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Stabilizing controllers, Lyapunov functions, and the inverse problem of optimal control

P. Rapisarda * **C. Kojima** **

* *Information: Signal, Images, Systems (ISIS) Research Group, School of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, United Kingdom, tel.: +44 (0)23 8059 3367, fax: +44 (0)23 80594498, e-mail:* `pr3@ecs.soton.ac.uk`

** *Department of Information Physics and Computing, Graduate School of Information Science and Technology, The University of Tokyo, Hongo, Bunkyo-Ku, Tokyo 113-0033, Japan tel/fax: +81-3-5841-6890; e-mail:* `chiaki_kojima@ipc.i.u-tokyo.ac.jp`

Abstract: We explore the connections of Margreta Kuijper's parametrization of stabilizing controllers, with some issues arising in inverse optimal control and in Lyapunov stability theory for higher-order linear time-invariant differential systems.

Keywords: Lyapunov functions, stationarity, quadratic differential forms, inverse problems, dissipative systems.

1. INTRODUCTION

The purpose of this paper is to study some consequences of the parametrization of controllers which interconnected with a given behavior $\mathfrak{B}$ yield a stable desired controlled behavior $\mathfrak{B}_{\rm des}$ (see Kuijper (1995)). We show that Kuijper's parametrization can be used in order to characterize the set of Lyapunov functions for $\mathfrak{B}_{\rm des}$; and to parametrize the solutions to the inverse optimal control problem for systems described by higher-order differential equations.

Some of the results presented in this note are reminiscent of those of Iwasaki et al. (1995); however, we operate in a different setting. While in Iwasaki et al. (1995) the starting point of the investigation is a state-space representation of a system, in this paper we consider systems described by higher-order linear, constant-coefficient, differential equations. This choice of setting is motivated by the fact that state-space representations are usually *derived a posteriori* from a model derived from first principles. Such a model consists of a set of differential equations of high order, usually involving also auxiliary variables, besides the variables that one is interested in modeling; and including algebraic constraints among the variables. It makes sense, consequently, to address issues for this type of representation, rather than restricting attention to more specific ones, such as the state-space, no matter how useful they are in other contexts.

The right language to formulate and solve problems defined at this level of generality is that of behavioral system and control theory, which we use extensively in this paper. We refer the reader unfamiliar with the concepts and terminology of the behavioral approach to the book Polderman et al. (1998). In this paper we will also use extensively the notion of *quadratic differential form*, and the related concepts and results; the reader is referred to the paper Willems et al. (1998).

The paper is organized as follows: in section 2 we recall some background material regarding linear differential systems, quadratic differential forms, and the parametrization of controllers introduced in Kuijper (1995). In section 3 we show how this parametrization can be used in order to characterize the set of Lyapunov functions associated with a stable closed-loop behavior. In section 4 we introduce the inverse problem of optimal control, and we show how Kuijper's result can be used in order to parametrize the solution to this problem.

Notation and terminology: The space of $\mathtt{n}$ dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, and the space of $\mathtt{m}\times\mathtt{n}$ real matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used: $\mathbb{R}^{\bullet\times\mathtt{n}}$ represents the set of real matrices with $\mathtt{n}$ columns and an unspecified (but finite) number of rows. Given two column vectors x and y, we denote with $\mathrm{col}(x,y)$ the vector obtained by stacking x over y; a similar convention holds for the stacking of matrices with the same number of columns.

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. The space of all $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that consisting of all $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta,\eta]$. Given a matrix $R\in\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, we define $R(\xi)^{\sim} := R(-\xi)^{\top}\in\mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$.

We denote with $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$; and with $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ the set of infinitely-differentiable compact support trajectories. The exponential function whose value at time t is $e^{\lambda t}$, will be denoted with $\exp_{\lambda}$.

2. BACKGROUND MATERIAL

2.1 Behavioral system theory

A linear differential behavior is a linear subspace $\mathfrak{B}$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, consisting of all solutions w of a given system of linear constant-coefficient differential equations. The set consisting of all linear differential behaviors whose trajectories w have dimension $\mathtt{w}$ is denoted with $\mathfrak{L}^{\mathtt{w}}$. A behavior in $\mathfrak{L}^{\mathtt{w}}$ can always be represented as

$$R(\frac{d}{dt})w = 0, \tag{1}$$

where $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$. The equation (1) is called a *kernel representation* of $\mathfrak{B}$, the latter defined as the set of all w's satisfying the equation. The variable w is called the *manifest variable* of $\mathfrak{B}$. Linear differential systems which are controllable (see Polderman et al. (1998) for the definition) can also be represented as follows. If $M \in \mathbb{R}^{\mathtt{w} \times \bullet}[\xi]$ and $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ s.t. } w = M(\frac{d}{dt})\ell\}$, then we call

$$w = M(\frac{d}{dt})\ell \tag{2}$$

an image representation of $\mathfrak{B}$. The image representation (2) of $\mathfrak{B}$ is called *observable* if $(M(\frac{d}{dt})\ell = 0) \Longrightarrow (\ell = 0)$. It can be shown that this is the case if and only if the matrix $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. The set of all controllable behaviors with manifest variable of dimension $\mathtt{w}$ is denoted with $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$.

