Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

STABILIZATION OF A CHAOTIC VAN DER POLE SYSTEM

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Abstract: An approach to stabilization problem of a chaotic Van der Pole system is presented. Control algorithm uses only measurements of output variable, not its derivatives or state vector of the system.

Keywords: Chaotic behaviour; nonlinear stabilization; nonlinear control; output control

1. INTRODUCTION

Problem of chaos control has been the area of intensive study for the last decade. Many papers, devoted to problem of chaos control, have been published and a number of practical tasks, in which chaotic regimes can arise (Andrievskii and Fradkov, 2003; Andrievskii and Fradkov, 2004), have been discovered. Theoretic and practical components of this problem are conditioned by the fact that oscillatory and chaotic processes are often found in nature and technics. Forms of their description are constantly being developed and improved. One of classical examples of differential models, describing oscillatory and chaotic processes is Van der Pole equation (Gilbert and Gammon, 2000).

In the given paper a problem of chaos control is considered for stabilization of chaotic processes arising in a Van der Pole system.

2. PROBLEM STATEMENT

Consider a nonlinear plant described by Van der Pole equation

$$\ddot{y} - \tau_1 (1 - y^2)\dot{y} + \tau_2 y = E\sin(\omega t) + u , \quad (1)$$

where $\tau_1 > 0$, $\tau_2 > 0$.

Let us transform model (1) in the following way

$$y = \frac{b(p)}{a(p)}u + \frac{d(p)}{a(p)}\varphi(y) + \frac{f(p)}{a(p)}w(t) , \quad (2)$$

where $p = d/dt$; $b(p) = 1$; $a(p) = p^2 - \tau_1 p + \tau_2$ is unstable polynomial ($\tau_1 > 0$ and $\tau_2 > 0$); $d(p) = d_1 p$, $d_1 = -\tau_1/3$; $f(p) = 1$; relative degree of the transfer function $b(p)/a(p)$ is $\rho = n - m = 2$; coefficients τ_1 and

τ_2 are assumed to be unknown; $w(t) = E\sin(\omega t)$ is an unknown, bounded disturbance; function $\varphi(y) = y^3$.

Purpose of control is to find a control law, which uses only measurements of the output variable of the model (2), ensuring convergence of output trajectory of the nonlinear system to an area ε_0. Boundaries of the area can be reduced by an appropriate selection of the controller coefficients.

3. DESIGN OF CONTROL LAW

Let us choose control law in the form

$$u = -\chi(p)(\mu + k)\xi , \tag{3}$$

where coefficient $\mu > 0$ (takes rather big value in common case) and any Hurwitz polynomial $\chi(p)$ are chosen such that polynomial $\gamma(p) = a(p) + \mu b(p)\chi(p)$ is Hurwitz; function ξ is formed by estimation algorithm

$$\dot{\xi} = \sigma(\nu - \xi) , \tag{4}$$

where function $\nu = y + y^5$; parameter $k > 0$ and function $\sigma > 0$ are chosen according to requirements presented below.

Substituting (3) in equation (2), we obtain

$$y = \frac{b(p)}{a(p)}[-\chi(p)(\mu + k)\nu + \chi(p)(\mu + k)\eta] + \\ + \frac{d(p)}{a(p)}\varphi(y) + \frac{f(p)}{a(p)}w(t) , \tag{5}$$

where $\eta = \nu - \xi$ is error.
After simple transformations, for model (5) we have

$$a(p)y + \mu\chi(p)b(p)y = b(p)\chi(p)[(\mu + k)\eta - \\ - ky - (\mu + k)y^5] + d(p)\varphi(y) + f(p)w(t).$$

Denoting

$$\gamma(p) = a(p) + \mu\chi(p)b(p) \text{ and } \beta(p) = \chi(p)b(p) ,$$

we obtain

$$y = \frac{\beta(p)}{\gamma(p)}[-ky - (\mu + k)y^5 + (\mu + k)\eta + \overline{w}(t)] + \\ + \frac{d(p)}{\gamma(p)}\varphi(y) , \tag{6}$$

where function $\overline{w}(t) = \dfrac{f(p)}{\beta(p)}w(t)$ is smooth and bounded according to the view of function $w(t)$.

