Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

State Based Self-triggered Feedback Control Systems with $\mathcal{L}_2$ Stability ⋆

Xiaofeng Wang * Michael D. Lemmon *

* *Electrical Engineering Department,University of Notre Dame, IN 46556 USA (e-mail: xwang13, lemmon@nd.edu).*

Abstract: This paper examines a class of real-time control systems in which each control task triggers its next release based on the value of the last sampled state. Prior work by Lemmon et al. (2007) used simulations to demonstrate that self-triggered control systems can be remarkably robust to task delay. This paper derives bounds on a task's sampling period and deadline to quantify how robust the control system's performance will be to variations in these parameters. In particular we establish inequality constraints on a control task's period and deadline whose satisfaction ensures that the closed loop system's induced $\mathcal{L}_2$ gain lies below a specified performance threshold. The results apply to linear time-invariant systems driven by external disturbances whose magnitude is bounded by a linear function of the system state's norm. The plant is regulated by a full-information $\mathcal{H}_\infty$ controller. These results can serve as the basis for the design of soft real-time systems that guarantee closed-loop control system performance at levels traditionally seen in hard real-time systems.

1. INTRODUCTION

Computer-controlled systems are often implemented using periodic tasks satisfying hard real-time constraints. Under a periodic task model, consecutive invocations (also called "jobs") of a task are released in a periodic manner. Periodic task models allow the control system designer to treat the computer-controlled system as a discrete-time system, for which there are a variety of mature controller synthesis methods.

Traditional methods by Astrom et al. (1990) for sample period selection are usually based on Nyquist sampling. Nyquist sampling ensures that the sampled signal can be perfectly reconstructed from its samples. In practice, however, feedback within the control system means the system's performance will be somewhat insensitive to errors in the feedback signal, so that perfect reconstruction is much more than we require in a feedback control system. An alternative approach to the sample period selection problem makes use of Lyapunov techniques. This was done by Zheng et al. (1990) for a class of nonlinear sampled-data system. Nesic et al. (1999) used input-to-state stability (ISS) techniques to bound the inter-sample behavior of nonlinear systems. Further work was done by Nesic et al. (2004), Carnevale et al. (2007) where upper bounds on the sampling periods were provided, known as the maximal allowable transfer interval (MATI).

However, periodic task models may be undesirable in many situations. Traditional approaches for estimating task periods and deadlines are very conservative, so the control task may have greater utilization than it actually needs. This results in significant over-provisioning of the real-time system hardware. With such high utilization, it may be difficult to schedule other tasks on the same processing system. Finally, it should be noted that real-time scheduling over networked systems may be poorly served by the periodic task model. In many networked systems, tasks are finished only after information has been successfully transported across the network. It is often unreasonable to expect hard real-time guarantees on message delivery in communication networks. This is particularly true for wireless sensor-actuator networks. In these applications, there may be good reasons to consider alternatives to periodic task models.

In recent years, a number of researchers have proposed aperiodic and sporadic task models in which tasks are event-triggered Arzen (1999). By event-triggering, we mean that the system state is sampled when some function of the system state exceeds a threshold. The idea of event-triggered feedback has appeared under a variety of names, such as interrupt-based feedback Hristu-Varsakelis (2002), Lebesgue sampling Astrom (1999), or state-triggered feedback Tabuada et al. (2006). Event triggering usually requires some form of hardware event detector to generate a hardware interrupt to release the control task. This can be done using either custom analog integrated circuits (ASIC's) or floating point gate array (FPGA) processors. Event-triggering provides a useful way of adaptively adjusting task periods at run time, provided the cost associated with using ASIC/FPGA hardware is acceptable. In some applications, however, it may be unreasonable or impractical to retrofit an existing system with such "event detectors". In these cases, it may be more appropriate to use a software approach such as self-triggering where each task determines the release of its next job.

There is relatively little prior work examining self-triggered feedback control. A self-triggered task model was introduced by Velasco et al. (2003) in which a heuristic rule was used to adjust task periods. A self-triggered task

⋆ The authors gratefully acknowledge the partial financial support of the National Science Foundation (grants NSF-ECS-0400479 and NSF-CNS-0410771).

model was also introduced by Lemmon et al. (2007) which chose task periods based on a Lyapunov-based technique. But the authors did not provide analytic bounds for task periods and the task delays were considered only in the simulation results. Different from the prior work, this paper is the first rigorous examination of what might be required to implement self-triggered feedback systems.

Our results pertain to linear time-invariant systems with state feedback. Since our controller seeks to ensure $\mathcal{L}_2$ stability, we use a full-information $\mathcal{H}_\infty$ controller in our analysis. We also assume that the system has a process noise whose magnitude is bounded by a linear function of the norm of the system state. Under these assumptions we obtain the state-based bounds for the task periods and deadlines, which are based on variations of the event-triggering conditions used by Tabuada et al. (2006). Taking advantage of these bounds, a state-based self-triggered scheme is presented where the periods and deadlines are strictly away from zero. On the basis of simulation results, these bounds appear to be tight and relatively easy to compute, so it may be possible to use them in actual real-time control systems.

