$

where $\delta = [\,\delta_1 \ \cdots \ \delta_s\,]^T$ represents the vector of uncertain parameters, which can vary in time arbitrarily. As a simple yet sufficiently representative example, let us consider

$$A_{\mathrm{e}} = \begin{bmatrix} 0 & -\varpi(t) \\ \varpi(t) & 0 \end{bmatrix}, \varpi(t) = (1+\delta(t))\omega_0, \tag{2}$$

where $\omega_0 > 0$ corresponds to a nominal frequency. With

$$\phi(t) = \int_0^t \varpi(\tau)d\tau = \omega_0 t + \omega_0 \int_0^t \delta(\tau)d\tau, \tag{3}$$

it is straightforward to verify for this example that

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} \cos(\phi(t)) & -\sin(\phi(t)) \\ \sin(\phi(t)) & \cos(\phi(t)) \end{bmatrix} \begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix}, \tag{4}$$

which reveals the motivations behind viewing the systems described by (1) as the generators of non-stationary sinusoidal disturbances. Systems that generate multi-sinusoidal disturbances can be obtained -for instance- by using block-diagonal system matrices with sub-blocks of the form given in (2). The uncertain parameters in our setting basically reflect the deviations of the frequencies from their nominal values and are assumed to vary in time arbitrarily in a compact region $\mathcal{R} \subset \mathbb{R}^s$ that contains the origin. The admissible parameter trajectories are hence identified as $\mathcal{T}_{\mathcal{R}} \triangleq \{\delta(\cdot) : [0,\infty) \to \mathcal{R}\}$. Note that, irrespective of the parameter trajectory, the state of the system in (1) evolves with a constant norm, i.e. $\|v(t)\|^2 \triangleq v(t)^T v(t) = \|v(0)\|^2, \forall t \geq 0$ (since $d\|v(t)\|^2/dt = v(t)^T \mathfrak{He}(A_{\mathrm{e}}(\delta(t)))v(t) = 0$, where $\mathfrak{He}\, A_{\mathrm{e}} \triangleq A_{\mathrm{e}} + A_{\mathrm{e}}^T$).

The problem is formulated for a plant whose dynamics might also depend on the uncertain parameters as

$$G : \begin{bmatrix} \dot{x} \\ e \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A(\delta) & B_{\mathrm{r}}(\delta) & B(\delta) \\ \hline C_{\mathrm{r}}(\delta) & D_{\mathrm{r}}(\delta) & D_{\mathrm{rc}}(\delta) \\ C(\delta) & D_{\mathrm{cr}}(\delta) & 0 \end{array}\right] \begin{bmatrix} x \\ v \\ u \end{bmatrix}, \tag{5}$$

where $x(t) \in \mathbb{R}^k$ denotes the state vector, while $u(t) \in \mathbb{R}^n$ is the vector of control inputs that are to be used to regulate the outputs $e(t) \in \mathbb{R}^r$ based on the measurements $y(t) \in \mathbb{R}^m$. We assume that:

A.1 The parameter dependencies of the system matrices are all of the form $T(\delta) = T^0 + T^1(\delta)$, with $T^1(\cdot)$'s being continuous maps that satisfy $T^1(0) = 0$;

A.2 $(A(\delta)\,|B(\delta))$ is quadratically stabilizable by a parameter-dependent feedback ($\exists Y \succ 0, F(\cdot) : \mathfrak{He}\,(A(\delta) + B(\delta)F(\delta))Y \prec 0, \forall \delta \in \mathcal{R}$);

A.3 $\begin{pmatrix} \tilde{A}(\delta) \\ \hline \tilde{C}(\delta) \end{pmatrix} \triangleq \begin{pmatrix} A(\delta) & B_{\mathrm{r}}(\delta) \\ 0 & A_{\mathrm{e}}(\delta) \\ \hline C(\delta) & D_{\mathrm{cr}}(\delta) \end{pmatrix}$ is quadratically detectable by a parameter-dependent observer gain ($\exists \tilde{X} \succ 0, \tilde{L}(\cdot) : \mathfrak{He}\,\tilde{X}(\tilde{A}(\delta) + \tilde{L}(\delta)\tilde{C}(\delta)) \prec 0, \forall \delta \in \mathcal{R}$).

