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Dynamically Scaled Generalized Inversion for Asymptotic Stabilization of Underactuated Spacecraft Dynamics

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Abstract: Novel concept of feedback linearization is introduced for smooth asymptotic stabilization of underactuated spacecraft equipped with one and two degrees of actuation. The concept is based on generalized inversion, and is aimed at asymptotically realizing a perturbation from the unrealizable feedback linearizing transformation. A desired stable second-order linear dynamics in a norm measure of the angular velocity components about the unactuated axes is prescribed. Evaluation of this dynamics along the vector field defined by the underactuated Euler's dynamical equations of angular motion results in a relation that is linear in the control variables. This relation is used to assess realizability of the desired unactuated dynamics, resulting in necessary and sufficient conditions for asymptotic stabilizability of underactuated spacecraft. Generalized inversion of the relation produces a control law that is composed of particular and auxiliary parts. The generalized inverse in the particular part is scaled by a dynamic factor that depends on the spacecraft angular velocity components about the spacecraft actuated axes, such that the generalized inverse converges uniformly to the standard Moore-Penrose generalized inverse as the transient response decays, resulting in asymptotic realization of the desired unactuated stable linear dynamics. The null-control vector in the auxiliary part of the control law is chosen for asymptotic stable perturbed feedback linearization of the actuated subsystem.

Keywords: asymptotic stabilization; asymptotic perturbed feedback linearization; underactuated spacecraft stabilization; generalized inversion control; controls coefficient generalized inversion; dynamic scaling factor; perturbed nullprojection; null-control vector.

1. INTRODUCTION

Controllability of underactuated spacecraft under different degrees and types of actuation was investigated in the seminal article [Crouch 1984], and it has continued to draw attention within the control system community during the following two decades.

An underlying feature of underactuated dynamics is that it is uncontrollable by feedback linearization, see for example [Bloch 2003]. An interesting result on feedback linearizability of underactuated spacecraft dynamics is found in [Bajodah 2007], stating that underactuation is feedback linearizable up to a perturbation from the feedback linearizing transformation. A stabilizing control law is derived in [Bajodah 2007] based on this fact, yielding arbitrarily small uniform ultimate bounds of the spacecraft angular velocity components. The primary tool used is the controls coefficient generalized inverse (CCGI) and the controls coefficient nullspace parametrization of redundancy in control authority, first introduced to control system design in [Bajodah et al. 2005].

The common factor between former control engineering applications of generalized inverses is that the systems are either fully- or over-determined, i.e., the numbers of independent solutions are either equal or exceed the numbers of constraint equations. In particular, the Moore-Penrose generalized inverse (MPGI) has been utilized in control system design for control variables allocation in overactuated control system. It is illustrated in [Bajodah 2007] that the MPGI can also be utilized to solve the counter problem, i.e., as a means of controlling underactuated systems, where the numbers of degrees of freedom to be controlled exceed the numbers of independent control variables. This is motivated by the fact that redundancy in control systems is ultimately in the control process itself rather than in the control variables, and a dynamical system may be *dynamically redundant*, although underactuated. That is, if the dynamical system is controllable then there exists no unique strategy to control it, regardless of its degree of actuation.

A well known obstacle in the way of employing the MPGI of matrices having variable elements is singularity of the generalized inverse. In this paper, we introduce a novel type of generalized inversion, based on scaling of the MPGI by a dynamic factor that depends on the vector norm of the angular velocity components about the actuated axes. The scaling factor vanishes as these components vanish, such that the modified generalized inverse uniformly converges to the standard MPGI, asymptotically realizing the desired unactuated dynamics.

We begin by partitioning the underactuated Euler's system of equations into actuated and unactuated subsystems, and we provide a condition the satisfaction of which guarantees the capability of the available control authority to realize a desired linear dynamics of the unactuated subsystem. The controls coefficient generalized inversion paradigm is used thereafter to design the control law, and the null-control vector is chosen to produce a perturbed feedback linearization of the actuated subsystem, in asymptotic feedback linearization of the unactuated subsystem, and in asymptotic stabilization of spacecraft equipped with one and two degrees of actuation.

The contribution of this paper is twofold. First, detailed necessary and sufficient asymptotic stabilizability conditions are derived for underactuated spacecraft having arbitrary inertia properties and equipped with one and two degrees of actuation. Second, the controls coefficient generalized inversion design methodology is modified via the dynamically scaled generalized inverse (DSGI) to yield asymptotic stabilization of underactuated spacecraft with one and two degrees of actuation.