The techniques used in this paper are similar to the input-to-state stability (ISS) methods used in Nesic et al. (2004), Carnevale et al. (2007) for bounding the MATI. However, our self-triggering scheme provides less conservative sampling periods than those obtained using the MATI estimates in Nesic et al. (2004). Another major contribution in this paper is an explicit state-dependent bound on the acceptable delay, something which is not found in either Nesic et al. (2004), Carnevale et al. (2007).

The remainder of this paper is organized as follows: Section 2 introduces the system model. Section 3 derives sufficient threshold condition that can serve as an event triggering state sampling. In section 4, the self-triggering scheme is presented and the system is shown to be $\mathcal{L}_2$ stable. Simulations are shown in section 5. Finally, conclusions and future work are presented in section 6.

2. SYSTEM MODEL

Consider a linear time-invariant system whose state $x : \Re \to \Re^n$ satisfies the initial value problem,

$$\dot{x}(t) = Ax(t) + B_1u(t) + B_2w(t),\ x(0) = x_0,$$

where $u : \Re \to \Re^m$ is a control input, $w : \Re \to \Re^l$ is an exogenous disturbance function in $\mathcal{L}_2$ space, and $A \in \Re^{n\times n}$, $B_1 \in \Re^{n\times m}$, and $B_2 \in \Re^{n\times l}$ are real matrices of appropriate dimensions.

Since we're interested in controllers that are finite-gain $\mathcal{L}_2$ stable, assume there exists a positive semi-definite matrix P satisfying the $\mathcal{H}_\infty$ algebraic Riccati equation (ARE),

$$0 = PA + A^TP - Q + R, \tag{1}$$

where $Q = PB_1B_1^TP$ and $R = I + \frac{1}{\gamma^2}PB_2B_2^TP$ for some real constant $\gamma > 0$.

If we consider the standard $\mathcal{L}_2$ storage function $V : \Re^n \to \Re$ given by $V(x) = x^TPx$, then the preceding assumptions about P allow us to show that the storage function's directional derivative satisfies the dissipative inequality,

$$\dot{V}(x(t)) \le -\|x(t)\|_2^2 + \gamma^2\|w(t)\|_2^2 \tag{2}$$

for all $t \ge 0$. Recall that a linear system, $\mathbf{T}$, is said to be finite gain $\mathcal{L}_2$ stable if $\mathbf{T}$ is a linear operator from $\mathcal{L}_2$ back into $\mathcal{L}_2$. The induced gain of $\mathbf{T}$ is

$$\|\mathbf{T}\| = \sup_{\|w\|_{\mathcal{L}_2}=1} \|\mathbf{T}w\|_{\mathcal{L}_2}.$$

Satisfaction of the dissipative inequality (2) is sufficient to show that the system $\mathbf{T}$ characterized by the state equation

$$\dot{x}(t) = (A - B_1B_1^TP)x(t) + B_2w(t) \tag{3}$$

is finite gain $\mathcal{L}_2$ stable from w to x with an induced gain less than γ. For notational convenience, let $A_{cl} = A - B_1B_1^TP$ and $K = -B_1^TP$.

This paper considers a sampled-data implementation of the closed loop system in (3). This means that the plant's control, u, is computed by a computer task. This task is characterized by two monotone increasing sequences of time instants; the release time sequence $\{r_k\}_{k=0}^\infty$ and the finishing time sequence $\{f_k\}_{k=0}^\infty$. We say these two sequences are admissible if $r_k \le f_k \le r_{k+1}$ for all $k = 0, \dots, \infty$. The time r_k denotes the time when the kth invocation of a control task (also called a job) is released for execution on the computer's central processing unit (CPU). At this time, we assume that the system state is sampled so that r_k also represents the kth sampling time instant. The time f_k denotes the time when then kth job has finished executing. Each job of the control task computes the control u based on the last sampled state. Upon finishing, the control job outputs this control to the plant. The control signal used by the plant is held constant by a zero-order hold (ZOH) until the next finishing time f_{k+1}. This means that the sampled-data system under study in this project satisfies the following set of state equations,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1u(t) + B_2w(t) \\ u(t) &= -B_1^TPx(r_k) \end{aligned} \tag{4}$$

for $t \in [f_k, f_{k+1})$ and all $k = 0, \dots, \infty$. The state trajectories x satisfying (4) are continuous so that the initial state at time f_k is simply $x(f_k) = \lim_{t\uparrow f_k} x(t)$.

We let $T_k = r_{k+1} - r_k$ denote the kth inter-release time (also called sampling or task period) and $D_k = f_k - r_k$ denote the time interval between the kth job's release and finishing time, which is called delay or jitter of the kth job. By construction of the control, we know that this original system is $\mathcal{L}_2$ stable with gain less than γ. This paper's main results establish nontrivial bounds on the sequence of sampling periods $\{T_k\}_{k=0}^\infty$ and delays $\{D_k\}_{k=0}^\infty$ such that the resulting release and finishing time sequences are admissible and the sampled-data system preserves the original system's $\mathcal{L}_2$ stability.

