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Boundedness of Multi-dimensional Systems over a Prescribed Frequency Domain

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Abstract: On the basis of a parameterization of the span of a multivariate matrix polynomial, a sufficient condition is derived in this paper for a multi-dimensional multi-input multi-output (MIMO) IIR system being upper bounded over a cuboid frequency domain. This condition is expressed through a linear matrix inequality (LMI) and can be computationally verified. Moreover, by means of parameter dependent LMIs, two necessary and sufficient conditions are also obtained for this boundedness verification problem, which are again expressed by LMIs. Furthermore, LMI based conditions are derived for system output matrix and direct transmission matrix. Numerical examples are included to illustrate the efficiency and characteristics of the derived theoretical results.

Keywords: IIR system, LMI, multi-dimensional system, MIMO system, spectral masks.

1. INTRODUCTION

In designing a linear time invariant (LTI) filter or control system, a frequently encountered requirement is that the magnitudes of its frequency response should fall into a specified range over some interesting frequency domains (Gonzalez and Woods [2002] and Parks and Burns [1987]). Generally, this requirement can be expressed as the boundedness of the mismatch between the frequency responses of the designed and the ideal systems. When this requirement is over the entire frequency range and only temporal operations are permitted, a necessary and sufficient condition has been established for the boundedness of infinite impulse response (IIR) systems, which is widely known as the Kalman-Yakubovich-Popov (KYP) lemma. Recently, some efforts have been seen in which magnitude constraints are put on a system only over parts of frequencies. Especially, in Davidson et al. [2002], a parameterization has been derived for all the bounded trigonometric polynomials over an interval $[\alpha,\ \beta] \subseteq [0,\ \pi]$. These results have been extended in Dumitrescu [2006] to multi-dimensional finite impulse response (FIR) systems and more complicated frequency domains. The success there, however, is mostly limited to single-input single-output (SISO) systems. On the other hand, for one dimensional multi-input multi-output (MIMO) IIR systems, the same problem is discussed in Iwasaki et al. [1994] using the so-called S-procedure. The efforts there are focused on causal and descriptor systems.

To describe a multi-dimensional dynamic system, many models have been suggested. Among them, it appears that the FIR model, regular/singular Roesser model, regular/singular Fornasini-Marchesini model, etc., are most widely adopted (D'andrea and Dullerud [2003], Gorinevsky and Stein [2003], Zhou [2006]). The attractive properties of the model of D'andrea and Dullerud [2003] include its physical relevance and simplicity. It can also be easily understood that all the other models can be regarded as its special cases. Moreover, in this model, the noncausalities of spatial operations have been explicitly taken into account, as well as the differences between temporal and spatial operations.

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In this paper, we investigate the boundedness of the system described in D'andrea and Dullerud [2003] over a prescribed cuboid frequency domain. On the basis of a parameterization for the span of a structured and mixed temporal-spatial operator, a sufficient condition is derived which is expressed by a linear matrix inequality (LMI) and can be computationally verified. Moreover, using the idea of parameter dependent LMIs, two necessary and sufficient conditions are also derived which are expressed again by LMIs. While the latter two conditions are theoretically interesting, there is still no method to determine the degree of the related multivariate matrix polynomials, as well as its upper bound. If this degree is fixed a priori, their necessities are generally violated. Based on these results, some conditions have been derived for the boundedness of the aforementioned system, which are expressed as LMIs of the system output matrix and the system direct transmission matrix. Numerical simulations show that the derived theoretical results are applicable to filter design.

The following symbols and notation are utilized in this paper. $\mathcal{P}_m$ and $\mathcal{H}_m$ are adopted to represent the sets of $m \times m$ dimensional positive semi-definite matrices and Hermitian matrices. The subscript m is often omitted when matrix dimensions are not very important or clear from context. When $\omega_l \leq \omega_h$ (or $v_a \geq 0$), $\Theta(\omega_l,\ \omega_h)$ (or $\Theta(v_a)$) is used to represent the set consisting of all real scalar pairs $(\alpha,\ \beta)$ satisfying $\alpha v^2 + \beta \geq 0$ if and only if $|v| \leq \left|tg\frac{\omega_h-\omega_l}{4}\right|$ (or $|v| \leq v_a$). For given integers $n(0)$ and $n(\pm, i)|_{i=1}^{L}$, J_i is defined as $I_{n(0)}$ when $i = 0$ and $diag\{\pm I_{n(\pm,i)}\}$ when $i = 1, 2, \cdots, L$. When

k and l are given, $v_i(k)$ and $v(k,l)$ denote respectively $\left[1;\ jv_i;\ \cdots;\ (jv_i)^{k-1}\right]$ and $v_0(k)\otimes v_1(k)\otimes\cdots\otimes v_l(k)$. For a positive integer l, $e_i(l)$, $i=1,2,\cdots,l$, stands for the i-th standard basis vector of $\mathcal{R}^l$, while $T_s(l)$ the $2l\times 2l$ dimensional matrix $[(e_i(2l)\ \ e_{l+i}(2l))|_{i=1}^{l}]$.

Due to space considerations, all the results are reported without proof. Also, A^HWA is sometimes written as $A^HW[\cdot]$, especially when A has a long expression.