Let us present input-output model (6) as input-state-output model

$$\dot{x} = Ax + b(-ky - (\mu + k)y^5 + (\mu + k)\eta + \\ + \overline{w}(t)) + q\varphi(y) , \; y = c^T x , \tag{7}$$

where $x \in \mathbf{R}^2$ is state of model (7), A, b, q и c are transformation matrices from input-output model (6) to input-state-output model (7). As polynomial $\gamma(p)$ is Hurwitz and model (6) is strictly minimum phase, according to consequence 3 (Bobtsov and Nikolaev, 2005), it is possible to determine a symmetric positively determined matrix P, satisfying two following matrix equations:

$$A^T P + PA = -Q_1 , \quad Pb = c , \tag{8}$$

where $Q_1 = Q_1^T > 0$, entries of matrix Q_1 depend on parameter μ and do not depend on parameter k.

Consider derivative of function of deviation η

$$\dot{\eta} = \dot{\nu} - \sigma(\nu - \xi) = \dot{y} + 5y^4\dot{y} - \sigma\eta = -\sigma\eta + \Omega\dot{y} , \tag{9}$$

where $\Omega = 1 + 5y^4$.

Let us present a theorem describing conditions for calculation of parameter k and function σ, ensuring accomplishing of purpose of control.

Theorem. There exist a parameter k and a function σ such that all trajectories of system (7), (9) can be localized in any small area by increasing of parameter k.

Proof. Consider a Lyapunov function of the form

$$V = V_1 + V_2 , \tag{10}$$

where

$$V_1 = x^T Px , \tag{11}$$

$$V_2 = \eta^2 . \tag{12}$$

Differentiating (11) with respect to time subject to equation (7) we obtain

$$\dot{V}_1 = x^T(A^T P + PA)x + 2(\mu + k)x^T Pb\eta - \\ - 2kx^T Pby - 2(\mu + k)x^T Pby^5 + \\ + 2x^T Pb\overline{w} + 2x^T Pq\varphi(y) , \tag{13}$$

Substituting equations (8) into (13) and considering the following equations

$$-2kx^T Pby = -2ky^2 ,$$

$$2(\mu + k)x^T Pb\eta = 2(\mu + k)y\eta \le y^2 + (\mu + k)^2\eta^2 ,$$

$$2x^T Pb\overline{w} = 2y\overline{w} \le ky^2 + \frac{1}{k}\overline{w}^2 ,$$

$$2x^T Pq\varphi(y) \le \delta x^T Pqq^T Px + \delta^{-1}[\varphi(y)]^2 ,$$

for derivative of Lyapunov function (11) we obtain

$$\dot{V}_1 \le x^T\left(-Q_1 + \delta Pqq^T P\right)x - (k-1)y^2 + (\mu+k)^2\eta^2 - \\ -2(\mu+k)y^6 + \delta^{-1}[\varphi(y)]^2 + \frac{1}{k}\overline{w}^2 , \qquad (14)$$

where small value $\delta > 0$.
Differentiating (12) with respect to time subject to equation (7) we receive

$$\dot{V}_2 = 2\eta\left(-\sigma\eta + \Omega\dot{y}\right) = -2\sigma\eta^2 + 2\Omega\eta c^T Ax - \\ -2k\Omega\eta c^T by + 2(\mu+k)\Omega c^T b\eta^2 + 2\Omega\eta c^T q\varphi(y) - \\ -2(\mu+k)\Omega\eta c^T by^5 + 2\eta\Omega c^T b\overline{w}, \qquad (15)$$

where in equation (15) $\dot{y}$ was substituted by summand

$$\dot{y} = c^T(Ax - kby - (\mu+k)by^5 + \\ +(\mu+k)b\eta + b\overline{w} + q\varphi(y)).$$