3. $\mathcal{L}_2$ STABILITY

Consider the sampled-data system in (4) with a set of admissible release and finishing time sequences. For all k, define the kth job's error function $e_k : [r_k, f_{k+1}) \to \Re^n$ by $e_k(t) = x(t) - x(r_k)$. This error represents the difference between the current system state and the system state at the last release time, r_k. This section presents two inequality constraints on $e_k(t)$ (see theorem 1 and corollary 2 below) whose satisfaction is sufficient to ensure that the sampled-data system's $\mathcal{L}_2$ gain is less than γ/β for some parameter $\beta \in (0,1]$. For notational convenience, we use x_r, x_{r^-}, x_{r^+}, x_t, w_t to represent $x(r_k)$, $x(r_{k-1})$, $x(r_{k+1})$, $x(t)$, $w(t)$, respectively.

Theorem 1. Consider the sampled-data system in (4) with admissible release and finishing time sequences. Let $x(r_0) = x_0$ and β be any real constant in the interval $(0,1]$ with $Q = PB_1B_1^TP$. If

$$e_k^T(t)Qe_k(t) < (1-\beta^2)\|x_t\|_2^2 + x_r^TQx_r \tag{5}$$

holds for all $t \in [f_k, f_{k+1})$ and $k = 0, \ldots, \infty$, then the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with a gain less than γ/β.

Proof. Consider $V(x) = x^TPx$ where P is defined in (1). By completing square, the directional derivative of V for $t \in [f_k, f_{k+1})$ satisfies $\dot{V} \leq -\|x_t\|_2^2 + e_t^TQe_t - x_r^TQx_r + \gamma^2\|w_t\|_2^2$. Applying (5) to this inequality, we have $\dot{V} \leq -\beta^2\|x_t\|_2^2 + \gamma^2\|w_t\|_2^2$, which ensures the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with a gain less than γ/β.

In our following work, we'll find it convenient to use a slightly weaker sufficient condition for $\mathcal{L}_2$ stability which is only a function of the state error $e_k(t)$. The following corollary states this result.

Corollary 2. Consider the sampled-data system in (4) with admissible sequences of release and finishing times. Let $x(r_0) = x_0$ and $Q = PB_1B_1^TP$. For any β in the interval $(0,1]$, let $M = (1-\beta^2)I + Q$. If the state error trajectory satisfies

$$e_k(t)^TMe_k(t) \leq x_r^TMx_r \tag{6}$$

for all $t \in [f_k, f_{k+1})$ and $k = 0, \ldots, \infty$, then the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with a gain less than γ/β.

Proof. Equation (6) implies $e_k(t)^TMe_k(t) \leq (1-\beta^2)\|x_r\|_2^2 + x_r^TQx_r$, which can be further developed as $e_k(t)^TQe_k(t) \leq (1-\beta^2)\|x_t\|_2^2 + x_r^TQx_r$. By applying theorem 1, we can conclude that the sampled-data system is $\mathcal{L}_2$ stable from w to x with a gain less than γ/β.

Remark 3. The inequalities in equations 5 or 6 can both be used as the basis for an event-triggered feedback control system, which is very similar to the state-triggering scheme proposed by Tabuada et al. (2006) for asymptotic stability. The main difference between that result and this one is that our proposed event-triggering condition provides a stronger assurance on the sampled-data system's performance as measured by its induced $\mathcal{L}_2$ gain.

4. ADMISSIBLE RELEASE AND FINISHING TIMES

This section introduces the self-triggering scheme to characterize the admissible sequences of release and finishing times that ensure the sampled data system in (4) is $\mathcal{L}_2$ stable with a specified gain.

For notational convenience, let $z_k : [r_k, f_{k+1}) \to \Re^n$ and $\rho : \Re^n \to \Re$ be given as

$$z_k(t) = \sqrt{(1-\beta^2)I + Q}\; e_k(t) = \sqrt{M}e_k(t) \tag{7}$$

$$\rho(x) = \sqrt{x^TMx} \tag{8}$$

where $\sqrt{M}$ is a matrix square root. So if we can guarantee for any $\delta \in (0,1]$ that

$$\|z_k(t)\|_2 \leq \delta\rho(x_r) \tag{9}$$

for all $t \in [f_k, f_{k+1})$ and $k = 0, \ldots, \infty$, then the hypotheses in corollary 2 are satisfied and we can conclude that the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with a gain less than γ/β.

With delays, we can partition the time interval $[r_k, f_{k+1})$ into two subintervals $[r_k, f_k)$ and $[f_k, f_{k+1})$, where the associated differential equations are

$$\dot{x}_t = Ax_t - B_1B_1^TPx_{r^-} + B_2w_t \quad \text{and}$$
$$\dot{x}_t = Ax_t - B_1B_1^TPx_r + B_2w_t,$$

respectively. We can use differential inequalities to bound $z_k(t)$ for all $t \in [r_k, f_{k+1})$ and thereby determine sufficient conditions assuring the admissibility of the release/finishing times while preserving the closed-loop system's $\mathcal{L}_2$-stability. The next two lemmas characterize the behavior of $z_k(t)$ over these two subintervals.