2. SYSTEM DESCRIPTION AND BOUNDEDNESS

Let t and $s=[s_1,\ s_2,\ \cdots,\ s_L]$ denote respectively the temporal and spatial variables for a discrete L-dimensional spatially distributed dynamic (SDD) system. Assume that the properties of its subsystems depend on neither time nor spatial positions. Then, by means of subsystem state vector $x(t,s)$, as well as internal subsystem input vector $v(t,s)$ and internal subsystem output vector $w(t,s)$, the relations between external subsystem output vector $y(t,s)$ and external subsystem input vector $d(t,s)$ of this system, can be described by the following state-space representation (D'andrea and Dullerud [2003]),

$$\begin{bmatrix} x(t+1,s)\\ w(t,s)\\ y(t,s)\end{bmatrix}=\begin{bmatrix} A_{TT} & A_{TS} & B_T\\ A_{ST} & A_{SS} & B_S\\ C_T & C_S & D\end{bmatrix}\begin{bmatrix} x(t,s)\\ v(t,s)\\ d(t,s)\end{bmatrix} \tag{1}$$

$$v(t,s)=Sw(t,s) \tag{2}$$

Here, S is a spatial operator which represents internal subsystem connections and usually consists of spatial shifting operations. Specifically, let z_i, $i=1,2,\cdots,L$, denote the forward shift operation in the i-th spatial direction. Moreover, assume that at a particular spatial position s, the numbers of "leaving from" and "entering into" components of $w(t,s)$ in this direction are respectively $n(+,i)$ and $n(-,i)$. Then, the operator S can be explicitly expressed as $S=diag\left\{diag\{z_i^{\pm 1}I_{n(\pm,i)}\}_{i=1}^{L}\right\}$.

For notational simplicity, abbreviate matrices $\begin{bmatrix} A_{TT} & A_{TS}\\ A_{ST} & A_{SS}\end{bmatrix}$, $\begin{bmatrix} B_T\\ B_s\end{bmatrix}$ and $[C_T\ \ C_s]$ respectively by A, B and C. With a little abuse of concepts, in this paper, matrices A, B, C and D are respectively called state transition matrix, system input matrix, system output matrix and system direct transmission matrix. Moreover, define operator Z as $Z=diag\{z_0I_{n(0)},\ \ S^{-1}\}$ with z_0 representing time domain forward shifting operations. Here, $n(0)$ stands for the dimension of the subsystem state vector $x(t,s)$. Using these symbols, the transfer function matrix (TFM) from the external input $d(t,s)$ to the external output $y(t,s)$ of the aforementioned multi-dimensional dynamic system, denote it by $G(Z)$, can be simply represented as $G(Z)=C(Z-A)^{-1}B+D$. Moreover, its frequency domain characteristics can be expressed as $G(\omega_i|_{i=0}^{L})=G(Z)|_{z_i=e^{j\omega_i}}$. To avoid introducing too many symbols, the capital letter G is used in this paper to denote both the system TFM and its frequency response. However, its actual meaning is clear from context.

In system design, there are quite a few of frequently encountered specifications that can be stated as or converted into the determination of whether $\bar{\sigma}(G(\omega_i|_{i=0}^{L}))<\gamma$ is satisfied for every $\omega_i|_{i=0}^{L}\in\mathcal{W}$ with a prescribed frequency domain $\mathcal{W}$ and a positive number γ. While the above problem formulation is quite general, our attention here is restricted to the case in which the set $\mathcal{W}$ takes a cuboid form. That is, $\mathcal{W}=\{\ \omega_i|_{i=0}^{L}\ |\ \omega_i\in[\omega_{il},\ \omega_{ih}],\ i=0,1,\cdots,L\ \}$, in which both ω_{il} and ω_{ih}, $i=0,1,\cdots,L$, are prescribed real numbers of $[-\pi,\ \pi]$. Although these constraints greatly violate the generality of the discussed problem, they are still widely adopted in system and filter design (D'andrea and Dullerud [2003], Gorinevsky and Stein [2003], Gonzalez and Woods [2002], Parks and Burns [1987], Davidson et al. [2002], Dumitrescu [2006]).

To solve this problem, we at first discuss the structure of the span of $[Z;\ \ I_n]$ with $z_i=e^{j\tilde{\omega}_i}$, $\tilde{\omega}_i=\omega_i-\omega_{im}$, $\omega_{im}=\frac{\omega_{il}+\omega_{ih}}{2}$, $i=0,1,\cdots,L$.

Lemma 1. Denote $n(+,i)+n(-,i)$ by $n(i)$, $i=1,2,\cdots,L$; and $\sum_{i=0}^{L}n(i)$ by n. Assume that $0\le\frac{\omega_{ih}-\omega_{il}}{2}\le\pi$ and $(\alpha_i,\ \beta_i)\in\Theta(\omega_{il},\ \omega_{ih})$, $i=0,1,\cdots,L$. Define vector sets Ξ and Π respectively as

$$\Pi=\left\{\xi\ \middle|\ \xi=\begin{bmatrix} Z\\ I_n\end{bmatrix}\eta,\ \begin{matrix} z_i=e^{j\tilde{\omega}_i},\ \eta\in\mathcal{C}^{n\times 1}\\ |\tilde{\omega}_i|\le(\omega_{ih}-\omega_{il})/2,\ i=0,\cdots,L\end{matrix}\right\}$$

$$\Xi=\left\{\xi\ \middle|\ \xi^H\tilde{\Phi}(X)\xi\ge 0,\ \forall P_i\in\mathcal{P}_{n(i)},\ \forall Q_i\in\mathcal{H}_{n(i)}\right\}$$

Then, $\Pi=\Xi$. Here, $X=(\alpha_i,\beta_i,P_i,Q_i)|_{i=0}^{L}$,

$$\tilde{T}=\left[diag\left\{\begin{bmatrix} I_{n(i)}\\ -I_{n(i)}\end{bmatrix}\bigg|_{i=0}^{L}\right\}\ \ -diag\left\{\begin{bmatrix} I_{n(i)}\\ I_{n(i)}\end{bmatrix}\bigg|_{i=0}^{L}\right\}\right]$$

$$\tilde{\Phi}(X)=\tilde{T}^H diag\left(\begin{bmatrix}\alpha_iJ_iP_iJ_i & J_iQ_i\\ Q_iJ_i & \beta_iP_i\end{bmatrix}\bigg|_{i=0}^{L}\right)\tilde{T}$$