1.1 Partitioned Form of Euler's Equations of Angular Motion

The Euler's model of underactuated spacecraft dynamics is given by the system of differential equations

$$\dot{\omega} = \mathcal{S}(\omega)\omega + \tau, \qquad \omega(0) = \omega_0 \tag{1}$$

where $\omega \in \mathbb{R}^{3\times 1}$ is the vector of angular velocities about the spacecraft's body-fixed axes, $\mathcal{S}(\omega) \in \mathbb{R}^{3\times 3}$ is given by

$$\mathcal{S}(\omega) = J^{-1}\widetilde{\omega}J \tag{2}$$

such that $J \in \mathbb{R}^{3\times 3}$ is the matrix containing the spacecraft's body moments of inertia, and is given by

$$J = \begin{bmatrix} J_{11} & -J_{12} & -J_{13} \\ -J_{12} & J_{22} & -J_{23} \\ -J_{13} & -J_{23} & J_{33} \end{bmatrix}, \tag{3}$$

$\widetilde{\omega}$ is a skew symmetric matrix of the form

$$\widetilde{\omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{4}$$

and $\tau \in \mathbb{R}^{3\times 1}$ is the scaled control vector. Let d be the degree of actuation of the spacecraft, i.e., number of independent gas jet actuator pairs. The vectors ω and τ can be put in the partitioned forms

$$\omega = \begin{bmatrix}\omega_u^T \ \omega_a^T\end{bmatrix}^T, \qquad \tau = \begin{bmatrix}\mathbf{0}^T \ u^T\end{bmatrix}^T \tag{5}$$

where $\omega_u \in \mathbb{R}^{(3-d)\times 1}$ is the vector of angular velocities about the unactuated spacecraft's body axes, $\omega_a \in \mathbb{R}^{d\times 1}$ is the vector of angular velocities about the actuated spacecraft's body axes, $u \in \mathbb{R}^{d\times 1}$ is the scaled vector of available control torques, and $\mathbf{0} \in \mathbb{R}^{(3-d)\times 1}$ contains zero elements. The matrix $\mathcal{S}(\omega)$ is partitioned compatibly as

$$\mathcal{S}(\omega) = \begin{bmatrix} \mathcal{S}_{11}(\omega) & \mathcal{S}_{12}(\omega) \\ \mathcal{S}_{21}(\omega) & \mathcal{S}_{22}(\omega) \end{bmatrix} \tag{6}$$

where $\mathcal{S}_{11} \in \mathbb{R}^{(3-d)\times(3-d)}$, $\mathcal{S}_{12} \in \mathbb{R}^{(3-d)\times d}$, $\mathcal{S}_{21} \in \mathbb{R}^{d\times(3-d)}$, and $\mathcal{S}_{22} \in \mathbb{R}^{d\times d}$. Hence, the Euler's system given by (1) splits into two coupled subsystems. The first one is unactuated, and is given by the equations

$$\dot{\omega}_u = \mathcal{S}_{11}(\omega)\omega_u + \mathcal{S}_{12}(\omega)\omega_a \tag{7}$$

and the second one is fully-actuated, and is given by the equations

$$\dot{\omega}_a = \mathcal{S}_{21}(\omega)\omega_u + \mathcal{S}_{22}(\omega)\omega_a + u. \tag{8}$$

2. REALIZABILITY OF LINEAR SPACECRAFT DYNAMICS

To set a feedback linearizing transformation for the unactuated subsystem, we define a function $\phi(\omega_u) : \mathbb{R}^{(3-d)\times 1} \to \mathbb{R}$ such that ϕ is at least globally twice continuously differentiable in ω_u and such that

$$\phi(\omega_u) = 0 \Leftrightarrow \omega_u = \mathbf{0}_{(3-d)\times 1}, \tag{9}$$

and we use it to specify the stable linear second-order dynamics

$$\ddot{\phi} + c_1\dot{\phi} + c_2\phi = 0, \ c_1, \ c_2 > 0. \tag{10}$$

The first two time derivatives of $\phi(\omega_u)$ along the vector field of Euler's equations of motion (1), $\dot{\phi}$ and $\ddot{\phi}$, are given by