A controller that is to be scheduled with the online measurements of the parameters can be realized in its most general form as

$$K : \begin{bmatrix} \dot{\xi}(t) \\ u(t) \end{bmatrix} = \left[\begin{array}{c|c} A_K(\delta_{[0\,,\,t]}, t) & B_K(\delta_{[0\,,\,t]}, t) \\ \hline C_K(\delta_{[0\,,\,t]}, t) & D_K(\delta_{[0\,,\,t]}, t) \end{array}\right] \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}, \tag{6}$$

where $\delta_{[0\,,\,t]}$ represents the portion of parameter trajectory $\delta(\cdot)$ in the time-interval $[0\,,\,t]$. When the feedback loop is closed with this controller, the dynamics of the system are modified to

$$\begin{aligned} \dot{\chi} &= \underbrace{\begin{bmatrix} A + BD_KC & BC_K \\ B_KC & A_K \end{bmatrix}}_{\mathcal{A}(\delta_{[0,\,t]},t)} \underbrace{\begin{bmatrix} x \\ \xi \end{bmatrix}}_{\chi} + \underbrace{\begin{bmatrix} B_{\mathrm{r}} + BD_KD_{\mathrm{cr}} \\ B_KD_{\mathrm{cr}} \end{bmatrix}}_{\mathcal{B}_{\mathrm{r}}(\delta_{[0,\,t]},t)} v \\ e &= \underbrace{[C_{\mathrm{r}} + D_{\mathrm{rc}}D_KC \ \ D_{\mathrm{rc}}C_K]}_{\mathcal{C}_{\mathrm{r}}(\delta_{[0,\,t]},t)}\chi + \underbrace{[D_{\mathrm{r}} + D_{\mathrm{rc}}D_KD_{\mathrm{cr}}]}_{\mathcal{D}_{\mathrm{r}}(\delta_{[0,\,t]},t)} v, \end{aligned} \tag{7}$$

where the dependencies of the plant and controller matrices on time and the parameter trajectory are suppressed to avoid notational clutter. In most parts of the paper, the explicit time dependencies or the dependencies on the past parameter trajectories will either be avoided or expressed as dependencies on only the uncertain parameter for notational simplicity. The problem that we consider in this paper is to synthesize a parameter-dependent controller of the form (6), such that:

C.1 **Internal Stability**: The autonomous closed-loop formed by G and K (i.e. $\dot{\chi}(t) = \mathcal{A}(\delta_{[0,t]}, t)\chi(t)$) is uniformly robustly asymptotically stable (i.e. $\|\chi(t)\| \triangleq \left(\chi(t)^T\chi(t)\right)^{1/2}$ is uniformly bounded and $\|\chi(t)\| \to 0$ as $t \to \infty$, $\forall \delta(\cdot) \in \mathcal{T}_{\mathcal{R}}$);

C.2 **Generalized Asymptotic Regulation of Level $\boldsymbol{\kappa > 0}$**: $\limsup_{t\to\infty} \|e(t)\| < \kappa\|v(0)\|$, $\forall \delta(\cdot) \in \mathcal{T}_{\mathcal{R}}$.

Remark 1. A larger class of disturbances can be considered with exo-systems for which there exists a positive-definite matrix P such that $A_{\mathrm{e}}^T(\delta)P + PA_{\mathrm{e}}(\delta) = 0, \forall \delta \in \mathcal{R}$. Such cases can easily be subsumed to our framework through the state transformation $\nu = P^{1/2}v$, since $P^{1/2}A_{\mathrm{e}}(\delta)P^{-1/2}$ is then skew-symmetric.

3. CONDITIONS FOR GENERALIZED ASYMPTOTIC REGULATION

In this section, we investigate the solvability of the generalized asymptotic regulation problem. For this we first obtain an alternative realization of the closed-loop of (7) by a state-transformation of the form $\varkappa = \chi + \Xi v$, where Ξ is a time-varying matrix that satisfies the differential Sylvester equation

$$\dot{\Xi} + \Xi A_{\mathrm{e}}(\delta) - \mathcal{A}(\delta)\Xi + \mathcal{B}_{\mathrm{r}}(\delta) = 0. \tag{8}$$

Provided that Ξ is bounded, the closed-loop dynamics can be represented alternatively as

$$\begin{aligned} \dot{\varkappa} &= \mathcal{A}(\delta)\varkappa, \\ e &= \mathcal{C}_{\mathrm{r}}(\delta)\varkappa - \underbrace{(\mathcal{C}_{\mathrm{r}}(\delta)\Xi - \mathcal{D}_{\mathrm{r}}(\delta))}_{\Lambda} v, \end{aligned} \tag{9}$$

where the state-evolution is autonomous. This means that, if the closed-loop is assured to be stable, the steady-state behavior of the output will be determined by Λ. With $\Phi_{A_e}(t,\tau)$ denoting the state transition matrix of $A_e(\delta(t))$, we can in fact express the output as

$$e(t) = \mathcal{C}_r(\delta(t))\varkappa(t) - \Lambda(t)\Phi_{A_e}(t,0)v(0). \quad (10)$$