Associated with a system in $\mathfrak{L}^{\mathtt{w}}$ there are a number of integer invariants (see Polderman et al. (1998)); in the following we will refer frequently to $\mathtt{p}(\mathfrak{B})$, the output cardinality of the behavior $\mathfrak{B}$; and to $\mathtt{m}(\mathfrak{B})$, the input cardinality of the behavior $\mathfrak{B}$. When it will be clear from the context which behavior is being referred to, we will drop the explicit dependence on $\mathfrak{B}$ in the invariants' symbols, and write $\mathtt{m}$ and $\mathtt{p}$ instead.

We now review those definitions and results regarding quadratic differential forms, which are used in this paper.

Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$; then Φ can be written in the form

$$\Phi(\zeta, \eta) = \sum_{h,k=0}^{N} \Phi_{h,k} \zeta^h \eta^k,$$

where $\Phi_{h,k} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ and N is an integer. The two-variable polynomial matrix Φ induces a quadratic functional acting on infinitely differentiable trajectories as follows:

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \longrightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}),$$
$$Q_\Phi(w) = \sum_{h,k=0}^{N} (\frac{d^h w}{dt^h})^T \Phi_{h,k} \frac{d^k w}{dt^k}.$$

Without loss of generality in the following we will assume that Φ is a *symmetric two-variable polynomial matrix*, i.e. $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$. We denote the set of all symmetric $\mathtt{w} \times \mathtt{w}$ two-variable polynomial matrices matrices by $\mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. The QDF Q_Φ is called nonnegative, denoted $Q_\Phi \geq 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Q_Φ is called positive if $Q_\Phi \geq 0$ and $[Q_\Phi(w) = 0] \Longrightarrow [w = 0]$. In the context of Lyapunov theory, the need arises to define nonnegativity and positivity along a behavior. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$; then Q_Φ is called nonnegative along $\mathfrak{B}$, denoted $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$. The notion of positivity along $\mathfrak{B}$ follows immediately.

We now illustrate a couple of features of the calculus of QDF's which will be used extensively in this paper. One of them is the notion of derivative of a QDF. Given a QDF Q_Φ we define its derivative as the QDF $\frac{d}{dt} Q_\Phi$ defined by $(\frac{d}{dt} Q_\Phi)(w) := \frac{d}{dt}(Q_\Phi(w))$. Q_Φ is called the derivative of Q_Ψ if $\frac{d}{dt} Q_\Psi = Q_\Phi$. In terms of the two-variable polynomial matrices associated with the QDF's, this relationship is equivalently expressed as $(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta)$.

A second notion of the calculus of QDF's which will be extensively used in the following is that of equivalence of QDF's modulo a behavior. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$; two QDF's Q_{Φ_1} and Q_{Φ_2} are called equivalent modulo $\mathfrak{B}$, denoted $Q_{\Phi_1} \overset{\mathfrak{B}}{=} Q_{\Phi_2}$, if $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$ for all $w \in \mathfrak{B}$. Let $\mathfrak{B} = \ker\ R\left(\frac{d}{dt}\right)$; then it can be shown (see Proposition 3.2 of Willems et al. (1998)) that $Q_{\Phi_1} \overset{\mathfrak{B}}{=} Q_{\Phi_2}$ if and only if there exists $F \in \mathbb{R}^{\bullet \times \bullet}[\zeta, \eta]$ such that $\Phi_1(\zeta, \eta) = \Phi_2(\zeta, \eta) + R(\zeta)^\top F(\zeta, \eta) + F(\eta, \zeta)^\top R(\eta)$. The class of equivalence of a given QDF modulo an autonomous behavior $\mathfrak{B}$ admits a canonical representative, defined as follows. Let $\mathfrak{B} = \ker\ R\left(\frac{d}{dt}\right)$, with $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ nonsingular; then it can be shown that for any $\Phi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ there exists $\Phi' \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ such that $Q_{\Phi'} \overset{\mathfrak{B}}{=} Q_\Phi$ and moreover $R(\zeta)^{-\top} \Phi'(\zeta, \eta) R(\eta)^{-1}$ is a matrix of strictly proper two-variable rational functions. Φ' is called the R-canonical representative of Φ.