For brevity, define matrices Δ_0 and $\hat{\Phi}(X)$ respectively as

$$\Delta_0=diag\left\{e^{-j\omega_{0m}}I_{n(0)},\ diag\left\{e^{\pm j\omega_{im}}I_{n(\pm,i)}\right\}\big|_{i=1}^{L}\right\}$$

$$\hat{\Phi}(X)=\begin{bmatrix}\Delta_0 & 0\\ 0 & I_n\end{bmatrix}^H\tilde{\Phi}(X)\begin{bmatrix}\Delta_0 & 0\\ 0 & I_n\end{bmatrix}$$

From Lemma 1, using the ideas of the so called S-procedure, a sufficient condition can be obtained for the boundedness of $G(\omega_i|_{i=0}^{L})$ over the frequency domain $\mathcal{W}$.

Theorem 1. Assume that $|\omega_{ih}-\omega_{il}|\le 2\pi$, $i=0,1,\cdots,L$. Moreover, assume that there exist $P_i\in\mathcal{P}_{n(i)}$ and $Q_i\in\mathcal{H}_{n(i)}$, such that

$$\begin{bmatrix} A & B\\ C & D\\ I_{n+q} & \end{bmatrix}^H T^H\Phi(X)[\cdot]<0,\quad T=\begin{bmatrix} I_n & 0 & 0 & 0\\ 0 & 0 & I_n & 0\\ 0 & I_p & 0 & 0\\ 0 & 0 & 0 & I_q\end{bmatrix} \tag{3}$$

is valid for some $(\alpha_i,\ \beta_i)\in\Theta(\omega_{il},\ \omega_{ih})$, $i=0,\cdots,L$. Then, whenever $\omega_i\in[\omega_{il},\ \omega_{ih}]$, $i=0,\cdots,L$, we have that $\bar{\sigma}(G(\omega_i|_{i=0}^{L})<\gamma$. Here, $\Phi(X)=diag\{\hat{\Phi}(X),I_p,-\gamma^2I_q\}$.

It is worthwhile to note that while the left hand side of Equation (3) is linear with respect to $Q_i|_{i=0}^{L}$, it is bilinear with respect to $P_i|_{i=0}^{L}$ and $(\alpha_i,\beta_i)|_{i=0}^{L}$. However, the feasibility of Inequality (3) does not depend on a particular selection of $(\alpha_i,\beta_i)|_{i=0}^{L}$. This property is clear from the definition of sets $\Theta(\omega_{il},\ \omega_{ih})$, $i=0,1,\cdots,L$.

When $n(0) > 0$ and $L = 0$ or $n(0) = 0$ and $L = 1$, from the properties of the S-procedure, it can be declared that the condition of Theorem 1 is also necessary.

To apply Theorem 1 to filter design, the following equality is useful.

$$\begin{bmatrix} C & D \\ 0 & I_q \end{bmatrix}^H \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} [\cdot] = [C \;\; D]^H [\cdot] - \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \quad (4)$$

On the basis of this relation and the well known Schur complement theorem, using direct algebraic operations, another LMI based sufficient condition can be derived from Theorem 1 for TFM boundedness.

Corollary 1. Inequality (3) is equivalent to

$$\begin{bmatrix} \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix}^H \hat{\Phi}(X)[\cdot] - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I_q \end{bmatrix} & \begin{bmatrix} C^H \\ D^H \end{bmatrix} \\ [C \;\; D] & -I_p \end{bmatrix} < 0 \quad (5)$$

From Corollary 1, it is clear that the sufficient condition of Theorem 1 can be written as an LMI of matrices C and D. This means that when matrices A and B are prescribed, Theorem 1 can also be used in verifying the existence of matrices C and D that satisfy the boundedness requirements.

Note that in filter design, both stop-bands and pass-bands are usually clear from the required specifications. This information is helpful in determining the appropriate system matrices A and B. For example, when a causal system is under investigation, this information can be used to generate an orthonormal basis for the desired filter. Compared with FIR filters, the resulting filter generally has a lower complexity (Gonzalez and Woods [2002], Heuberger et al. [1995], Parks and Burns [1987]). However, for the system discussed in D'andrea and Dullerud [2003], it is still not known how to incorporate this information into filter design. But it appears that the ideas of causal systems are helpful, which is illustrated by numerical simulations of Section IV.

3. BOUNDEDNESS CONDITION BASED ON PARAMETER DEPENDENT LMIS

In the previous section, a sufficient condition has been derived for the boundedness of a multi-dimensional IIR system over a prescribed cuboid frequency domain. To make the condition also necessary, it is required that a so called lossless condition is satisfied. However, it is still not very clear how to express this condition in a mathematically verifiable formula when $n(0) > 0$ and $L > 1$ or $n(0) = 0$ and $L > 2$ (Derinkuyu and Pınar [2006]).

To reduce the conservatism of Theorem 1, we re-investigate the above boundedness problem using ideas of parameter dependent LMIs, which extends the results on stability analysis for additively perturbed matrices of Apkarian et al. [2000], Bliman [2004] to the boundedness of a matrix with linear fractional perturbations. Fundamentally, this approach utilizes the continuation properties of the feasibility of an LMI. More precisely, the following results can be proved using similar arguments as those of Bliman [2004] in which the variables are restricted to be real.