Lemma 4. Consider the sampled-data system in (4). Assume that M has full rank and $\|w_t\|_2 \leq W\|x_t\|_2$ holds for all $t \in \Re$ with some non-negative real W. For any non-negative integer k and some $\epsilon \in (0,1)$, if the kth release time r_k and finishing time f_k satisfy

$$0 \leq D_k = f_k - r_k \leq L_1(x_r, x_{r^-}; \epsilon) \tag{10}$$

for all $t \in [r_k, f_k)$, then the kth trigger signal, z_k, satisfies

$$\|z_k(t)\|_2 \leq \phi(x_r, x_{r^-}; t - r_k) \leq \epsilon\rho(x_r) \tag{11}$$

for all $t \in [r_k, f_k)$, where

$$L_1(x_r, x_{r^-}; \epsilon) = \frac{1}{\alpha}\ln\left(1 + \epsilon\alpha\frac{\rho(x_r)}{\mu_1(x_r, x_{r^-})}\right), \tag{12}$$

$$\alpha = \left\|\sqrt{M}A\sqrt{M}^{-1}\right\| + W\left\|\sqrt{M}B_2\right\|\left\|\sqrt{M}^{-1}\right\|,$$

$$\phi(x_r, x_{r^-}; t - r_k) = \frac{\mu_1(x_r, x_{r^-})}{\alpha}\left(e^{\alpha(t-r_k)} - 1\right),$$

$$\mu_1(x_r, x_{r^-}) = \left\|\sqrt{M}\left(Ax_r - B_1B_1^TPx_{r^-}\right)\right\|_2 + W\left\|\sqrt{M}B_2\right\|\|x_r\|_2.$$

Proof. For $t \in [r_k, f_k)$, the derivative of $\|z_k(t)\|_2$ satisfies the differential inequality $\frac{d}{dt}\|z_k(t)\|_2 \leq \alpha\|z_k(t)\|_2 + \mu_1(x_r, x_{r^-})$. Solving this differential inequality with the initial condition $z_k(r_k) = 0$, we have $\|z_k(t)\|_2 \leq \phi(x_r, x_{r^-}; t - r)$ for all $t \in [r_k, f_k)$. Combining this inequality with the inequality $\phi(x_r, x_{r^-}; D_k) \leq \epsilon\rho(x_r)$ developed from (10) yields $\|z_k(t)\|_2 \leq \phi(x_r, x_{r^-}; t - r) \leq \phi(x_r, x_{r^-}; D_k) \leq \epsilon\rho(x_r)$, which leads to (11) holding for all $t \in [r_k, f_k)$.

Lemma 5. Consider the sampled-data system in (4). Assume that M has full rank and $\|w_t\|_2 \leq W\|x_t\|_2$ holds for all $t \in \Re$ with some non-negative real W. For a given integer k and some $\epsilon \in (0,1)$, assume that $r_{k-1} \leq f_{k-1} \leq r_k$. For any $\eta \in (\epsilon, 1]$, let $d_\eta = f_k + L_2(x_r, x_{r^-}; D_k, \eta)$, where $L_2 : \Re^n \times \Re^n \times \Re \times (0,1] \to \Re$ is given by

$$
\begin{aligned}
&L_2(x_r, x_{r^-}; D_k, \eta) \\
&\quad = \frac{1}{\alpha} \ln \left(1 + \alpha \frac{\eta\rho(x_r) - \phi(x_r, x_{r^-}; D_k)}{\mu_0(x_r) + \alpha\phi(x_r, x_{r^-}; D_k)} \right) \\
&\mu_0(x_r) = \left\| \sqrt{M} A_{\mathrm{cl}} x_r \right\|_2 + W \left\| \sqrt{M} B_2 \right\| \|x_r\|_2 .
\end{aligned}
\quad (13)
$$

If $0 \leq D_k \leq L_1(x_r, x_{r^-}; \epsilon)$, then $d_\eta > f_k$ and $\|z_k(t)\|_2 \leq \eta\rho(x_r)$ for all $t \in [f_k, d_\eta]$.

Proof. The hypotheses of this lemma also satisfy the hypotheses of lemma 4 so we know that

$$\|z_k(f_k)\|_2 \leq \phi(x_r, x_{r^-}; D_k) \leq \epsilon\rho(x_r) \leq \eta\rho(x_r). \quad (14)$$

By (13) and (14), we have $L_2(x_r, x_{r^-}; D_k, \eta) > 0$ which implies $d_\eta > f_k$. Assume the system state x_t satisfies the differential equation $\dot{x}_t = Ax_t - B_1 B_1^T P x_r + B_2 w_t$ for $t \in [f_k, d_\eta]$. Using an argument similar to that in lemma 4, we can show that $\|z_k(t)\|_2$ satisfies the differential inequality $\frac{d}{dt}\|z_k(t)\|_2 \leq \alpha\|z_k(t)\|_2 + \mu_0(x_r)$. Solving this differential inequality using (14) as the initial condition, we know $\|z_k(t)\|_2 \leq e^{\alpha(t-f_k)}\phi(x_r, x_{r^-}; D_k) + \frac{\mu_0(x_r)}{\alpha}\left(e^{\alpha(t-f_k)} - 1\right)$ for all $t \in [f_k, d_\eta]$. Because the right side of the equation above is an increasing function of t, we get $\|z_k(t)\|_2 \leq \eta\rho(x_r)$ for all $t \in [f_k, d_\eta]$.