In this paper, we also use integrals of QDF's. In order to make sure that the integrals exist, we assume that the trajectory w on which the QDF acts is of compact support; that is, w belongs to $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Given a QDF Q_Φ, we define its integral as the functional

$$\int Q_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \longrightarrow \mathbb{R},$$
$$\int Q_\Phi(w) = \int_{-\infty}^{+\infty} Q_\Phi(w) dt.$$

Questions such as when the integral of a QDF is a positive semidefinite operator arise naturally in the study of dissipativity. We call a QDF Q_Φ *average nonnegative*, if $\int Q_\Phi \geq 0$, i.e., $\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

2.2 Lyapunov stability

We begin by recalling the definition of asymptotic stability of a behavior. A behavior $\mathfrak{B}$ is asymptotically stable if $(w \in \mathfrak{B}) \Longrightarrow (\lim_{t \to \infty} w(t) = 0)$. It can be shown that if $\mathfrak{B}$ is asymptotically stable, then there exists a nonsingular matrix $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ such that $\mathfrak{B} = \ker\ R\left(\frac{d}{dt}\right)$ and $\det(R)$ is a Hurwitz polynomial.

The following result from Willems et al. (1998) holds.

Proposition 1. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. The following statements are equivalent:

(1) $\mathfrak{B}$ is asymptotically stable;
(2) For every $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ such that $Q_\Phi \overset{\mathfrak{B}}{<} 0$ there exists $\Psi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ such that $Q_\Psi \overset{\mathfrak{B}}{>} 0$ and $\frac{d}{dt} Q_\Psi \overset{\mathfrak{B}}{=} Q_\Phi$.

In practical situations it is of interest to compute a polynomial matrix Ψ from a given Φ, such that $Q_\Phi \overset{\mathfrak{B}}{<} 0$ and $\frac{d}{dt}Q_\Psi \overset{\mathfrak{B}}{=} Q_\Phi$. Let $\mathfrak{B} = \ker\ R\left(\frac{d}{dt}\right)$; then from Theorem 3.2 of Willems et al. (1998) it follows that the last equation can be equivalently written as

$$(\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) + Y(\eta,\zeta)^\top R(\eta) + R(\zeta)^\top Y(\eta,\zeta) \tag{3}$$

for some $Y \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$. In particular, it can be shown (see Peeters et al. (2001)) that if $\mathfrak{B}$ is asymptotically stable and Φ is R-canonical, then a R-canonical solution Ψ to (3) (i.e. a Ψ such that (3) holds for some Y) always exists, and it is unique. Indeed, in order to compute Ψ from Φ the following procedure can be used. Consider the polynomial matrix equation

$$X(-\xi)^\top R(\xi) + R(-\xi)^\top X(\xi) = \Phi(-\xi,\xi) \tag{4}$$

in the unknown R-canonical matrix $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. Then under any of the conditions of Proposition 1, equation (4) has a unique R-canonical solution X; moreover,

$$\Psi(\zeta,\eta) := \frac{X(\zeta)^\top R(\eta) + R(\zeta)^\top X(\eta) - \Phi(\zeta,\eta)}{\zeta+\eta}$$

satisfies (3) and is R-canonical. *Vice versa*, if Φ and Ψ are R-canonical, and Ψ satisfies (3), then a R-canonical X satisfying (4) can be found from Ψ as

$$X(\xi) := -\lim_{|\mu|\to\infty} \mu R(\mu)^{-\top}\Psi(\mu,\xi)$$

These results show that there is a one-one correspondence between R-canonical solutions of (3) and R-canonical solutions of (4).

The equations (3) and (4) will play an important role in the following; they are called the *two-variable* and *one-variable polynomial Lyapunov equation*, respectively.