Recall that $\Phi_{A_e}(t,\tau)$ is the unique matrix-valued function that satisfies $d\left(\Phi_{A_e}(t,\tau)\right)/dt = A_e(\delta(t))\Phi_{A_e}(t,\tau)$ and $\Phi(\tau,\tau) = I$. Since $A_e(\delta)$ is skew-symmetric, we have

$$d\left(\Phi_{A_e}(t,\tau)^T\Phi_{A_e}(t,\tau)\right)/dt = \Phi_{A_e}(t,\tau)^T\mathfrak{He}(A_e(\delta(t)))\Phi_{A_e}(t,\tau) = 0, \quad (11)$$

which means

$$\Phi_{A_e}(t,\tau)^T\Phi_{A_e}(t,\tau) = \Phi_{A_e}(\tau,\tau)^T\Phi_{A_e}(\tau,\tau) = I, \quad (12)$$

or equivalently $\Phi_{A_e}(t,\tau)\Phi_{A_e}(t,\tau)^T = I$. We then have

$$\Lambda(t)\Phi_{A_e}(t,0)\Phi_{A_e}(t,0)^T\Lambda(t)^T = \Lambda(t)\Lambda(t)^T, \quad (13)$$

which implies that $\|\Lambda(t)\Phi_{A_e}(t,0)\| = \|\Lambda(t)\|$. We can thus relate the solvability of the generalized asymptotic regulation problem to the existence of a bounded solution to the differential equation (8) with which Λ satisfies an asymptotic norm bound, as described precisely in the following lemma:

Theorem 2. There exists a linear time-varying controller which guarantees C.1 and C.2 for a fixed $\delta(\cdot) \in \mathcal{T}_\mathcal{R}$, if and only if there exist bounded Π and Γ that satisfy

$$\begin{aligned} \dot{\Pi} &= A(\delta)\Pi - \Pi A_e(\delta) + B(\delta)\Gamma - B_r(\delta), \\ \Lambda &= C_r(\delta)\Pi + D_{rc}(\delta)\Gamma - D_r(\delta), \end{aligned} \quad (14)$$

$$\limsup_{t\to\infty} \|\Lambda(t)\| < \kappa. \quad (15)$$

Proof. In order to prove the necessity of (14) and (15), we first consider a fixed trajectory $\delta(\cdot) \in \mathcal{T}_\mathcal{R}$ and assume that C.1 as well as C.2 are satisfied by a certain controller K. We then infer from the stability of $\dot{\varkappa} = \mathcal{A}(\delta)\varkappa$ that the unique solution of (8) is bounded for any bounded initial condition $\Xi(0)$, thanks to $A_e(\delta)$ being skew-symmetric (see Ichikawa and Katayama [2006]). Assuming a partition of the form $\Xi^T = \left[\,\Pi^T\ \Sigma^T\,\right]$ compatible with $\mathcal{A}$, we thus infer that there exist bounded Π, Σ and Γ that satisfy (14) and

$$\begin{aligned} \dot{\Sigma} &= A_K\Sigma - \Sigma A_e + B_K(C\Pi - D_{cr}), \\ \Gamma &= C_K\Sigma + D_K(C\Pi - D_{cr}). \end{aligned}$$

Since $\dot{\varkappa} = \mathcal{A}(\delta)\varkappa$ is stable, we have $\lim_{t\to\infty}\|\varkappa(t)\| = 0$. It then follows from C.2 that

$$\limsup_{t\to\infty}\|e(t)\| = \limsup_{t\to\infty}\|\Lambda(t)\Phi_{A_e}(t,0)v(0)\| < \kappa\|v(0)\|,$$

which means $\limsup_{t\to\infty}\|\Lambda(t)\Phi_{A_e}(t,0)\| < \kappa$. We conclude the necessity proof by recalling (13), which implies $\|\Lambda(t)\Phi_{A_e}(t,0)\| = \|\Lambda(t)\|$. The sufficiency of the conditions will be established with the design of an observer-based controller in the next section. □

Although the conditions provided by Theorem 2 are exact, they are typically untractable. In order to derive tractable conditions as well as a controller synthesis procedure, we view (14) as a parameter-dependent system with a matrix-valued state. The state is required to remain bounded and the matrix-valued output is required to satisfy (15). The problem is then reformulated as a search for a suitable initial condition $\Pi(0) = \Pi^0$ and a matrix-valued and bounded input function Γ. The initial condition Π^0 is viewed also as a nominal value for Π, which is to be generated online in forward time as

$$\Pi(t) = \Pi^0 + \Pi^1(t),\ \Pi^1(0) = 0, \quad (16)$$

by updating the values of Π^1 using the available values of the uncertain parameter. Since this means that Π^1 will be at our disposal, we can rely on a parameter-dependent state-feedback-like synthesis of the form

$$\Gamma = \digamma(\delta)\Pi^1 + \Psi(\delta), \quad (17)$$

where $\digamma(\cdot)$ and $\Psi(\cdot)$ are to be designed. Introducing

$$B_a(\Pi^0,\Psi(\delta),\delta) \triangleq B_r(\delta) + \Pi^0 A_e(\delta) - A(\delta)\Pi^0 - B(\delta)\Psi(\delta), \quad (18)$$

$$D_a(\Pi^0,\Psi(\delta),\delta) \triangleq D_r(\delta) - C_r(\delta)\Pi^0 - D_{rc}(\delta)\Psi(\delta), \quad (19)$$

we can the represent the system of (14) as

$$\begin{aligned} \dot{\Pi}^1 &= (A(\delta) + B(\delta)\digamma(\delta))\,\Pi^1 - \Pi^1 A_e(\delta) - B_a(\Pi^0,\Psi(\delta),\delta), \\ \Lambda &= (C_r(\delta) + D_{rc}(\delta)\digamma(\delta))\,\Pi^1 - D_a(\Pi^0,\Psi(\delta),\delta). \end{aligned} \quad (20)$$