According to lemma 5, for a constant $\delta \in (\epsilon, 1)$, if $D_k \leq L_1(x_r, x_{r^-}; \epsilon)$, then $r_{k+1} = f_k + L_2(x_r, x_{r^-}; D_k, \delta)$, and $f_{k+1} \leq f_k + L_2(x_r, x_{r^-}; D_k, 1)$ imply $\|z_k(r_{k+1})\|_2 \leq \delta\rho(x_r)$ and $\|z_k(f_{k+1})\|_2 \leq \rho(x_r)$, respectively. We will use this fact below to characterize a self-triggering scheme that preserves the sampled-data system induced $\mathcal{L}_2$ gain. Theorem 7 formally states this self-triggering scheme. The proof of theorem 7 requires the following lemma showing that the bound for delays given in lemma 4 is bounded below by a positive function of x_{r^-}.

Lemma 6. Consider the sampled-data system in (4). Assume that M has full rank and $\|w_t\|_2 \leq W\|x_t\|_2$ holds for all $t \in \Re$ with some non-negative real W. If for a constant $\delta \in (\epsilon, 1)$, the release time r_{k-1} and r_k satisfy $\|z_{k-1}(r_k)\|_2 \leq \delta\rho(x_{r^-})$ for all k, then L_1 given by (12) satisfies $L_1(x_r, x_{r^-}; \epsilon) \geq \xi(x_{r^-}; \epsilon, \delta) > 0$, where

$$\xi(x_{r^-}; \epsilon, \delta) = \frac{1}{\alpha} \ln \left(1 + \frac{\epsilon(1-\delta)\rho(x_{r^-})}{\delta\rho(x_{r^-}) + \mu_0(x_{r^-})/\alpha} \right) \quad (15)$$

Proof. A lower bound on $\rho(x_r)$ is obtained by $\rho(x_r) \geq \|\sqrt{M}x_{r^-}\|_2 - \|z_{k-1}(r_k)\|_2 \geq \rho(x_{r^-}) - \delta\rho(x_{r^-})$. Similarly, an upper bound on $\mu_1(x_r, x_{r^-})$ is obtained: $\mu_1(x_r, x_{r^-}) \leq \mu_0(x_{r^-}) + \alpha\delta\rho(x_{r^-})$. Putting both inequalities together we see that $L_1(x_r, x_{r^-}; \epsilon) \leq \xi(x_{r^-}; \epsilon, \delta) > 0$.

With the preceding technical lemma we can now state a self-triggered feedback scheme which can guarantee the sampled-data system's induced $\mathcal{L}_2$ gain.

Theorem 7. Consider the sampled-data system in (4). Assume that M has full rank and $\|w_t\|_2 \leq W\|x_t\|_2$ holds for all $t \in \Re$ with some non-negative real W. For given $\epsilon \in (0, 1)$ and $\delta \in (\epsilon, 1)$, we assume that

- The initial release and finishing times satisfy

$$r_{-1} = r_0 = f_0 = 0$$

- For any non-negative integer k, the release times are generated by the following recursion,

$$r_{k+1} = f_k + L_2(x_r, x_{r^-}; D_k, \delta) \quad (16)$$

and the finishing times satisfy

$$r_{k+1} \leq f_{k+1} \leq r_{k+1} + \xi(x_r; \epsilon, \delta). \quad (17)$$

where L_2 and ξ are given in (13) and 15, respectively. Then the sequence of release times, $\{r_k\}_{k=0}^\infty$, and finishing time, $\{f_k\}_{k=0}^\infty$, are admissible and the sampled-data system is finite gain $\mathcal{L}_2$ stable from w to x with an induced gain less than γ/β.

Proof. By the definition of ξ in (15), we can easily see that $\xi(x_r; \epsilon, \delta) > 0$ for any non-negative integer k and therefore the interval $[r_{k+1}, r_{k+1} + \xi(x_r; \epsilon, \delta)]$ is nonempty for all k. Next, we insert (16) into (17) to show that for all k,

$$f_{k+1} \leq f_k + L_2(x_r, x_{r^-}); D_k, 1). \quad (18)$$

With the preceding two preliminary results, we now use mathematical induction to show that under the theorem's hypotheses, the following statement holds for all k:

$$
\begin{aligned}
&r_k \leq f_k \leq r_{k+1} \\
&\|z_k(t)\|_2 \leq \delta\rho(x_r) \text{ for all } t \in [f_k, r_{k+1}] \\
&\|z_k(t)\|_2 \leq \rho(x_r) \quad \text{for all } t \in [f_k, f_{k+1}].
\end{aligned}
\quad (19)
$$