2.3 Stationarity

Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, and consider the corresponding QDF $Q_\Phi(w)$ on $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. For a given w we define the *cost degradation* of adding the compact-support function $\delta \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ to w as

$$J_w(\delta) := \int_{-\infty}^{+\infty} (Q_\Phi(w+\delta) - Q_\Phi(w))dt.$$

The cost degradation equals $J_w(\delta) = \int_{-\infty}^{+\infty} Q_\Phi(\delta)dt + 2\int_{-\infty}^{+\infty} L_\Phi(w,\delta)dt$, where L_Φ is the *bilinear differential form* associated with Φ, defined as the functional from $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ to $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$, defined as:

$$L_\Phi(w_1,w_2) = \sum_{h,k=0}^{N} (\frac{d^h w_1}{dt^h})^T \Phi_{h,k} \frac{d^k w_2}{dt^k}.$$

We call the integral $2\int_{-\infty}^{+\infty} L_\Phi(w,\delta)dt$ the *variation associated with* w. It defines a linear functional which associates with every $\delta \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ a real number $2\int_{-\infty}^{+\infty} L_\Phi(w,\delta)dt$. We call w a *stationary trajectory* of Q_Φ if the variation associated with w is the zero functional. The following proposition (for a proof, see Proposition 4.1 of Rapisarda et al. (2004)) establishes a representation of all stationary trajectories of given QDF Q_Φ. In the following, for a given two-variable polynomial matrix $\Phi(\zeta,\eta)$, $\partial\Phi(\xi)$ is defined as the one-variable polynomial matrix $\Phi(-\xi,\xi)$.

Proposition 1. Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be symmetric. Then $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is a stationary trajectory of the QDF Q_Φ if and only if w satisfies the system of differential equations

$$\partial\Phi\left(\frac{d}{dt}\right) w = 0. \tag{5}$$

Often a $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ and a Q_Φ are given, and it is desired to characterize the set of stationary trajectories of $\mathfrak{B}$ with respect to Q_Φ. Let $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ an observable image representation $\mathfrak{B} = \mathrm{im}\ M(\frac{d}{dt})$; in this case, the result of Proposition 1 can be applied to the QDF induced by the two-variable polynomial matrix

$$\Phi'(\zeta,\eta) := M(\zeta)^\top \Phi(\zeta,\eta) M(\eta) \tag{6}$$

2.4 Kuijper's parametrization of controllers

In Kuijper's work an essential role is played by the image and kernel representation of the behavior, respectively equation (2) and (1). In the following we assume that the kernel representation is "minimal", and that the image representation is "observable"; this implies $\mathtt{p}+\mathtt{m} = \mathtt{w}$. Observe that the assumption of controllability of $\mathfrak{B}$ and of minimality of (1) also implies that $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

In Kuijper's framework, the desired sub-behavior $\mathfrak{B}_{\mathrm{des}} \subset \mathfrak{B}$ which one aims to achieve after interconnection of $\mathfrak{B}$ with a controller, is assumed to be autonomous, and consequently representable in the form

$$\begin{aligned} w &= M(\frac{d}{dt})\ell \\ 0 &= D(\frac{d}{dt})\ell \end{aligned} \tag{7}$$

where $D \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ is nonsingular.

Instrumental in Kuijper's parametrization is a doubly coprime factorization over the polynomials, i.e.

$$\begin{bmatrix} R \\ C_0 \end{bmatrix} [N\ M] = \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & I_{\mathtt{m}} \end{bmatrix} \tag{8}$$

where $C_0 \in \mathbb{R}^{\mathtt{m}\times\mathtt{w}}[\xi]$, $N \in \mathbb{R}^{\mathtt{w}\times\mathtt{p}}[\xi]$. The existence of this factorization is guaranteed by standard polynomial algebra arguments.

For the purposes of this paper, the most important result of Kuijper's work is the following (see Theorem 3.3 p. 624 of Kuijper (1995)).

Theorem 2. Let $\mathfrak{B}_{\mathrm{des}}$ be described as in (7), and let (8) be a doubly-coprime factorization. A controller behavior $\mathfrak{B}_c$ is such that $\mathfrak{B} \cap \mathfrak{B}_c = \mathfrak{B}_{des}$, if and only if there exist $G \in \mathbb{R}^{\mathtt{m}\times\mathtt{p}}[\xi]$, and $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ unimodular, such that $\mathfrak{B}_c = \ker\ C(\frac{d}{dt})$, where

$$C = GR + UDC_0 \tag{9}$$

In the next section we show how the result of Theorem 2 relates to Lyapunov functions.