Lemma 2. Assume that every element of finite dimensional matrices $H_i(\delta)$, $i = 0, 1, \cdots, l$, is a continuous function of $\delta = [\delta_0 \; \delta_1 \; \cdots \; \delta_m]^H$ over a compact set $\mathcal{K} \subseteq \mathcal{C}^m$. Moreover, assume that for every $\delta \in \mathcal{K}$, there exists a $x = [x_1; \; x_2; \; \cdots; x_l] \in \mathcal{C}^l$ such that $H(x, \delta) = H_0(\delta) + \sum_{i=1}^{p} x_i H_i(\delta)$ is negative definite. Then, there exists a vector valued polynomial function $x^*(\delta)$, such that for every $\delta \in \mathcal{K}$, $H(x^*(\delta), \delta)$ is negative definite.

The following result is a special case of Theorem 1.

Lemma 3. Assume that $j\nu E - F$ is invertible for every $|\nu| \leq \nu_a$. Moreover, assume that $(\alpha, \; \beta) \in \Theta(\nu_a)$. Then, $\bar{\sigma}(V(j\nu E - F)^{-1}N + O) < \gamma$ whenever $|\nu| \leq \nu_a$, if and only if there are $P \in \mathcal{P}$ and $Q \in \mathcal{H}$, such that the following matrix inequality is feasible.

$$\begin{bmatrix} F & N \\ E & 0 \end{bmatrix}^H \begin{bmatrix} \alpha P & Q \\ Q & \beta P \end{bmatrix} [\cdot] + \begin{bmatrix} V & O \\ 0 & I \end{bmatrix}^H \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} [\cdot] < 0$$

Having these preparations, it is ready to investigate the boundedness problem using parameter dependent LMIs. For notational simplicity, denote $diag\{ \; jv_i J_i|_{i=0}^{L} \; \}$ by U. Then, it is not difficulty to show that

$$C(Z - A)^{-1}B + D|_{z_i = e^{j\omega_i}} = \hat{C}(U - \hat{A})^{-1}\hat{B} + \hat{D} \quad (6)$$

in which $\hat{A} = (I_n - \Delta_0 A)(I_n + \Delta_0 A)^{-1}$, $\hat{B} = (I_n + \Delta_0 A)^{-1}\Delta_0 B$, $\hat{C} = -2C(I_n + \Delta_0 A)^{-1}$ and $\hat{D} = D - C(I_n + \Delta_0 A)^{-1}\Delta_0 B$.

Assume that for every $i = 0, 1, \cdots, L-1$, both ω_{il} and ω_{ih} belong to $(-\pi, \; \pi)$. In this case, we have that $\left|\frac{\omega_{ih} - \omega_{il}}{4}\right| < \frac{\pi}{2}$. This means that $tg\frac{\tilde{\omega}_i}{2}$ is of a finite value and therefore $v_i|_{i=0}^{L-1}$ belongs to a compact set. Define matrices E_i, $\hat{E}_i$ and $\tilde{A}(v_i|_{i=0}^{L-1})$ respectively as $E_i = diag\{ \; 0_{n(\kappa)\times n(\kappa)}|_{\kappa=0}^{i-1} \;\; J_i \;\; 0_{n(\kappa)\times n(\kappa)}|_{\kappa=i+1}^{L} \; \}$, $\hat{E}_i = [\; 0_{n(i)\times n(\kappa)}|_{\kappa=0}^{i-1} \;\; J_i^H \;\; 0_{n(i)\times n(\kappa)}|_{\kappa=i+1}^{L-1} \;\; 0_{n(i)\times(n+n(L))} \;]^H$, $i = 0, 1, \cdots, L$, and $\tilde{A}(v_i|_{i=0}^{L-1}) = \hat{A} - \sum_{i=0}^{L-1} jv_i E_i$. Then, it is obvious that $U - \hat{A} = jv_L E_L - \tilde{A}(v_i|_{i=0}^{L-1})$.

Consider the boundedness of $G(Z)$ when $\omega_L \in [\omega_{Ll}, \; \omega_{Lh}]$. As every element of $\tilde{A}(v_i|_{i=0}^{L-1})$ is a continuous function of $jv_i|_{i=0}^{L-1}$, it can be declared from Lemmas 2 and 3 that this boundedness property is equivalent to the existence of a semi-definite positive multivariate matrix polynomial $P_L(v_i|_{i=0}^{L-1})$ and a Hermitian multivariate matrix polynomial $Q_L(v_i|_{i=0}^{L-1})$, such that for some $(\alpha_L, \; \beta_L) \in \Theta(\omega_{Ll}, \; \omega_{Lh})$ and every $|v_i| \leq tg\frac{\omega_{ih} - \omega_{il}}{4}$, $i = 0, 1, \cdots, L-1$, the following matrix inequality is feasible,

$$\begin{bmatrix} \tilde{A}(v_i|_{i=0}^{L-1}) & \hat{B} \\ E_L & 0 \end{bmatrix}^H \begin{bmatrix} \alpha_L P_L(v_i|_{i=0}^{L-1}) & Q_L(v_i|_{i=0}^{L-1}) \\ Q_L(v_i|_{i=0}^{L-1}) & \beta_L P_L(v_i|_{i=0}^{L-1}) \end{bmatrix} [\cdot] + \begin{bmatrix} \hat{C} & \hat{D} \\ 0 & I_q \end{bmatrix}^H \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} [\cdot] < 0 \quad (7)$$

On the other hand, from the parametrization of a multivariate matrix polynomial (Apkarian et al. [2000], Dumitrescu [2006], Bliman [2004]), it can be declared that the existence of the desirable $P_L(v_i|_{i=0}^{L-1})$ and $Q_L(v_i|_{i=0}^{L-1})$ is equivalent to the existence of a positive integer k, a semi-definite positive matrix P_L, and a Hermitian matrix Q_L, such that the corresponding matrix inequality is feasible with $P_L(v_i|_{i=0}^{L-1}) = [v(k,L-1)\otimes I_n]^H P_L[v(k,L-1)\otimes I_n]$ and $Q_L(v_i|_{i=0}^{L-1}) = [v(k,L-1)\otimes I_n]^H Q_L[v(k,L-1)\otimes I_n]$.