Based on an analysis of this system, we obtain the following key result that will facilitate the solution of the robust generalized asymptotic regulation problem:

Theorem 3. There exist bounded Π and Γ with which (14) and (15) are satisfied for all $\delta(\cdot) \in \mathcal{T}_\mathcal{R}$, if there exist: $\eta \in \mathbb{R}_+$, $\Pi^0 \in \mathbb{R}^{k\times l}$, $0 \prec P = P^T \in \mathbb{R}^{k\times k}$, $0 \succcurlyeq S = S^T \in \mathbb{R}^{l\times l}$ and $Q : \mathcal{R} \to \mathbb{R}^{n\times k}$, $\Psi : \mathcal{R} \to \mathbb{R}^{n\times l}$ such that, for all $\delta \in \mathcal{R}$

$$\mathcal{L}_s(\delta) \triangleq \mathfrak{He}\begin{bmatrix} A(\delta)P + B(\delta)Q(\delta) + \eta P & B_a(\Pi^0,\Psi(\delta),\delta) \\ 0 & SA_e(\delta) + \eta S \end{bmatrix} \preccurlyeq 0, \quad (21)$$

$$\mathcal{L}_r(\delta) \triangleq \mathfrak{He}\begin{bmatrix} \frac{P}{2} & 0 & 0 \\ 0 & \frac{\kappa I + S}{2} & 0 \\ C_r(\delta)P + D_{rc}(\delta)Q(\delta) & D_a(\Pi^0,\Psi(\delta),\delta) & \frac{\kappa}{2}I \end{bmatrix} \succ 0, \quad (22)$$

Such Π and Γ can be generated online using the available values of the uncertain parameter as in Figure 1, where

$$\digamma(\delta) = Q(\delta)P^{-1}. \quad (23)$$

When Π and Γ are scheduled with this system starting from $\Pi^1(0) = 0$, we will have at any time instant that

$$\begin{bmatrix} -S & \Pi^1(t)^T \\ \Pi^1(t) & P \end{bmatrix} \succcurlyeq \begin{bmatrix} -e^{-2\eta t}\Phi_{A_e}(t,0)S\Phi_{A_e}(t,0)^T & 0 \\ 0 & 0 \end{bmatrix}, \quad (24)$$

$$\Lambda(t)^T\Lambda(t) \prec \kappa^2 I + \kappa e^{-2\eta t}\Phi_{A_e}(t,0)S\Phi_{A_e}(t,0)^T, \quad (25)$$

where $\Phi_{A_e}(t,\tau)$ represents the state-transition matrix of the exogenous system (1) for the considered $\delta(\cdot) \in \mathcal{T}_\mathcal{R}$.

Proof. We first introduce a matrix-valued function as $\mathcal{V}(\Pi^1) = (\Pi^1)^T P^{-1}\Pi^1 + S$ and infer from (21) that, along the state trajectories of (14) for any $\delta(\cdot) \in \mathcal{T}_\mathcal{R}$

$$\frac{d}{dt}\left(e^{2\eta t}\Phi_{A_e}(t,0)^T\mathcal{V}(\Pi^1(t))\Phi_{A_e}(t,0)\right) = \Theta(t)^T\mathcal{L}_s(\delta(t))\Theta(t) \preccurlyeq 0,$$

where $\Theta(t)^T = e^{\eta t}\left[\,\left(P^{-1}\Pi^1(t)\Phi_{A_e}(t,0)\right)^T\ \ -\Phi_{A_e}(t,0)^T\,\right]$.

We thus conclude with $\Pi^1(0) = 0$ (and hence $\mathcal{V}(0) = S$) that $\mathcal{V}(\Pi^1(t)) \preccurlyeq e^{-2\eta t}\Phi_{A_e}(t,0)S\Phi_{A_e}(t,0)^T$, which reads via the Schur-complement lemma as the inequality of (24). With $\|\Phi_{A_e}(t,0)\| = 1$, this clearly implies that Π^1 and hence Π as well as Γ will remain bounded. On the other hand, Λ will respect (25) as follows from

$$\mathcal{V}(\Pi^1(t)) + \kappa I - \kappa^{-1}\Lambda(t)^T\Lambda(t) = \Omega(t)^T\mathcal{L}_r(\delta(t))\Omega(t) \succ 0,$$

where $\Omega(t)^T = [\,\left(P^{-1}\Pi^1(t)\right)^T\ \ -I\ \ -\kappa^{-1}\Lambda(t)^T\,]$. □

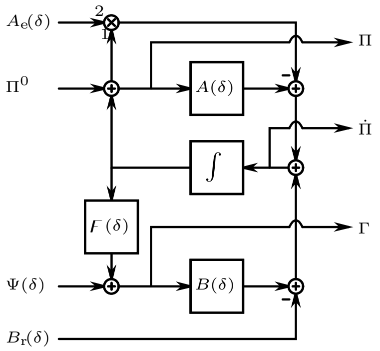

Fig. 1. System for scheduling of Π and Γ (Π^0, Γ^0: matrix-valued constant inputs; $A_e(\delta), B_r(\delta), \Psi(\delta)$: matrix-valued parameter-dependent inputs; $A(\delta), B(\delta), F(\delta)$: parameter-dependent gain matrices; $\int$: integrator).