3. STABILIZING CONTROLLERS AND LYAPUNOV FUNCTIONS

In the rest of this paper, we will make use of the following rather straightforward consequence of the material presented in the previous section, namely that the autonomous interconnected system with behavior $\mathfrak{B}_{\text{des}} = \mathfrak{B} \cap \mathfrak{B}_c$ is described in kernel form by the matrix

$$\hat{R} := \begin{bmatrix} R \\ C \end{bmatrix} = \begin{bmatrix} I_{\mathtt{p}} & 0 \\ G & U \end{bmatrix} \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} R \\ C_0 \end{bmatrix} \tag{10}$$

Observe that the first- and the last matrix in the factorization (10) are unimodular. Observe also that $\text{col}(R, C_0)$ defines a one-one correspondence between $\mathfrak{B}_{\text{des}}$ and the set

$$\left\{ \text{col}(0_{\mathtt{p}}, \ell) \mid \ell \in D\left(\frac{d}{dt}\right) \right\} \tag{11}$$

This correspondence implies that if $\mathfrak{B}_{\text{des}}$ is asymptotically stable, then also the set (11) is asymptotically stable. Moreover, we can associate in a one-one way a Lyapunov function Q_Ψ for the behavior

$$\mathfrak{B}' := \ker R'(\frac{d}{dt}) := \ker \begin{bmatrix} I_{\mathtt{p}} & 0 \\ G\left(\frac{d}{dt}\right) & U\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D\left(\frac{d}{dt}\right) \end{bmatrix} \tag{12}$$

with a Lyapunov function $Q_{\hat{\Psi}}$ for $\ker \hat{R}(\frac{d}{dt})$ in the following way:

$$\Psi(\zeta, \eta) \to \hat{\Psi}(\zeta, \eta) := \begin{bmatrix} R(\zeta)^\top & C_0(\zeta)^\top \end{bmatrix} \Psi(\zeta, \eta) \begin{bmatrix} R(\eta) \\ C_0(\eta) \end{bmatrix}$$

An analogous relation holds for the derivative $\hat{\Phi}$ of $\hat{\Psi}$.

Now consider the $\text{col}(I_p, D)$-canonical representatives of $\hat{\Psi}$ and $\hat{\Phi}$. Observe that in this case there exists $\hat{Q} \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that the $\text{col}(I_p, D)$-canonical representative of $\hat{\Phi}(\zeta, \eta)$ is $-\hat{Q}(\zeta)^\top \hat{Q}(\eta)$ (see Willems et al. (1998)). Observe also that since $\hat{Q}$ is $\text{col}(I_p, D)$-canonical, it is of the form

$$\hat{Q} := \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix}$$

with L such that LD^{-1} is strictly proper. Moreover, from $Q_{\hat{\Phi}} \overset{\mathfrak{B}}{<} 0$ it follows that $\text{col}(L(\lambda), D(\lambda))$ is of full column rank for all $\lambda \in \mathbb{C}$.

From the one-one correspondence existing between R-canonical solutions of the two-variable polynomial Lyapunov equation and R-canonical solutions of the one-variable polynomial equation illustrated in section 2.2, it follows that the $\text{col}(I_p, D)$-canonical representative of $\hat{\Psi}$ is associated with a unique $\text{col}(I_p, D)$-canonical solution $\hat{X} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ to the univariate polynomial Lyapunov equation

$$\text{col}(I_p, D)^\sim X + X^\sim \text{col}(I_p, D) = -\hat{Q}^\sim \hat{Q} \tag{13}$$

Observe that since X is $\text{diag}(I_{\mathtt{p}}, D)$-canonical, it is of the form $\text{diag}(0_{\mathtt{p}}, Z)$ for some D-canonical $Z \in \mathbb{R}^{\mathtt{m} \times \mathtt{m}}[\xi]$ satisfying $D^\sim Z + Z^\sim D = -L^\sim L$. It follows from this discussion that the $\text{diag}(I_{\mathtt{p}}, D)$-canonical representative of $\hat{\Psi}$ is

$$\frac{1}{\zeta + \eta} \left[\begin{bmatrix} 0 \\ Z(\zeta) \end{bmatrix}^\top \begin{bmatrix} I_{\mathtt{p}} \\ D(\eta) \end{bmatrix} + \begin{bmatrix} I_{\mathtt{p}} \\ D(\eta) \end{bmatrix}^\top \begin{bmatrix} 0 \\ Z(\eta) \end{bmatrix} + \begin{bmatrix} 0 \\ L(\zeta) \end{bmatrix}^\top \begin{bmatrix} 0 \\ L(\eta)^\top \end{bmatrix} \right]$$