$$\dot{\phi} = L_f\phi(\omega_u) \ , \ \ \ddot{\phi} = L_f^2\phi(\omega_u) + \frac{\partial}{\partial\omega_a}\left[L_f\phi(\omega_u)\right]u \tag{11}$$

where $L_f\phi(\omega_u)$ and $L_f^2\phi(\omega_u)$ are the first and second Lie derivatives [Slotine 1991] of $\phi(\omega_u)$ along $f(\omega) := \mathcal{S}(\omega)\omega$. With $\dot{\phi}$ and $\ddot{\phi}$ given by (11), it is possible to write (10) in the pointwise-linear form

$$\mathcal{A}(\omega)u = \mathcal{B}(\omega), \tag{12}$$

where $\mathcal{A}(\omega) \in \mathbb{R}^{1\times d}$ is given by

$$\mathcal{A}(\omega) = \frac{\partial}{\partial\omega_a}L_f\phi(\omega_u) \tag{13}$$

and $\mathcal{B}(\omega) \in \mathbb{R}$ is given by

$$\mathcal{B}(\omega) = -L_f^2\phi(\omega_u) - c_1L_f\phi(\omega_u) - c_2\phi(\omega_u). \tag{14}$$

The row vector $\mathcal{A}(\omega)$ is the *controls coefficient relative to* $\phi(\omega_u)$ *of the dynamics given by (10) along the spacecraft trajectories*, and the scalar $\mathcal{B}(\omega)$ is the corresponding *controls load*.

Definition 1. The dynamics given by (10) is said to be realizable by the underactuated Euler's system of equations at some value of ω if there exists a control vector u that solves (12) at that value of ω. If this is true for all $\omega \in \mathbb{R}^{3\times 1} \neq \mathbf{0}_{3\times 1}$, then the dynamics given by (10) is said to be globally realizable by the underactuated Euler's system of equations.

Definition 2. The zero actuated state Jacobian of the controls coefficient is defined as the square matrix resulting from differentiating the controls coefficient with respect to ω_a, evaluated at $\omega_a = \mathbf{0}_{d\times 1}$

$$\mathcal{J}_a(\omega_u) = \left[\frac{\partial\mathcal{A}(\omega)}{\partial\omega_a}\right]_{\omega_a = \mathbf{0}_{d\times 1}}. \tag{15}$$

For proofs of the following two propositions and the following theorem, the reader is referred to [Bajodah 2007].

Proposition 1. Let $\mathcal{A}(\omega)$ be the controls coefficient relative to $\phi(\omega_u)$ of a dynamics given by (10) that is globally realizable by the underactuated Euler's system of equations. Then

$$\mathcal{A}(\omega) = \mathbf{0}_{1\times d} \Leftrightarrow \omega = \mathbf{0}_{3\times 1}. \tag{16}$$

Proposition 2. The unactuated dynamics given by (10) is globally realizable by the underactuated Euler's system of equations *if and only if*

$$\det\left[\mathcal{J}_a(\omega_u)\right] \neq 0 \quad \forall\, \omega_u \neq \mathbf{0}_{(3-d)\times 1}. \tag{17}$$

Theorem 1. If the linear unactuated dynamics given by (10) along the underactuated Euler's system given by (1) has a nonsingular zero actuated state Jacobian $\mathcal{J}_a(\omega_u)$ of the controls coefficient $\mathcal{A}(\omega)$ for all $\omega_u \neq \mathbf{0}_{(3-d)\times 1}$, then the family of *all* controllers that realize the unactuated dynamics by the underactuated Euler's equations of motion are given by

$$u = \mathcal{A}^+(\omega)\mathcal{B}(\omega) + \mathcal{P}(\omega)y \tag{18}$$

where "$\mathcal{A}^+$" stands for the MPGI of the controls coefficient, and is given by

$$\mathcal{A}^{+}(\omega)=\begin{cases}\dfrac{\mathcal{A}^{T}(\omega)}{\mathcal{A}(\omega)\mathcal{A}^{T}(\omega)}, & \mathcal{A}(\omega)\neq\mathbf{0}_{1\times d}\\ \mathbf{0}_{d\times 1}, & \mathcal{A}(\omega)=\mathbf{0}_{1\times d}\end{cases} \quad (19)$$

and $\mathcal{P}(\omega)\in\mathbb{R}^{d\times d}$ is the corresponding controls coefficient nullprojector (CCNP), given by

$$\mathcal{P}(\omega)=I_{d\times d}-\mathcal{A}^{+}(\omega)\mathcal{A}(\omega) \quad (20)$$

and $y\in\mathbb{R}^{d\times 1}$ is an arbitrarily chosen *null-control vector*.