Remark 4. Conditions (21) and (22) read as infinitely many matrix inequalities once the parameter dependencies of Q and Ψ are fixed. Such conditions can be rendered tractable by employing suitable relaxations from the robust optimization literature (e.g. convex-hull or Pólya relaxation when $\mathcal{R}$ is a polytope; sum-of-squares relaxation if $\mathcal{R}$ is described by polynomial matrix inequalities; see Apkarian and Tuan [2000], Scherer [2006] and the references therein). In this fashion, one can obtain sufficient conditions in terms of finitely many LMIs by simply fixing the value of η. Once sufficient conditions are relaxed into finitely many LMIs, the best level of generalized asymptotic regulation achievable according to these conditions can then be obtained through a line search over η. When the exogenous system as well as plant matrices have affine dependence on the uncertain parameters (i.e. $A_e = A_e^0 + A_e^1 \delta \otimes I, ...$) and $\mathcal{R}$ is a polytope $\mathcal{R} = \left\{ \sum_{j=1}^q \alpha_j \delta^j : \sum_{j=1}^q \alpha_j = 1, \alpha_j \geq 0 \right\}$ with extreme points $\{\delta^1 \ldots, \delta^q\}$, one can choose Q and Ψ simply as matrix variables. In this case, conditions (21) and (22) are satisfied throughout $\mathcal{R}$ if and only if they are satisfied for all $\delta^j, j = 1, \ldots, q$. If, moreover, B and D_{rc} are constant, one can employ the same approach with Q and Ψ that have affine parameter dependence. In the case of general polynomial parameter dependence, we can employ the multi-convexity approach of Gahinet et al. [1996] or resort to more advanced relaxation techniques.

Remark 5. In retrospect to the feedback synthesis in (17) with (23), we realize that the matrix inversion might lead to numerical troubles. This can be avoided by imposing a uniform upper bound on the condition number of P or alternatively on the norm of $F(\delta)$ at the expense of some conservativeness. As a means to improve the numerical reliability of the synthesis, we consider here imposing a uniform upper bound on $\|\Gamma(t)\|$. This is also expected to have curing effects against the undesirable outcomes of large control inputs, which are in part shaped by Γ (see Figure 1). By a simple adaptation of (22), we can derive a constraint that guarantees $\|\Gamma(t)\| \leq \varrho, \forall t \geq 0$ as

$$\begin{bmatrix} P & 0 & Q(\delta)^T \\ 0 & \varrho I + S & \Psi(\delta)^T \\ Q(\delta) & \Psi(\delta) & \varrho I \end{bmatrix} \succcurlyeq 0, \ \forall \delta \in \mathcal{R}. \tag{26}$$

With a fixed level of κ that is known to be achievable, the feedback synthesis can be constructed as in (17) by minimizing ϱ subject to some relaxed versions of (21), (22) and (26) that are tractable. In order to obtain a uniform bound on the norm of Π as $\|\Pi(t)\| \leq \lambda$, we can add a constraint of the form

$$\begin{bmatrix} P & 0 & P \\ 0 & \lambda I + S & (\Pi^0)^T \\ P & \Pi^0 & \lambda I \end{bmatrix} \succcurlyeq 0. \tag{27}$$

Remark 6. If a certain level of disturbance attenuation, say κ^0, is required for the nominal case in which $\delta(t) = 0, \forall t \geq 0$, this can be achieved simply by imposing the convex constraints $B_a(\Pi^0, \Psi(0), 0) = 0$ and $\|D_a(\Pi^0, \Psi(0), 0)\| < \kappa_0$, at the cost of possible degradation in the worst-case disturbance attenuation performance in the face of parameter variations.