It is easy to show the inductive statement hold for $k = 0$. We now turn to the general case for any k. For a given k, assume the statement in (19) hold and consider the $k+1$st job. Since the hypothesis of lemma 6 is satisfied, we have $\xi(x_r); \epsilon, \delta) \leq L_1(x_{r^+}, x_r; \epsilon)$. We can use it in (17) to obtain

$$0 \leq D_{k+1} \leq L_1(x_{r^+}, x_r; \epsilon). \quad (20)$$

From (20) and the fact that $\delta \in (0, 1)$ we know that the hypotheses of lemma 5 hold and we can conclude that

$$
\begin{aligned}
&f_{k+1} \leq r_{k+2} \\
&\|z_{k+1}(t)\|_2 \leq \delta\rho(x_{r^+}) \text{ for } t \in [f_{k+1}, r_{k+2}]
\end{aligned}
\quad (21)
$$

Combining (17) with the above (21) yields $r_{k+1} \leq f_{k+1} \leq r_{k+2}$. Therefore, the first two parts of the statement are established for the case $k+1$. Let $d_1^{k+1} = f_{k+1} + L_2(x_{r^+}, x_r; D_{k+1}, 1)$. According to (18), $f_{k+2} \leq d_1^{k+1}$. Combining this and (17), (21) yields $[f_{k+1}, f_{k+2}] \subseteq [f_{k+1}, d_1^{k+1}]$. We know that the validity of (20) satisfies the hypotheses of lemma 5 and therefore conclude that $\|z_{k+1}(t)\|_2 \leq \rho(x_{r^+})$ for all $t \in [f_{k+1}, f_{k+2}] \subseteq [f_{k+1}, d_1^{k+1}]$, which completes the third part of the statement for $k+1$.

We may therefore use mathematical induction to conclude that the inductive statement holds for all non-negative integers k. The first part of the statement simply means that the sequences $\{r_k\}_{k=0}^\infty$ and $\{f_k\}_{k=0}^\infty$ are admissible. The third part of the inductive statement implies that the hypotheses of corollary 2 are satisfies, thereby ensuring that the system's induced $\mathcal{L}_2$ gain is less than γ/β.

Remark 8. $\xi(x_r; \epsilon, \delta)$ is the deadline for the delay D_k.

Remark 9. By the way we construct δ, we see that it controls when the next job's finishing time. We might therefore expect to see a larger δ result in larger sampling periods. This is indeed confirmed by the analysis. Since

$$T_k \geq r_{k+1} - f_k = L_2(x_r, x_{r^-}; D_k, \delta)$$

and since L_2 is an increasing function of δ we can see that larger δ result in larger sampling periods.

Remark 10. By our construction of ϵ, we see that it controls the current job's finishing time. Since this

$$D_k = f_k - r_k \leq \xi(x_{r^-};\epsilon,\delta)$$

and since ξ is an increasing function of ϵ, we can expect to see the allowable delay increase as we increase ϵ. Note also that ξ is a decreasing function of δ so that adopting a longer sampling period by increasing δ will have the effect of reducing the maximum allowable task delay.

The following corollary to the above theorem shows that the task periods and deadlines generated by our self-triggered scheme are all bounded away from zero. This is important in establishing that our scheme does not generate infinite sampling frequencies.

Corollary 11. Assume the assumptions in theorem 7 hold. Then there exist two positive constants $\zeta_1, \zeta_2 > 0$ such that $T_k \geq \zeta_1$ and $\xi(x_r;\epsilon,\delta) \geq \zeta_2$.

Proof. From theorem 7, we know $D_k \leq \xi(x_r;\epsilon,\delta) \leq L_1(x_r, x_{r^-};\epsilon)$. Therefore, by lemma 4, $\phi(x_r, x_{r^-}; D_k) \leq \epsilon\rho(x_r)$. Then it is easy to show
$T_k \geq \frac{1}{\alpha}\ln\left(1+\frac{\alpha(\delta-\epsilon)\underline{\lambda}(\sqrt{M})}{\|\sqrt{M}A_{\text{cl}}\|+W\|\sqrt{M}B_2\|+\alpha\epsilon\overline{\lambda}(\sqrt{M})}\right)$ and
$\xi(x_r;\epsilon,\delta) \geq \frac{1}{\alpha}\ln\left(1+\frac{\epsilon\alpha(1-\delta)\underline{\lambda}(\sqrt{M})}{\|\sqrt{M}A_{\text{cl}}\|+W\|\sqrt{M}B_2\|+\delta\alpha\overline{\lambda}(\sqrt{M})}\right)$.

5. SIMULATION

The following simulation results were generated for self-triggered feedback systems. The plant was an inverted pendulum on top of a moving cart with state equations

$$\dot{x}_t = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/\ell & 0 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(M\ell) \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} w_t$$

where M was the cart mass, m was the mass of the pendulum bob, ℓ was the length of the pendulum arm, and g was gravitational acceleration. For these simulations, we let $M = 10$, $\ell = 3$, $g = 10$, $\gamma = 200$, and $\beta = 0.5$. The system's initial state was the vector $x_0 = [\,0.98\ 0\ 0.2\ 0\,]^T$. The control gain is $K = [\,2\ 12\ 378\ 210\,]$.