Then, considering inequalities

$$2\Omega\eta c^T Ax \le \delta^{-1}c^T AA^T c\Omega^2\eta^2 + \delta x^T x ,$$
$$-2k\Omega\eta c^T by \le k^2\left(c^T b\right)^2\Omega^2\eta^2 + y^2 ,$$
$$2\Omega\eta c^T q\varphi(y) \le k\left(c^T q\right)^2\Omega^2\eta^2 + k^{-1}[\varphi(y)]^2 ,$$
$$2\eta\Omega c^T b\overline{w} \le k(c^T b)^2\Omega^2\eta^2 + k^{-1}\overline{w}^2 ,$$
$$-2(\mu+k)\Omega\eta c^T by^4 y \le \\ \le (\mu+k)^2(c^T b)^2\Omega^2 y^8\eta^2 + y^2 ,$$

for derivative of function (12)

$$\dot{V}_2 \le -2\sigma\eta^2 + \delta^{-1}c^T AA^T c\Omega^2\eta^2 + \\ + \delta x^T x + k^2\left(c^T b\right)^2\Omega^2\eta^2 + y^2 + \\ + 2(\mu+k)\Omega c^T b\eta^2 + \\ + k\left(c^T q\right)^2\Omega^2\eta^2 + k^{-1}[\varphi(y)]^2 + \\ + (\mu+k)^2(c^T b)^2\Omega^2 y^8\eta^2 + y^2 + \\ + k(c^T b)^2\Omega^2\eta^2 + k^{-1}\overline{w}^2 \qquad (16)$$

for derivative of Lyapunov function (10) we obtain

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \le x^T\left(-Q_1 + \delta Pqq^T P + \delta I\right)x - (k-3)y^2 - \\ -2(\mu+k)y^6 + \delta^{-1}[\varphi(y)]^2 + \frac{2}{k}\overline{w}^2 + \\ +(-2\sigma + (\mu+k)^2 + \delta^{-1}c^T AA^T c\Omega^2 + \\ + k^2\left(c^T b\right)^2\Omega^2 + 2(\mu+k)\Omega c^T b + \\ + k\left(c^T q\right)^2\Omega^2 + (\mu+k)^2(c^T b)^2\Omega^2 y^8 + \\ + k(c^T b)^2\Omega^2)\eta^2 + k^{-1}[\varphi(y)]^2 . \qquad (17)$$

Let us choose $\delta > 0$ such a way that the following inequality holds

$$\left(-Q_1 + \delta Pqq^T P + \delta I\right) \le -Q_2 , \qquad (18)$$

where $Q_2 = Q_2^T$ is a positively determined matrix.

Choose function σ , so the following inequality holds

$$-2\sigma + (\mu+k)^2 + \delta^{-1}c^T AA^T c\Omega^2 + \\ + k^2\left(c^T b\right)^2\Omega^2 + 2(\mu+k)\Omega c^T b + \\ + k\left(c^T q\right)^2\Omega^2 + (\mu+k)^2(c^T b)^2\Omega^2 y^8 + k\left(c^T b\right)^2\Omega^2 + \\ + k\left(c^T b\right)^2\Omega^2 \le -\lambda , \qquad (19)$$

where $\lambda > 0$.
Evidently that inequality (19) holds if the parameter σ would be greater or equal than some number σ_0 :

$$\sigma \ge \sigma_0 = \frac{1}{2}(\lambda + (\mu+k)^2 + \delta^{-1}c^T AA^T c\Omega^2 + \\ + k^2\left(c^T b\right)^2\Omega^2 + 2(\mu+k)\Omega c^T b + k\left(c^T q\right)^2\Omega^2 + \\ + (\mu+k)^2(c^T b)^2\Omega^2 y^8 + k\left(c^T b\right)^2\Omega^2)$$

Then, according to restrictions on the nonlinearity for derivative of Lyapunov function we obtain

$$\dot{V} \le -x^T Q_2 x - \lambda\eta^2 - (k-3)y^2 - \\ -2(\mu+k)y^6 + \left(\frac{1}{\delta} + \frac{1}{k}\right)\cdot y^6 + \frac{2}{k}\overline{w}^2 . \qquad (20)$$

Choosing number $k > 3$ as

$$k > \frac{1}{\delta} + \frac{1}{k} , \qquad (21)$$

we obtain

$$\dot{V} \le -x^T Q_2 x - \lambda\eta^2 + \frac{2}{k}\overline{w}^2 . \qquad (22)$$

From inequality (22) as disturbance $|w(t)| \le w_0 < \infty$ is bounded, it follows that there exists such $k > 3$, that trajectories of system (7), (9) can be localized in any given area ε_0 , which was to be proved.