4. A SCHEDULED OBSERVER-BASED CONTROLLER FOR GENERALIZED ASYMPTOTIC REGULATION

We provide in this section a solution to the robust generalized asymptotic regulation problem by a scheduled observer-based controller of the form

$$\left[\begin{array}{c} \dot{\xi} \\ \hline u \end{array}\right] = \left[\begin{array}{c|c} \tilde{A}(\delta) + \tilde{U} B(\delta) \tilde{F}(\delta) + \tilde{L}(\delta) \tilde{C}(\delta) & -\tilde{L}(\delta) \\ \hline \tilde{F}(\delta) & 0 \end{array}\right] \left[\begin{array}{c} \xi \\ \hline y \end{array}\right], \tag{28}$$

where $\tilde{U} = [\, I_k \ 0 \,]^T$ and $\tilde{F}(\cdot), \tilde{L}(\cdot)$ are to be designed. This controller is commonly employed in classical regulation problems, Saberi et al. [2000], and can also be applied for regulation in linear time-varying systems, Ichikawa and Katayama [2006]. Note, however, that it differs from the standard observer-based controllers in that it replicates the dynamics of the exo-system as well as that of the plant. In other words, ξ represents the estimate of the combined vector $\left[\, x^T \ \ v^T \,\right]^T$. Assuming for a moment that ξ is indeed equal to this combined vector, we can observe by the help of a state transformation of the form

$$\varsigma = x + \Pi v, \tag{29}$$

that, the state and disturbance feedback achieved by $u = \tilde{F}(\delta)\xi = F(\delta)x + F_e(\delta)v$ will guarantee internal stability as well as generalized asymptotic regulation, provided that: (i) $\dot{\varsigma} = (A(\delta) + B(\delta)F(\delta))\,\varsigma$ is stable; (ii) $F_e = F\Pi - \Gamma$ with Π and Γ being related as in (14) and bounded; and (iii) Λ satisfies (15). On the other hand, if ξ corresponds to an estimate provided by an observer, the estimation error

$$\zeta = -\xi + \tilde{U}x + [\, 0 \ I \,]^T v, \tag{30}$$

evolves according to $\dot{\zeta} = (\tilde{A}(\delta) + \tilde{L}(\delta)\tilde{C}(\delta))$ and can be steered asymptotically to zero by a suitable choice of $\tilde{L}(\cdot)$. In this case, we can express the dynamics of the closed-loop system as

$$\left[\begin{array}{c} \dot{\varsigma} \\ \dot{\zeta} \\ \hline e \end{array}\right] = \left[\begin{array}{cc|c} A(\delta) + B(\delta)F(\delta) & -B(\delta)\tilde{F}(\delta) & 0 \\ 0 & \tilde{A}(\delta) + \tilde{L}(\delta)\tilde{C}(\delta) & 0 \\ \hline C_r(\delta) + D_{rc}(\delta)F(\delta) & -D_{rc}(\delta)\tilde{F}(\delta) & -\Lambda \end{array}\right] \left[\begin{array}{c} \varsigma \\ \zeta \\ \hline v \end{array}\right]. \tag{31}$$

We can clearly guarantee the internal stability of the closed-loop for all admissible parameter trajectories by an off-line design of $F(\cdot)$ and $\tilde{L}(\cdot)$ without requiring the knowledge of Π and Γ or uniform bounds on the norms thereof (cf. Köroğlu and Scherer [2007]). On the other hand, generalized asymptotic regulation can be achieved by basing the design of $F_e(\delta) = F(\delta)\Pi - \Gamma$ on the synthesis described in Theorem 3.

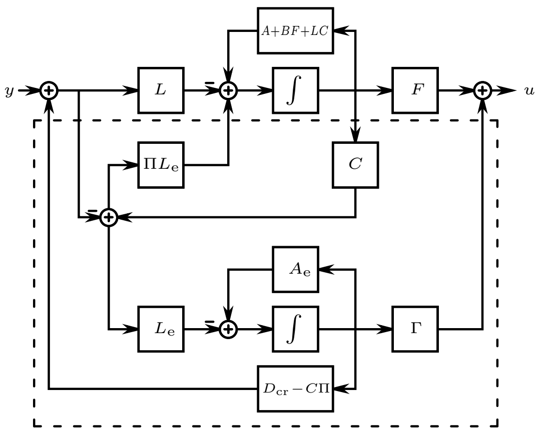

Fig. 2. Implementation of the scheduled observer-based controller (All rectangular blocks, except for the integrator $\int$, represent time-varying gain matrices).

The observer-based design we have thus sketched allows us to develop the following scheduled controller synthesis procedure for generalized asymptotic regulation:

Theorem 7. There exists a controller that solves the generalized asymptotic regulation problem as formulated in Section 2, if the conditions in Theorem 3 are satisfied and there exist $Y = Y^T \in \mathbb{R}^{k\times k}, N : \mathcal{R} \to \mathbb{R}^{n\times k}$ and $\tilde{X} = \tilde{X}^T \in \mathbb{R}_+^{(k+l)\times(k+l)}, \tilde{M} : \mathcal{R} \to \mathbb{R}^{(k+l)\times r}$ with which

$$\mathfrak{He}\,((A(\delta) + \rho I)Y + B(\delta)N(\delta)) \preccurlyeq 0, \forall\delta \in \mathcal{R}, \quad (32)$$

$$Y - \sigma I \preccurlyeq 0 \text{ and } \begin{bmatrix} Y & I \\ I & \sigma I \end{bmatrix} \succcurlyeq 0, \quad (33)$$