5.1 Self-triggered Feedback

The simulations in this subsection examined the self-triggering feedback scheme in theorem 7. In this case we assumed $w_t = 0$ and set $\epsilon = 0.65$, $\delta = 0.7$. The task release time r_{k+1} was computed at f_k using (16) and the finishing times were assumed to satisfy $f_{k+1} = r_{k+1} + \xi(x_r;\epsilon,\delta)$.

Let x_t^s denote the self-triggered system's response and x_t^c the continuous-time system's response. The top plot of Fig. 1 plots the error signal $\|x_t^s - x_t^c\|_2$ as a function of time. The error signal is small over time, thereby suggesting that the continuous-time and self-triggered systems have nearly identical impulse responses

The bottom plot of Fig. 1 plots the task periods, T_k, (crosses) and deadlines, ξ, (dots) generated by the self-triggered scheme. The sampling periods range between 0.027 to 0.187. These sampling periods show significant variability. The shortest and most aggressive sampling

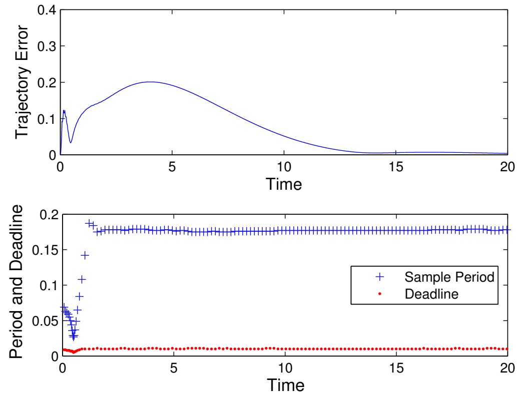

Fig. 1. A self-triggered system ($\delta = 0.7$, $\epsilon = 0.65$, $w_t = 0$)

periods occurred in response to the system's non-zero initial condition. Longer and relatively constant sampling periods were generated once the system state has returned to the neighborhood of the system's equilibrium point. This seems to confirm the conjecture that self-triggering can effectively adjust task periods in response to changes in the control system's external inputs.

5.2 Self-triggered versus Periodically Triggered Control

The simulations in this subsection directly compare the performance of self-triggered and "comparable" periodically triggered feedback control systems. These simulations were done on the inverted pendulum system described above. The self-triggered simulations assumed that $\epsilon = 0.65$ and $\delta = 0.7$ and task delays were set equal to the deadlines given by the function ξ.

The state trajectories were compared against periodically triggered systems with a "comparable" task period and delays. The comparable task period was the mean sample period, 0.0673, over the interval when the system is near its equilibrium point generated by a self-triggered system whose exogenous inputs were chosen to be a noise process in which $\|w_t\|_2 \leq 0.01\|x_t^s\|_2$. The delay was set equal to the minimum predicted deadline, 0.004. Fig. 2 plots the sample periods, T_k, and predicted deadlines generated by such a self-triggered system.

We compared the self-triggered and periodically triggered system's performance by examining their normalized trajectory errors, $E(t|x_t)$, given by

$$E(t|x_t) = \frac{\left|\sqrt{V(x_t)} - \sqrt{V(x_t^c)}\right|}{\sqrt{V(x_t^c)}},$$

where $V(x) = x^T P x$ and P satisfying (1). This normalization of the trajectory error allows us to fairly compare those states (i.e. the pendulum bob angle) that are most directly affected by input disturbances. Let x_t^p denote the periodically triggered system's response. Fig. 3 plots the time history of the normalized errors, $E(t|x_t^s)$ and $E(t|x_t^p)$, for the inverted pendulum using the input signal, $w_t = \mu_t + \nu_t$ where ν is the disturbance satisfying $\|\nu_t\|_2 \leq 0.01\|x_t^s\|_2$ and $\mu : \Re \to \Re$ takes the values

$$\mu_t = \begin{cases} \text{sgn}(\sin(0.7t)) & \text{if } 0 \leq t < 10 \\ 0 & \text{otherwise} \end{cases}.$$

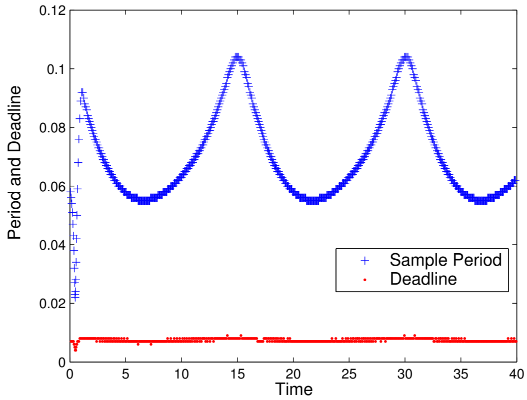

Fig. 2. Sample periods and deadlines for a self-triggered system ($\epsilon = 0.65$, $\delta = 0.7$, $\|w_t\|_2 \leq 0.01\|x_t^s\|_2$)