This expression can be rewritten as

$$\frac{1}{\zeta + \eta} \left[\begin{bmatrix} 0 \\ Z(\zeta) \end{bmatrix}^\top \begin{bmatrix} I_{\mathtt{p}} & 0 \\ G(\eta) & U(\eta) \end{bmatrix}^{-1} \begin{bmatrix} I_{\mathtt{p}} & 0 \\ G(\eta) & U(\eta) \end{bmatrix} \begin{bmatrix} I_{\mathtt{p}} \\ D(\eta) \end{bmatrix} + \begin{bmatrix} I_{\mathtt{p}} \\ D(\zeta) \end{bmatrix}^\top \begin{bmatrix} I_{\mathtt{p}} & 0 \\ G(\zeta) & U(\zeta) \end{bmatrix}^\top \begin{bmatrix} I_{\mathtt{p}} & 0 \\ G(\zeta) & U(\zeta) \end{bmatrix}^{-\top} \begin{bmatrix} 0 \\ Z(\eta) \end{bmatrix} + \begin{bmatrix} 0 \\ L(\zeta) \end{bmatrix}^\top \begin{bmatrix} 0 \\ L(\eta) \end{bmatrix} \right]$$

By pre- and post-multiplying this expression by

$$\text{col}(R(\zeta), C_0(\zeta))^\top$$

and

$$\text{col}(R(\eta), C_0(\eta))$$

respectively, we obtain

$$\frac{1}{\zeta + \eta} \left[Y(\eta, \zeta)^\top \hat{R}(\eta) + \hat{R}(\zeta)^\top Y(\zeta, \eta) + \hat{R}(\zeta)^\top \text{col}(0, L(\zeta)^\top) \text{col}(0, L(\eta)^\top) \hat{R}(\eta) \right] \tag{14}$$

where

$$Y(\zeta, \eta) := \begin{bmatrix} I_{\mathtt{p}} & 0 \\ G(\zeta) & U(\zeta) \end{bmatrix}^{-\top} \begin{bmatrix} 0 \\ Z(\eta) \end{bmatrix} \begin{bmatrix} R(\eta) \\ C_0(\eta) \end{bmatrix} \tag{15}$$

The above argument proves the following statement.

Proposition 3. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, and let $\mathfrak{B}_{\text{des}}$ be described as in (7), with (8) a doubly-coprime factorization. Let $\mathfrak{B}_c$ be such that $\mathfrak{B} \cap \mathfrak{B}_c = \mathfrak{B}_{\text{des}}$, and let (10) be a kernel representation of $\mathfrak{B}_{\text{des}}$. A two-variable symmetric matrix $\Psi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ induces a Lyapunov function for $\mathfrak{B}_{\text{des}}$ if and only it is $\mathfrak{B}_{\text{des}}$-equivalent to a QDF induced by an expression (14), where

(1) $L \in \mathbb{R}^{\mathtt{m} \times \mathtt{m}}[\xi]$ is such that $\text{col}(D(\lambda), L(\lambda))$ has full column rank for all $\lambda \in \mathbb{C}$;
(2) $Z \in \mathbb{R}^{\mathtt{m} \times \mathtt{m}}[\xi]$ is a D-canonical matrix satisfying $Z^\sim D + D^\sim Z = -L^\sim L$;
(3) Y is defined as in (15).

4. PARAMETRIZATION OF ALL SOLUTIONS TO THE INVERSE PROBLEM OF OPTIMAL CONTROL

In this section we first formulate the inverse optimal control problem in the behavioral framework; and then proceed to show how the result of Kuijper (1995) can be used in order to parametrize its solutions.

Problem 1 (inverse optimal control problem) *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be controllable, and let $\mathfrak{B}_s \subset \mathfrak{B}$ be asymptotically stable. Find a quadratic differential form Q_Φ acting on the variables of $\mathfrak{B}$ such that*

(1) $\int Q_\Phi \overset{\mathfrak{B}}{\geq} 0$;

(2) *$\mathfrak{B}_s$ consists of all stable trajectories of $\mathfrak{B}$ stationary with respect to Q_Φ.*

The main result of this section is the following.

Proposition 4. Let a doubly coprime factorization (8) be given, and assume that $\mathfrak{B}_s$ of the statement of Problem 1 is described as in (7). Then Q_Φ solves the inverse optimal control problem if and only if there exist $F \in$

$\mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\zeta,\eta]$, $G \in \mathbb{R}^{\mathtt{m}\times\mathtt{p}}[\xi]$, a symmetric $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, and a unimodular matrix $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ such that

$$\begin{aligned}\Phi(\zeta,\eta) &= (\zeta+\eta)\Psi(\zeta,\eta)\\ &+ \begin{bmatrix} R(\zeta) \\ C_0(\zeta)\end{bmatrix}^\top \begin{bmatrix} I_{\mathtt{p}} & G(\zeta)^\top \\ 0 & D(\zeta)^\top U(\zeta)^\top\end{bmatrix}\\ &\cdot \begin{bmatrix} I_{\mathtt{p}} & 0 \\ G(\eta) & U(\eta)D(\eta)\end{bmatrix}\begin{bmatrix} R(\eta) \\ C_0(\eta)\end{bmatrix} \qquad (16)\end{aligned}$$