When $P_L(v_i|_{i=0}^{L-1})$ and $Q_L(v_i|_{i=0}^{L-1})$ take these parameterizations, it can be proved that

$$\begin{bmatrix} \alpha_L P_L(v_i|_{i=0}^{L-1}) & Q_L(v_i|_{i=0}^{L-1}) \\ Q_L(v_i|_{i=0}^{L-1}) & \beta_L P_L(v_i|_{i=0}^{L-1}) \end{bmatrix} = [I_2\otimes(v(k,L-1)\otimes I_n)]^H \begin{bmatrix} \alpha_L P_L & Q_L \\ Q_L & \beta_L P_L \end{bmatrix} [\cdot] \quad (8)$$

Moreover, from the definition of matrix $T_s(l)$, direct algebraic operations show that

$$I_2\otimes(v(k,L-1)\otimes I_n) = (T_s(k^L)\otimes I_n)(v(k,L-1)\otimes I_{2n}) \quad (9)$$

Define matrices S_f and S_l respectively as $S_f = [I_k \ \ 0_{k\times 1}]$ and $S_l = [0_{k\times 1} \ \ I_k]$. Moreover, define matrix S_i, $i = 0,1,\cdots,L$, as $S_i = \underbrace{S_f\otimes\cdots\otimes S_f}_{i\ times}\otimes S_l\otimes\underbrace{S_f\otimes\cdots\otimes S_f}_{L-i-1\ times}$. Furthermore, denote $\underbrace{S_f\otimes\cdots\otimes S_f}_{L\ times}$ by S_{-1}. Then, it is apparent that $v(k,L-1) = S_{-1}v(k+1,L-1)$ and $(jv_i)v(k,L-1) = S_i v(k+1,L-1)$, $i=0,1,\cdots,L$. From these relations, it can be directly proved that

$$(v(k,L-1)\otimes I_{2n})\begin{bmatrix} \tilde{A}(v_i|_{i=0}^{L-1}) & \hat{B} \\ E_L & 0 \end{bmatrix} = \left(S_{-1}\otimes\begin{bmatrix} \hat{A} & \hat{B} \\ E_L & 0\end{bmatrix} - [\ S_i\otimes \hat{E}_i|_{i=0}^{L-1} \ \ 0\ ]\right) M[v(k+1,L-1)\otimes I_{n+q}] \quad (10)$$

in which

$$M = \left[\left[diag\left\{\left[\begin{array}{c} 0_{m\hat{n}(i)\times\hat{n}(i)} \\ I_{\hat{n}(i)} \\ 0_{((k+1)^L-1-m)\hat{n}(i)\times\hat{n}(i)} \end{array}\right]\Bigg|_{i=0}^{L}\right\}\right]\Bigg|_{m=0}^{(k+1)^L-1}\right]$$

and $\hat{n}(i) = n(i)$ for $i=0,\cdots,L-1$, while $\hat{n}(L) = n(L)+q$.

Note that $I_{n+q} = [I_{n+q} \quad 0_{(n+q)\times[(k+1)^L-1](n+q)}](v(k+1,L-1)\otimes I_{n+q})$. From Equations (8)-(10), we have that Inequality (7) is equivalent to

$$[v(k+1,L-1)\otimes I_{n+q})^H\left\{R_1^H\begin{bmatrix} \alpha_L P_L & Q_L \\ Q_L & \beta_L P_L\end{bmatrix}R_1 + R_2^H\begin{bmatrix} C & D \\ 0 & I_q\end{bmatrix}^H\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q\end{bmatrix}\begin{bmatrix} C & D \\ 0 & I_q\end{bmatrix}R_2\right\}[\cdot] < 0 \quad (11)$$

in which $R_1 = (T_s(k^L)\otimes I_n)R_{10}$, $R_2 = R_{20}[I_{n+q}\ 0]$,

$$R_{10} = \left(S_{-1}\otimes\begin{bmatrix} \hat{A} & \hat{B} \\ E_L & 0\end{bmatrix} - [\ S_i\otimes\hat{E}_i|_{i=0}^{L-1}\ \ 0\ ]\right)M$$

$$R_{20} = \begin{bmatrix} -2(I_n+\Delta_0 A)^{-1} & -(I_n+\Delta_0 A)^{-1}\Delta_0 B \\ 0 & I_q\end{bmatrix}$$

When the requirement $\bar{\sigma}(G(\omega_i|_{i=0}^{L})) < \gamma$, $\forall\omega_i|_{i=0}^{L}\in\mathcal{W}$, is expressed by Inequality (11), the method suggested in Bliman [2004] can be applied to derive a constant matrix inequality based necessary and sufficient condition.

For every $i=0,1,\cdots,L-1$, denote $(k+1)^{L-i-1}(q+n)$ by $m(k,i)$, and define matrix $H(k,i)$ as

$$H(k,i) = \begin{bmatrix} 0_{km(k,i)\times m(k,i)} & I_{km(k,i)} \\ I_{km(k,i)} & 0_{km(k,i)\times m(k,i)}\end{bmatrix}$$

Then, on the basis of Equation (11), a necessary and sufficient condition can be derived from Lemmas 2 and 4 for the boundedness of TFM $G(Z)$ over the cuboid frequency domain $\mathcal{W}$.