$$\mathfrak{He}\,(\tilde{X}(A(\delta) + \rho I) + \tilde{M}(\delta)C(\delta)) \preccurlyeq 0, \forall\delta \in \mathcal{R}, \quad (34)$$

$$\tilde{X} - \alpha I \preccurlyeq 0 \text{ and } \begin{bmatrix} \tilde{X} & I \\ I & \alpha I \end{bmatrix} \succcurlyeq 0, \quad (35)$$

for some $\rho, \sigma, \alpha \in \mathbb{R}_+$. A scheduled observer-based controller can then be constructed in terms of

$$F(\delta) = N(\delta)Y^{-1}, \tilde{L}(\delta) = \begin{bmatrix} L(\delta) \\ L_{\mathrm{e}}(\delta) \end{bmatrix} = \tilde{X}^{-1}\tilde{M}(\delta), \tilde{\Pi} = [I\ \Pi], (36)$$

as a controller with realization

$$\begin{bmatrix} \dot{\xi}_{\mathrm{a}} \\ \dot{\xi}_{\mathrm{i}} \\ u \end{bmatrix} = \left[\begin{array}{cc|c} A+BF+\tilde{\Pi}\tilde{L}C & \tilde{\Pi}\tilde{L}(D_{\mathrm{cr}} - C\Pi) & \tilde{\Pi}\tilde{L} \\ L_{\mathrm{e}}C & A_{\mathrm{e}} + L_{\mathrm{e}}(D_{\mathrm{cr}} - C\Pi) & L_{\mathrm{e}} \\ \hline -F & \Gamma & 0 \end{array}\right] \begin{bmatrix} \xi_{\mathrm{a}} \\ \xi_{\mathrm{i}} \\ y \end{bmatrix}, (37)$$

where the parameter dependencies of the terms are suppressed. Implementable as in Figure 2 with the unit in the dashed box to be scheduled by Π and Γ that are obtained as in Figure 1, this controller guarantees $\forall\delta(\cdot) \in \mathcal{T}_{\mathcal{R}}$ that

$$\|e(t)\| < \left(\alpha\phi_3\|\zeta(0)\| + \sigma\phi_2\left(\|\varsigma(0)\| + \alpha\phi_1\|\zeta(0)\|t\right)\right)e^{-\rho t} + \kappa\|v(0)\|, \quad (38)$$

for some positive scalars $\phi_1, \phi_2, \phi_3 \in \mathbb{R}_+$. Possible choices for these scalars are: $\phi_1 = b(f(\lambda + 1) + \varrho)$, where $b \triangleq \sup_{\delta\in\mathcal{R}} \|B(\delta)\|$ and $f \triangleq \sup_{\delta\in\mathcal{R}} \|F(\delta)\|$ whereas λ and ϱ are uniform bounds on $\|\Pi(t)\|$ and $\|\Gamma(t)\|$ respectively, as inherited from (27) and (26); $\phi_2 = \sup_{\delta\in\mathcal{R}} \|C_{\mathrm{r}}(\delta) + D_{\mathrm{rc}}(\delta)F(\delta)\|$; and $\phi_3 = d(f(\lambda + 1) + \varrho)$, where $d = \sup_{\delta\in\mathcal{R}} \|D_{\mathrm{rc}}(\delta)\|$.

Proof. The explicit expressions for the states of the closed-loop system in (31) are given by

$$\varsigma(t) = \Phi_{A+BF}(t, 0)\varsigma(0) - \vartheta(t),$$
$$\zeta(t) = \Phi_{\tilde{A}+\tilde{L}\tilde{C}}(t, 0)\zeta(0),$$
$$\vartheta(t) = \int_0^t \Phi_{A+BF}(t, \tau)B(\delta(\tau))\tilde{F}(\delta(\tau))\Phi_{\tilde{A}+\tilde{L}\tilde{C}}(\tau, 0)\zeta(0)d\tau,$$

where the dependencies of state-transition matrices on the considered parameter trajectory is suppressed for simplicity. It follows from (34) and the choice of $\tilde{L}$ as in (36) that a positive-definite function of the form $\mathcal{V}(\zeta) = \zeta^T\tilde{X}\zeta$ satisfies $d\left(\mathcal{V}(\zeta(t))\right)/dt + 2\rho\mathcal{V}(\zeta(t)) \le 0$ and hence $\mathcal{V}(\zeta(t)) \le \mathcal{V}(\zeta(\tau))e^{-2\rho(t-\tau)}$ along the trajectories of the closed-loop system. This reads as $\Phi_{\tilde{A}+\tilde{L}\tilde{C}}(t,\tau)^T\tilde{X}\Phi_{\tilde{A}+\tilde{L}\tilde{C}}(t,\tau) - e^{-2\rho(t-\tau)}\tilde{X} \preccurlyeq 0$, which implies together with (35) (i.e. $\alpha^{-1}I \preccurlyeq \tilde{X} \preccurlyeq \alpha I$) that $\|\Phi_{\tilde{A}+\tilde{L}\tilde{C}}(t,\tau)\| \le \alpha e^{-\rho(t-\tau)}, \forall\delta(\cdot) \in \mathcal{T}_{\mathcal{R}}$. Setting $\zeta(0) = 0$, we can establish along similar lines that $\|\Phi_{A+BF}(t,\tau)\| \le \sigma e^{-\rho(t-\tau)}, \forall\delta(\cdot) \in \mathcal{T}_{\mathcal{R}}$. Using the bounds on the state transition matrices and applying the usual norm bounding, we conclude that $\|\zeta(t)\| \le \alpha\|\zeta(0)\|e^{-\rho t}$, $\|\vartheta(t)\| \le \sigma\alpha\phi_1\|\zeta(0)\|te^{-\rho t}$ and $\|\varsigma(t)\| \le (\sigma\|\varsigma(0)\| + \sigma\alpha\phi_1\|\zeta(0)\|t)e^{-\rho t}$, which clearly imply internal stability as well as (38). With $\xi_{\mathrm{a}} = -\tilde{\Pi}\xi$, $\xi_{\mathrm{i}} = -[\,0\ I_l\,]\xi$, we can realize the controller of (28) as in (37). □