Proof. We prove necessity first. Let $\Phi(\zeta,\eta) \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be a solution to the inverse optimal control problem. Observe that condition 2) of the inverse optimal control problem and the result of Proposition 1 implies that there exists a polynomial matrix $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that

$$\partial\Phi = SR'$$

where R' induces a kernel representation of $\mathfrak{B}_{\rm des}$. Moreover, from the fact that $\partial\Phi$ is para-Hermitian it follows that $\partial\Phi = R'^{\sim}S^{\sim}$. A simple argument shows that there exists a para-Hermitian, unimodular matrix $V \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that $\partial\Phi = R'^{\sim}VR$.

Now from $\partial\Phi = R'^{\sim}VR'$ it follows (see Theorem 3.1 p. 1711 of Willems et al. (1998)) that there exists a $\Psi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that

$$\Phi(\zeta,\eta) = R'^\top(\zeta)\Phi'(\zeta,\eta)R'(\eta) + (\zeta+\eta)\Psi(\zeta,\eta)$$

where $\Phi'(\zeta,\eta) \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is such that $\partial\Phi' = V$. Observe that for example, $\Phi'(\zeta,\eta) = \frac{1}{2}V(\zeta)^\top + \frac{1}{2}V(\eta)$ is such a matrix.

We now investigate further the structure of $\Phi'(\zeta,\eta)$. Apply the result of Theorem 2 in order to conclude that the polynomial matrix R' of the kernel representation of $\mathfrak{B}_{\rm des}$ can be chosen as

$$R' = \begin{bmatrix} I_{\mathtt{p}} & 0 \\ G' & U'D\end{bmatrix}\begin{bmatrix} R'' \\ C_0\end{bmatrix}$$

for some minimal kernel representation R'' of $\mathfrak{B}$, some $G' \in \mathbb{R}^{\mathtt{m}\times\mathtt{p}}[\xi]$, and unimodular $U' \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$. Partition $\Phi'(\zeta,\eta)$ as

$$\Phi'(\zeta,\eta) = \begin{bmatrix} \Phi'_{11}(\zeta,\eta) & \Phi'_{12}(\zeta,\eta) \\ \Phi'_{12}(\eta,\zeta)^\top & \Phi'_{22}(\zeta,\eta)\end{bmatrix}$$

with $\Phi'_{11} \in \mathbb{R}_s^{\mathtt{p}\times\mathtt{p}}[\zeta,\eta]$, $\Phi'_{12} \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\zeta,\eta]$, and $\Phi'_{22} \in \mathbb{R}_s^{\mathtt{m}\times\mathtt{m}}[\zeta,\eta]$.

We now prove that since the set of stable stationary trajectories of Q_Φ is $\mathfrak{B}_{\rm des}$, it follows that $\partial\Phi'_{22}$ is unimodular. Indeed, assume by contradiction that this is not the case; then there exists a nonzero trajectory $w'_{22} \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}})$, such that $\lim_{t\to\infty} w'_{22}(t) = 0$, which is a stationary trajectory for $Q_{\Phi'_{22}}$. Recall that R' is nonsingular, and consequently it represents a surjective polynomial differential operator from $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ to itself. Consequently there exists a trajectory $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ such that $R'\left(\frac{d}{dt}\right)w = \mathrm{col}(0_{\mathtt{p}}, w'_{22})$. It is easy to see that since $\mathfrak{B}_{\rm des}$ is asymptotically stable, and consequently $R'(\lambda)$ loses rank only for $\lambda \in \mathbb{C}_-$, it must hold that $\lim_{t\to\infty} w(t) = 0$. Evidently $w \notin \mathfrak{B}_{des} = \ker\ R'\left(\frac{d}{dt}\right)$, since $w'_{22} \neq 0$; moreover, w is also stationary for $Q_{\Phi'}$, since it belongs to $\ker\partial\Phi'\left(\frac{d}{dt}\right)$. This contradicts the fact that the set of stable stationary trajectories of $Q_{\Phi'}$ equals $\mathfrak{B}_{\rm des}$, and proves that $\partial\Phi'_{22}$ is unimodular.