Theorem 2. Assume that $(\alpha_i,\ \beta_i)\in\Theta(\omega_{il},\ \omega_{ih})$, $i=0,1,\cdots,L$; and $|\omega_{ih}-\omega_{il}| < 2\pi$, $i=0,1,\cdots,L-1$. Then, $\bar{\sigma}(G(\omega_i|_{i=0}^{L})) < \gamma$ whenever $\omega_i|_{i=0}^{L}\in\mathcal{W}$, if and only if there exist a positive integer k, positive semi-definite matrices $P_i|_{i=0}^{L}$ and Hermitian matrices $Q_i|_{i=0}^{L}$, such that the following LMI is feasible,

$$R_1^H\begin{bmatrix} \alpha_L P_L & Q_L \\ Q_L & \beta_L P_L\end{bmatrix}R_1 + R_2^H\begin{bmatrix} C & D \\ 0 & I_q\end{bmatrix}^H\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q\end{bmatrix}\times \begin{bmatrix} C & D \\ 0 & I_q\end{bmatrix}R_2 + \sum_{i=0}^{L-1}W_i^H\begin{bmatrix} \alpha_i P_i & Q_i \\ Q_i & \beta_i P_i\end{bmatrix}W_i < 0 \quad (12)$$

in which $W_i = (T_s((k+1)^i)\otimes I_{(k+1)^{L-i}(q+n)})(I_{(k+1)^i}\otimes H(k,i))$.

While Theorem 2 provides a necessary and sufficient condition for the boundedness of a multi-dimensional IIR filter $G(Z)$, there is still no method to determine a priori the value of k, as well as its upper bound. When k is fixed, the necessity of this condition is generally lost. It can be, however, proved that if Inequality (12) is feasible for a particular k, say k_0, then, it is also feasible for all $k\geq k_0$. Moreover, if the sufficient condition of Theorem 1 is satisfied, then, Inequality (12) is certainly feasible with $k=1$. These observations imply that the condition of Theorem 2 is generally less conservative than that of Theorem 1 even if k is fixed. Moreover, the conservatism of this condition can be reduced monotonically through increasing k. But it is worthwhile to emphasize that a large k is generally not appreciative in actual filter design, as the dimensions of $P_i|_{i=0}^{L}$ and $Q_i|_{i=0}^{L}$ increase very fast with the increment of k, which leads to a rapid increment of computational burden.

On the basis of the Schur complement theorem and Equation (4), the following results can be obtained through direct algebraic operations.

Corollary 2. Denote matrix

$$R_1^H\begin{bmatrix} \alpha_L P_L & Q_L \\ Q_L & \beta_L P_L\end{bmatrix}R_1 + \sum_{i=0}^{L-1}W_i^H\begin{bmatrix} \alpha_i P_i & Q_i \\ Q_i & \beta_i P_i\end{bmatrix}W_i - \gamma^2([0_{q\times n}\ I_q]R_2)^H([0_{q\times n}\ I_q]R_2)$$

by $W(X)$. Then, Inequality (12) is equivalent to

$$\begin{bmatrix} W(X) & ([C\ \ D]R_2)^H \\ [C\ \ D]R_2 & -I_p\end{bmatrix} < 0 \quad (13)$$

Note that matrix $W(X)$ does not depend on matrices C and D. Therefore, the left hand side of the above inequality is a linear matrix valued function of both C and D. Hence, when matrices A and B are known, the desirable matrices C and D can be obtained through solving an LMI.

When $n(0)>0$ and $L=1$ or $n(0)=0$ and $L=2$, using the above procedure and the properties of linear fractional transformations, another necessary and sufficient condition can be derived for the boundedness of $G(Z)$ over $\mathcal{W}$. Compared with Theorem 2 or Corollary 2, a nice property of this condition is that the dimensions are much smaller of both the involved matrix variables and the related matrix inequalities, which is attractive in actual computations. Only results with $n(0) > 0$ and $L = 1$ is reported here. The results remain valid with $n(0) = 0$ and $L = 2$ if J_0 is replaced by J_2 and the corresponding matrices are modified correspondingly. In principle, the results can be extended to other cases through a repeated utilization of the following procedure. However, for the general case, the expressions are quite complicated and the derivations are quite tedious.

Partition matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ as $\hat{A} = \begin{bmatrix} \hat{A}_{00} & \hat{A}_{01} \\ \hat{A}_{10} & \hat{A}_{11} \end{bmatrix}$, $\hat{B} = \begin{bmatrix} \hat{B}_0 \\ \hat{B}_1 \end{bmatrix}$ and $\hat{C} = \begin{bmatrix} \hat{C}_0 & \hat{C}_1 \end{bmatrix}$. Here, $\hat{A}_{00} \in \mathcal{C}^{n0\times n0}$, $\hat{B}_0 \in \mathcal{C}^{n0\times q}$, $\hat{C}_0 \in \mathcal{C}^{p\times n0}$, and the other matrices have compatible dimensions. Then, from the definition of matrix U, we have

$$\hat{C}(U - \hat{A})^{-1}\hat{B} + \hat{D} = \tilde{C}_1(jv_1J_1 - \tilde{A}_1)^{-1}\tilde{B}_1 + \tilde{D} \quad (14)$$

in which $[\tilde{A}_1 \quad \tilde{B}_1] = [\hat{A}_{11} \quad \hat{B}_1] + \hat{A}_{10}(jv_0J_0 - \hat{A}_{00})^{-1}[\hat{A}_{01} \quad \hat{B}_0]$, and $[\tilde{C}_1 \quad \tilde{D}] = [\hat{C}_1 \quad \hat{D}] + \hat{C}_0(jv_0J_0 - \hat{A}_{00})^{-1}[\hat{A}_{01} \quad \hat{B}_0]$.