Consider results of computer simulation of Van der Pole system. First let us assume that $w(t) = 0$ and there appear stable oscillations in system (1) (system possesses a stable limit cycle). Then, with harmonic disturbance there arise chaotic phenomena in the system. And finally, on the last step of simulation of system (1) with control law (3), (4), we discover accomplishment of purpose of control.

Figures 1 and 2 show results of computer simulation of undisturbed system (1) for $\tau_1 = 0{,}5$, $\tau_2 = 2$, $w(t) = 0$ and disturbed system (1) for $\tau_1 = 0{,}5$, $\tau_2 = 2$, $w(t) = \sin(0{,}5t)$ respectively. Results of computer simulations (Figures 1 and 2) show, that system (1) possesses stable limit cycle in absence of disturbance ($w(t) = 0$) and that in presence of harmonic disturbance ($w(t) = \sin(0{,}5t)$) there appear chaotic processes in system (1).

To stabilize system (1) let us choose control algorithm (3). Choose polynomial $\chi(p) = p + 1$, then

$$u = -(p+1)(\mu + k)\xi = -(\mu + k)(\xi + \dot{\xi}), \qquad (23)$$

where function ξ is formed by estimation algorithm (4).

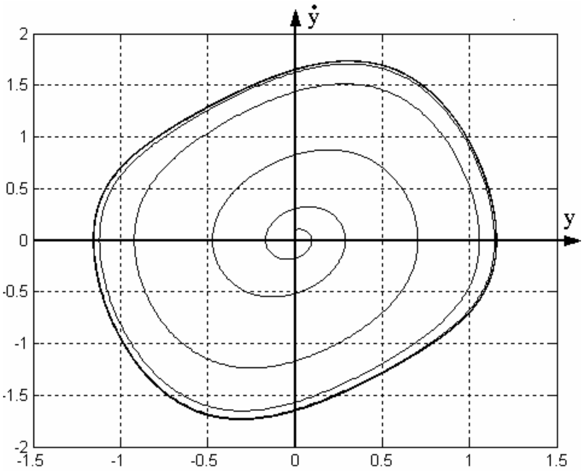

Fig. 1. Phase portrait of system (1) for $\dot{y}(0) = 0.1$, $w(t) = 0$.

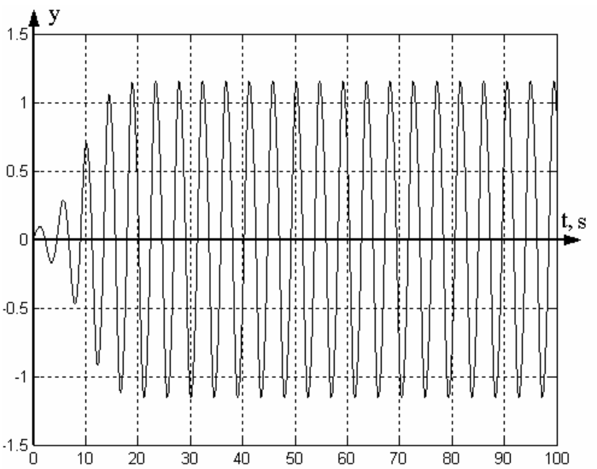

Fig. 2. Transients of system (1) for $\dot{y}(0) = 0.1$, $w(t) = 0$.

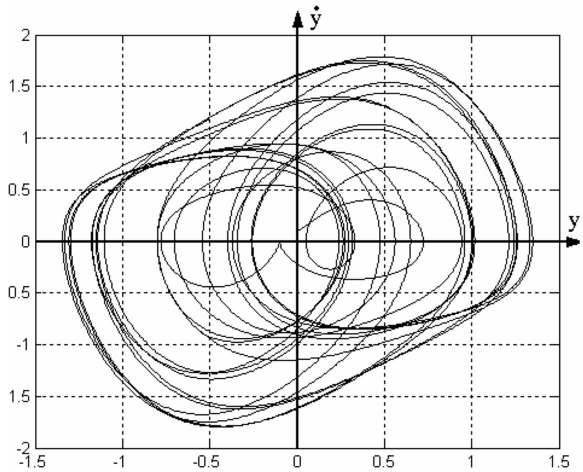

Fig. 3. Phase portrait of system (1) for $\dot{y}(0) = 0.1$, $w(t) = \sin(0.5t)$.