Now observe that from $\partial\Phi'(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ it follows that there exists a unimodular matrix $U'' \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ such that $\partial\Phi'_{22} = U''^{\sim}U''$. Observe that

$$\begin{aligned}\partial\Phi'(\xi) = &\begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & U''^{\sim}(\xi)\end{bmatrix}\\ &\begin{bmatrix} \partial\Phi'_{11}(\xi) & \partial\Phi'_{12}(-\xi,\xi)U''(\xi)^{-1} \\ (U''^{\sim}(\xi))^{-\top}\partial\Phi'_{12}(\xi,-\xi)^\top & I_{\mathtt{m}}\end{bmatrix}\\ &\begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & U''(\xi)\end{bmatrix}\end{aligned}$$

and moreover that

$$\begin{aligned}&\begin{bmatrix} \partial\Phi'_{11}(\xi) & \partial\Phi'_{12}(\xi)U''(\xi)^{-1} \\ (U''^{\sim}(\xi))^{-\top}\partial\Phi'_{12}(\xi)^\top & I_{\mathtt{m}}\end{bmatrix}\\ &= \begin{bmatrix} I_{\mathtt{p}} & \Phi'_{12}(\xi,-\xi)^\top U''(\xi)^{-1} \\ 0 & I_{\mathtt{m}}\end{bmatrix}\\ &\begin{bmatrix} \partial\Phi'_{11}(\xi) - (U''^{\sim}(\xi))^{-\top}\partial\Phi'_{12}(\xi)\partial\Phi'_{12}(\xi)^\top U''(\xi)^{-1} & 0 \\ 0 & I_{\mathtt{m}}\end{bmatrix}\\ &\begin{bmatrix} I_{\mathtt{p}} & 0 \\ (U''^{\sim}(\xi))^{-1}\partial\Phi'_{12}(\xi)^\top & I_{\mathtt{m}}\end{bmatrix}\end{aligned}$$

Observe that since $\partial\Phi'$ is unimodular, also the matrix

$$\partial\Phi'_{11}(\xi) - (U''^{\sim}(\xi))^{-\top}\partial\Phi'_{12}(\xi)\partial\Phi'_{12}(-\xi)^\top U''(\xi)^{-1}$$

is unimodular. Moreover, since $\partial\Phi'(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, the matrix

$$\partial\Phi'_{11}(\xi) - (U''^{\sim}(\xi))^{-\top}\partial\Phi'_{12}(\xi)\partial\Phi'_{12}(-\xi)^\top U''(\xi)^{-1}$$

is also nonnegative definite on the imaginary axis. It follows by standard results on spectral factorization that there exists a unimodular matrix $V' \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ such that

$$\begin{aligned}&V'^{\sim}(\xi)V'(\xi) =\\ &\partial\Phi'_{11}(\xi) - (U''^{\sim}(\xi))^{-1}\partial\Phi'_{12}(-\xi)^\top\Phi'_{12}(\xi,-\xi)^\top U''(\xi)^{-1}\end{aligned}$$

Now define

$$\begin{aligned}R &:= V'R''\\ G(\xi) &:= (U''^{\sim}(\xi))^{-1}\Phi'_{12}(\xi,-\xi)^\top G'(\xi)\\ U(\xi) &:= U''(\xi)V'(\xi)\end{aligned}$$

and conclude that equation (16) holds. This concludes the proof of necessity.

Sufficiency follows easily by computing $\partial\Phi$ for the two-variable polynomial matrix Φ defined in (16). ■

Remark 2. Using the framework developed in Willems et al. (2005) it can be shown that the inverse problem of optimal control as stated in this section is equivalent to the following problem in the theory of dissipativity.

Problem 1 (dissipativity version)
Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be controllable, and let $\mathfrak{B}_s \subset \mathfrak{B}$ be asymptotically stable. Find a supply rate defined by a quadratic differential form Q_Φ acting on the variables of $\mathfrak{B}$, such that

(1) *$\mathfrak{B}$ is dissipative with respect to Q_Φ;*

(2) *$\mathfrak{B}_s$ consists of all stable trajectories of $\mathfrak{B}$ with zero-dissipation respect to Q_Φ.*

The result of Proposition 4 can be used in order to parametrize the solutions to this problem. We will not enter into this ramification of our results here.

CONCLUSIONS

The main results of this paper are Propositions 3 and 4, which use the parametrization of controllers introduced in Kuijper (1995) in order to investigate the structure respectively of the set of Lyapunov functions for a given stable behavior obtained by interconnection of a plant and a controller; and the solutions to the inverse problem of optimal control defined in section 4 of this paper.

Current research is concentrated on extending the application of Kuijper's results in the development of a unifying point of view on dissipativity and stabilization of linear, time-invariant systems described by higher-order differential equations.

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