When $|\omega_{0h} - \omega_{0l}| < 2\pi$, through similar arguments as those for Equation (7), we have that $G(Z)$ is bounded over $\mathcal{W}$ if and only if for an arbitrary $(\alpha_1,\ \beta_1) \in \Theta(\omega_{1h},\ \omega_{1l})$, there exist a positive integer k, a semi-definite positive matrix P_1 and a Hermitian matrix Q_1, such that $P_1(v_0) = (v_0(k) \otimes I_{n1})^H P_1(v_0(k) \otimes I_{n1})$ and $Q_1(v_0) = (v_0(k) \otimes I_{n1})^H Q_1(v_0(k) \otimes I_{n1})$ satisfy

$$\begin{bmatrix} J_1\tilde{A}_1 & J_1\tilde{B}_1 \\ I_{n1} & 0 \end{bmatrix}^H \begin{bmatrix} \alpha_1P_1(v_0) & Q_1(v_0) \\ Q_1(v_0) & \beta_1P_1(v_0) \end{bmatrix} \begin{bmatrix} J_1\tilde{A}_1 & J_1\tilde{B}_1 \\ I_{n1} & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}_1 & \tilde{D} \\ 0 & I_q \end{bmatrix}^H \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & \tilde{D} \\ 0 & I_q \end{bmatrix} < 0 \quad (15)$$

Define matrix $\tilde{H}_k$ as

$$\tilde{H}_k = \begin{bmatrix} 0 & 0 & \cdots & I_{n0} \\ J_0 & 0 & \cdots & J_0\hat{A}_{00} \\ J_0\hat{A}_{00} & J_0 & \cdots & (J_0\hat{A}_{00})^2 \\ \vdots & \vdots & \ddots & \vdots \\ J_0(\hat{A}_{00}J_0)^{k-2} & J_0(\hat{A}_{00}J_0)^{k-3} & \cdots & (J_0\hat{A}_{00})^{k-1} \end{bmatrix} \times \begin{bmatrix} I_{k-1} \otimes [\hat{A}_{01} \quad \hat{B}_0] & 0 \\ 0_{n0\times k(n1+q)} & I_{n0} \end{bmatrix}$$

Then, based on Equation (8), direct algebraic operations show that Inequality (15) is equivalent to

$$\begin{bmatrix} v_0(k) \otimes I_{n1+q} \\ (jv_0J_0 - \hat{A}_{00})^{-1})[\hat{A}_{01} \quad \hat{B}_0] \end{bmatrix}^H \left\{ \tilde{R}_1^H \begin{bmatrix} \alpha_1P_1 & Q_1 \\ Q_1 & \beta_1P_1 \end{bmatrix} \tilde{R}_1 + ([C \quad D]\tilde{R}_2)^H([C \quad D]\tilde{R}_2) - \gamma^2\tilde{R}_3^H\tilde{R}_3 \right\} [\cdot] < 0 \quad (16)$$

Here, $\tilde{R}_3 = [0_{q\times n1} \quad I_q \quad 0]$, $\tilde{R}_1 = (T_s(k) \otimes I_{n1})\tilde{R}_{10}$ and

$$\tilde{R}_2 = R_{20} \begin{bmatrix} 0_{n0\times k(n1+q)} & I_{n0} \\ I_{n1+q} & 0 \end{bmatrix}$$

$$\tilde{R}_{10} = \left[I_k \otimes \begin{bmatrix} J_1\hat{A}_{11} & J_1\hat{B}_1 \\ I_{n1} & 0 \end{bmatrix} \quad 0 \right] + \left(I_k \otimes \begin{bmatrix} J_1\hat{A}_{10} \\ 0 \end{bmatrix} \right) \tilde{H}_k$$

On the other hand, note that

$$[I \quad -jv_0I]\,\tilde{W}_k \begin{bmatrix} v_0(k) \otimes I_{n1+q} \\ (jv_0J_0 - \hat{A}_{00})^{-1}[\hat{A}_{01} \quad \hat{B}_0] \end{bmatrix} \equiv 0 \quad (17)$$

in which

$$\tilde{W}_k = \begin{bmatrix} 0_{(k-1)(n1+q)\times(n1+q)} & I_{(k-1)(n1+q)} & 0 \\ J_0[\hat{A}_{01} \quad \hat{B}_0] & 0_{n0\times(k-1)(n1+q)} & \hat{A}_{00} \\ I_{(k-1)(n1+q)} & 0_{(k-1)(n1+q)\times(n1+q+n0)} & \\ 0_{n0\times k(n1+q)} & & I_{n0} \end{bmatrix}$$

From these equations, using completely the same arguments as those in the proof of Theorem 1 and the lossless property of the S-procedure, the following results can be proved. The details are omitted due to its obviousness.

Corollary 3. Denote

$$\tilde{R}_1^H \begin{bmatrix} \alpha_1P_1 & Q_1 \\ Q_1 & \beta_1P_1 \end{bmatrix} \tilde{R}_1 + \tilde{W}_k^H \begin{bmatrix} \alpha_0P_0 & Q_0 \\ Q_0 & \beta_0P_0 \end{bmatrix} \tilde{W}_k - \gamma^2\tilde{R}_3^H\tilde{R}_3$$

by $\tilde{W}(X)$. Assume that $|v_{0h} - v_{0l}| < 2\pi$ and $(\alpha_i,\ \beta_i) \in \Theta(\omega_{il},\ \omega_{ih})$, $i = 0, 1$. Then, $\bar{\sigma}(G(\omega_i|_{i=0}^1)) < \gamma$ whenever $\omega_i|_{i=0}^1 \in \mathcal{W}$, if and only if there exist positive integer k, semi-definite positive matrices $P_i|_{i=0}^1$ and Hermitian matrices $Q_i|_{i=0}^1$, such that the following LMI is feasible,

$$\begin{bmatrix} \tilde{W}(X) & ([C \quad D]\tilde{R}_2)^H \\ [C \quad D]\tilde{R}_2 & -I_p \end{bmatrix} < 0 \quad (18)$$

It is worthwhile to note that while Corollary 1 only gives a sufficient condition for the boundedness of a multi-dimensional TFM, it has a lower computational complexity. It is not difficult to understand that the computational complexity of Corollary 2 increases very fast with increasing either the spatial dimension L or the multivariate matrix polynomial degree k. Moreover, Corollary 3 is generally more computationally efficient than Corollary 2. However, when k is fixed, it is still not clear that between these two corollaries, which one is less conservative.