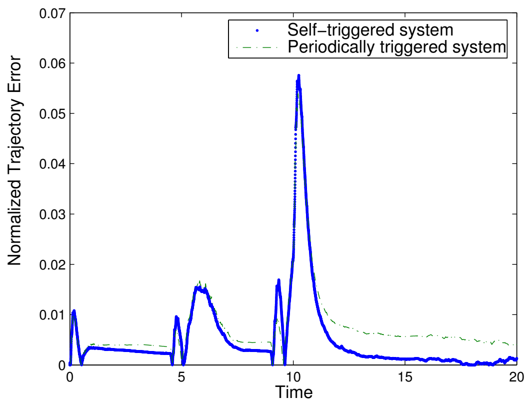

Fig. 3. Normalized trajectory errors versus time for a self-triggered system ($\epsilon = .65$ and $\delta = 0.7$) and a periodically triggered system

Fig. 3 clearly shows that the self-triggered error is significantly smaller than the error of the periodically triggered system. This error is a direct result of the self-triggered system's ability to adjust its sample period. Fig. 4 plots the sampling periods generated by the self-triggered system for the preceding system. This plot shows that the sampling period readjusts and gets smaller when the square wave input hits the system over the time interval $[0, 10]$. These results again demonstrate the ability of self-triggering to successfully adapt to changes in the system's input disturbances.

We then compare the sampling period in the self-triggered system with the bound of MATI in Nesic et al. (2004) given by $\tau_{\text{MATI}} = \frac{1}{L}\ln\frac{L+\bar{\gamma}}{\bar{\rho}L+\bar{\gamma}} = 0.0112$, where, for the inverted pendulum model, $\bar{\rho} = 0$, $L = \max(0.5\lambda_{\max}(-B_1K - K^TB_1^T), 0)$, and $\bar{\gamma}$ is the $\mathcal{L}_2$ gain for the closed-loop system ($\dot{x} = A_{cl}x + B_1Ke + B_2w$) from (e, w) to $-A_{cl}x$. Clearly the average period, 0.0673, generated by the self-triggered scheme is longer than the bound of MATI.

6. CONCLUSION

This paper has presented a self-triggered feedback scheme with guaranteed $\mathcal{L}_2$ stability. Simulation results show that the proposed self-triggered scheme perform better than comparable periodically triggered feedback controllers. The results in this paper, therefore, appear to provide a solid analytical basis for the development of aperiodic sampled-data control systems that adjust their periods and deadlines to variations in the system's external inputs.

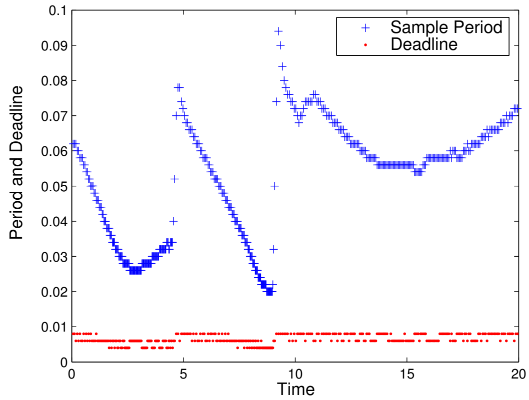

Fig. 4. Sampling period versus time for the self-triggered system ($\epsilon = 0.65$, $\delta = 0.7$, $w_t = \mu_t + \nu_t$)

REFERENCES

M.D. Lemmon, T. Chantem, X. Hu and M. Zyskowski. On Self-triggered full information H-infinity controllers. *Hybrid Systems: computation and control*, 2007.

M. Velasco, P. Marti and J.M. Fuertes. The Self Triggered Task Model for Real-Time Control Systems. *Work-in-Progress Session of the 24th IEEE Real-Time Systems Symposium (RTSS03)*, 2003.

K.J. Astrom and B. Wittenmark. *Computer-Controlled Systems: theory and design* 2nd ed. Prentice-Hall, 1990.

Y. Zheng, D.H. Owens and S.A. Billings. Fast Sampling and Stability of Nonlinear Sampled-Data Systems: Part 2. Sampling Rate Estimations. *IMA Journal of Mathematical Control and Information*, volume 7, pages 13–33. 1990.

D. Nesic, A.R. Teel and E.D. Sontag. Formulas relating KL stability estimates of discrete-time and sampled-data nonlinear systems. *Systems and Control Letters*, volume 38, pages 49–60, 1999.

D. Nesic and A.R. Teel. Input-output Stability Properties of Networked Control Systems. *IEEE Transactions on Automatic Control*, volume 49, pages 1650–1667, 2004.

D. Carnevale and A.R. Teel and D. Nesic. Further results on stability of networked control systems: a Lyapunov approach. *IEEE Transactions on Automatic Control*, volume 52, pages 892–897, 2007.

P. Tabuada and X. Wang. Preliminary results on state-triggered scheduling of stabilizing control tasks. *IEEE Conference on Decision and Control*, 2006.

K.E. Arzen. A simple event-based PID controller. *Proceedings of the 14th IFAC World Congress*, 1999.

D. Hristu-Varsakelis and P.R. Kumar. Interrupt-based feedback control over a shared communication medium. *Proceedings of the IEEE Conference on Decision and Control*, 2002.

K.J. Astrom and B.M. Bernhardsson. Comparison of Riemann and Lebesgue sampling for first order stochastic systems. *Proceedings of the IEEE Conference on Decision and Control*, 1999.