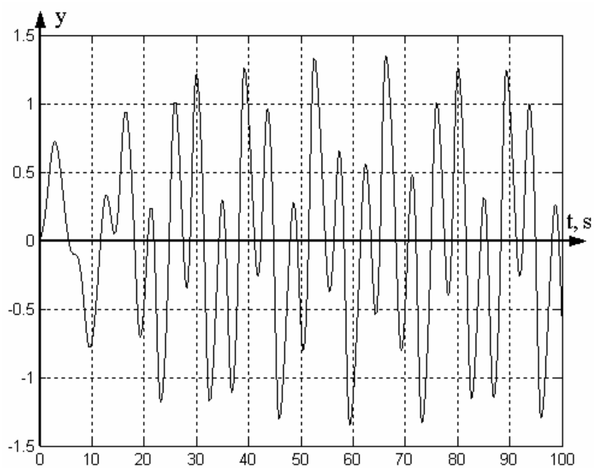

Fig. 4. Transients of system (1) for $\dot{y}(0) = 0.1$, $w(t) = \sin(0.5t)$.

Function σ is chosen so that inequality (19) holds, i.e.

$$\sigma = (\mu + k)^2 + \left(1 + 2k + k^2 + (\mu + k)^2 y^8\right)\Omega^2 + 2(\mu + k)\Omega. \qquad (24)$$

Let us choose parameter $\mu = 2$, and simulate control system for different values of parameter k. Transients of closed loop system for nonzero initial conditions ($\dot{y}(0) = 0{,}1$) and values $k = 10$ and $k = 25$ are shown on Figure 5 and 6.

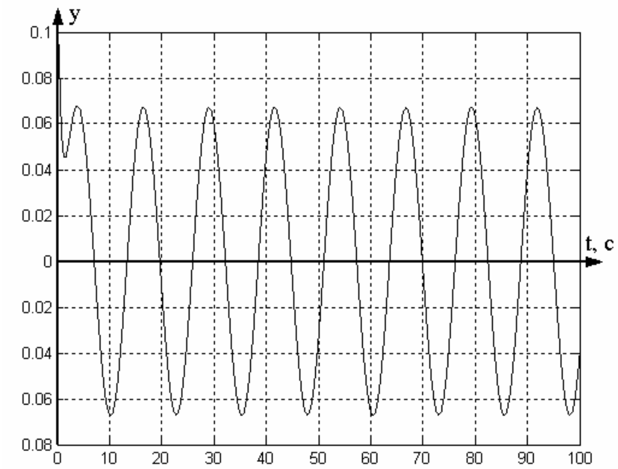

Fig. 5. Transients of system (1), (23), (24) for $k = 10$, $w(t) = \sin(0.5t)$.

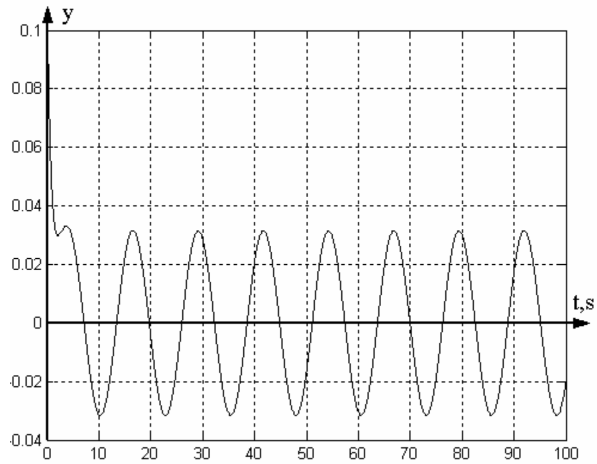

Fig. 6. Transients of system (1), (23), (24) for $k = 25$, $w(t) = \sin(0.5t)$.

CONCLUSION

Problem of stabilization of a chaotic Van der Pole system is solved. Control law uses only measurements of output variable, not its derivatives or state vector of the system. In comparison with known results proposed scheme has smaller dimension equal to one.

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