4. NUMERICAL SIMULATIONS

In this section, the derived theoretical results are applied to filter design. Assume that only real coefficient filters are of interest and a two-dimensional filter with 2 inputs and 2 outputs is required to meet the following specifications.

- $\bar{\sigma}(G(\omega_i|_{i=1}^2) - I_2) < 0.1$ when both ω_1 and ω_2 belong to $[-0.4\pi \quad 0.4\pi]$;
- For prescribed matrices A and B, find matrices C and D which minimize the following cost function

$$\max\left\{ \max_{0.9\pi\le|\omega_1|\le\pi} \bar{\sigma}(G(\omega_i|_{i=1}^2)),\ \max_{0.9\pi\le|\omega_2|\le\pi} \bar{\sigma}(G(\omega_i|_{i=1}^2)) \right\}$$

In filter design, the input matrix B is designated to be $B = [\underbrace{I_2 \ \cdots \ I_2}_{n/2\ blocks}]^H$. To investigate the influences of matrix A on filter performances, both FIR and IIR filters are designed. In designing a FIR filter, the state transition

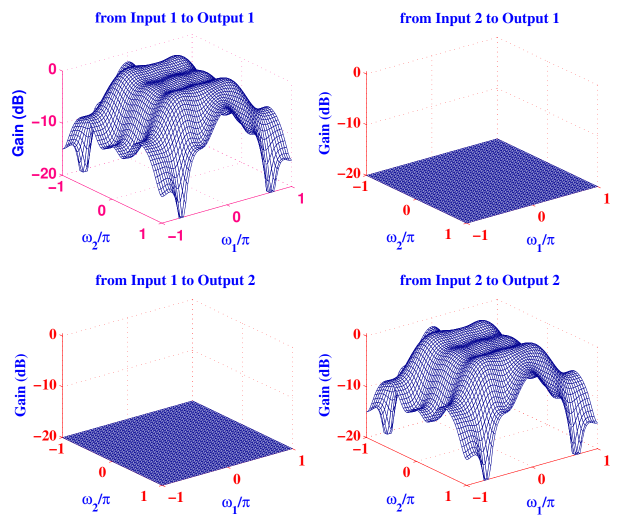

Fig. 1. Frequency Response Magnitude of the Designed FIR Filter.

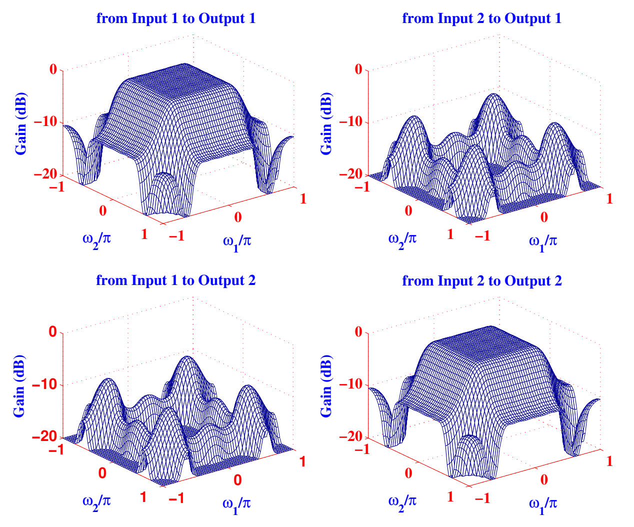

Fig. 2. Frequency Response Magnitude of the Designed IIR Filter.

matrix A is selected as $A = \begin{bmatrix} 0_{2\times(n-2)} & 0_{2\times 2} \\ I_{n-2} & 0_{(n-2)\times 2} \end{bmatrix}$. In designing an IIR filter, A is determined by the pass-band and stop-band of the desirable filter.

In optimizing the cost function, a bisection procedure is utilized.

The magnitudes of the designed FIR and IIR filters are respectively shown in Figures 2 and 3, in case that $n(\pm, i) = 8$, $i = 1, 2$ and Corollary 3 with $k = 1$ is adopted. To make the illustrations clearer, a magnitude is replaced by -20dB if it is smaller than that value. It can be seen from these figures that when both ω_1 and ω_2 belong to $[\ -0.4\pi,\ 0.4\pi\ ]$, frequency response magnitude of the IIR filter is much flatter than that of the FIR filter, which is a widely appreciative property in filter design. On the other hand, with respect to reducing influences between different channels, the FIR filter appears better.

5. CONCLUDING REMARKS

In this paper, on the basis of the structure of the span of a matrix polynomial and parameter dependent LMIs, one sufficient condition and two necessary and sufficient conditions have been derived for the boundedness of a multi-dimensional MIMO IIR system over a cuboid frequency domain. Generally, these conditions can be computationally verified and applicable to multi-dimensional system analysis. Two of these conditions will lose their necessity if the degree of the related multivariate matrix polynomials is fixed, but the conservatism of the corresponding conditions can be reduced sequentially through increasing this degree. Moreover, if this degree is sufficiently large, then, the corresponding sufficient conditions become "almost" necessary.

From these conditions, three conditions have been derived which are expressed by LMIs of system output matrix and system direct transmission matrix. These conditions can be directly used in filter design.

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