Remark 8. Fixing ρ as a sufficiently large value and minimizing σ, α will be helpful for synthesizing F and $\tilde{L}$ reliably as well as for obtaining a desirable transient response.

5. ILLUSTRATIVE EXAMPLE

In this section, we consider a mass-spring damper system whose dynamics are described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \hline e \\ y \end{bmatrix} = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{b}{m_1} & \frac{k_1}{m_1} & \frac{b}{m_1} & 0 & \frac{1}{m_1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{m_2} & \frac{b}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{b}{m_2} & \frac{k_2}{m_2} & -\frac{1}{m_2} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \hline d \\ u \end{bmatrix},$$

where the disturbance affecting the system is assumed to be of the form $d(t) = \sin(\omega_0(1 + \delta(t)))$, $\delta(t) \in [-\beta, \beta]$, as is the first state of (2) for the initial condition $v(0) = [\,0\ \ -1\,]^T$. For a set of parameters given by $m_1 = 2, m_2 = 0.5, k_1 = 100, k_2 = 150, b = 10, \omega_0 = 8$, we synthesized a scheduled observer-based controller according to Theorem 3 and Theorem 7. With parameter-independent variables, we obtained an upper bound on the minimum level of κ as 0.0646 (for $\eta = 0.0483$). We then obtained $\digamma$, Ψ and Π^0 all as constant matrices given by

$$\digamma = [\,-89340\ \ -535\ \ -57\ \ -36\,], \qquad \Pi^0 = \begin{bmatrix} 0.0004 & 0.0000 \\ 0.0003 & -0.0031 \\ -1.2747 & 0.0000 \\ -0.0017 & 10.0755 \end{bmatrix},$$
$$\Psi = [\,127.51\ \ -100.77\,],$$

in a way to guarantee generalized asymptotic regulation below $\kappa = 0.0653$, with the particular approach described in Remark 4. A constant state-feedback and a constant observer gain are then designed with $\rho = 0.5$ as

$$F = [99.3\ 8.1\ -159.3\ -5.3], \ \tilde{L} = [-43.0\ 70.6\ -10.7\ 144.2\ 7.7\ -11.9]^T.$$

Figure 3 presents the simulation results obtained with this controller, starting from zero initial plant and controller states. The disturbance is generated by a particular parameter trajectory given in Figure 3-a. The parameter

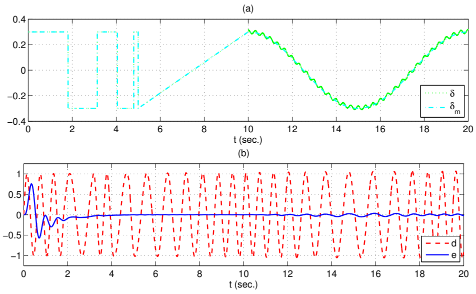

Fig. 3. Example simulation: (a) The uncertain parameter, δ, and the measurement, δ_{m}, used to schedule the controller; (b) The disturbance and the output.

first switches between -0.3 and 0.3 with an increasing frequency, then increases from -0.3 to 0.3 with a constant rate and in the last phase it shows a small and fast sinusoidal variations around a low frequency sinusoidal curve. In the first two phases, the controller is scheduled with exact values of the parameter, whereas in the last phase the scheduling variable simply chosen to follow the slow sinusoidal variation, thus averaging out the fast variations in the uncertain parameter with which the disturbance is generated. As is visible from Figure 3, the controller attenuates the disturbance significantly if scheduled with the correct values of the parameter. On the other hand, the performance degrades noticeably when there is measurement error.

6. CONCLUDING REMARKS

We have developed a novel procedure to synthesize an observer-based LPV controller for robust attenuation of non-stationary sinusoidal disturbances. It is of much interest to develop methods for synthesizing controllers that guarantee additional performance objectives and in particular robustness against possible noise that effects the online measurements of the uncertain parameters.

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