3. SPACECRAFT STABILIZABILITY ASSESSMENT

3.1 Case 1: Two Degrees of Actuation $(d=2)$

The actuated body axes in this case are the ones about which the spacecraft angular velocity components are ω_2 and ω_3, and the unactuated body axis is the one about which the spacecraft angular velocity component is ω_1. Therefore, $\mathcal{S}_{11}\in\mathbb{R}$, $\mathcal{S}_{12}\in\mathbb{R}^{1\times 2}$, $\mathcal{S}_{21}\in\mathbb{R}^{2\times 1}$, $\mathcal{S}_{22}\in\mathbb{R}^{2\times 2}$, $\omega_u=\omega_1$, and $\omega_a=[\omega_2\ \omega_3]^T$, and $\tau=[0\ u]^T$, where $u\in\mathbb{R}^{2\times 1}$. The function ϕ may be chosen to be

$$\phi(\omega_u)=\omega_1^k \quad (21)$$

where k is any positive real number. For $k=1$, the resulting zero actuated state Jacobian $\mathcal{J}_a(\omega_u)\in\mathbb{R}^{2\times 2}$ has the elements:

$$\mathcal{J}_{a_{(1,1)}}=2\frac{I_{12}(I_{12}I_{23}+I_{13}I_{22})-I_{23}(I_{22}I_{33}-I_{23}^2)}{D} \quad (22)$$

$$\begin{aligned}\mathcal{J}_{a_{(1,2)}}=\mathcal{J}_{a_{(2,1)}}=&\\ -\frac{I_{12}(I_{12}I_{33}+I_{13}I_{23})-I_{13}(I_{12}I_{23}+I_{13}I_{22})}{D}&\\ -\frac{I_{22}(I_{22}I_{33}-I_{23}^2)-I_{33}(I_{22}I_{33}-I_{23}^2)}{D}&\end{aligned} \quad (23)$$

$$\mathcal{J}_{a_{(2,2)}}=-2\frac{I_{13}(I_{12}I_{33}+I_{13}I_{23})-I_{23}(I_{22}I_{33}-I_{23}^2)}{D} \quad (24)$$

where

$$\begin{aligned}D=&\,I_{13}(I_{12}I_{23}+I_{13}I_{22})\\ &+I_{23}(I_{11}I_{23}+I_{12}I_{13})-I_{33}(I_{11}I_{22}-I_{12}^2).\end{aligned} \quad (25)$$

Therefore, the condition given by (17) on $\mathcal{J}_a(\omega_u)$ implies that

$$\begin{aligned}&[I_{12}(I_{12}I_{33}+I_{13}I_{23})-I_{13}(I_{12}I_{23}+I_{13}I_{22})\\ &\quad+I_{22}(I_{22}I_{33}-I_{23}^2)-I_{33}(I_{22}I_{33}-I_{23}^2)]^2\\ &\quad+4[I_{13}(I_{12}I_{33}+I_{13}I_{23})-I_{23}(I_{22}I_{33}-I_{23}^2)]\\ &\quad[I_{12}(I_{12}I_{23}+I_{13}I_{22})-I_{23}(I_{22}I_{33}-I_{23}^2)]\neq 0.\end{aligned} \quad (26)$$

In particular,

(1) The spacecraft is asymptotically stabilizable by two torque actuators that are mounted in a common body fixed plane or in distinct body fixed planes, provided that the condition given by (26) is satisfied.
(2) The spacecraft is asymptotically stabilizable by a pair of torque actuators that are mounted on two arbitrarily chosen body fixed axes in a common principal plane of inertial symmetry, i.e., $I_{12}=I_{13}=0$ and $I_{22}=I_{33}$, unless $I_{23}(I_{22}^2-I_{23}^2)=0$ (equivalently $I_{23}(I_{33}^2-I_{23}^2)=0$).
(3) The spacecraft is asymptotically stabilizable by two torque actuators that are mounted along two axes that belong to a principal system of axes, i.e., $I_{12}=I_{13}=I_{23}=0$, unless the third (unactuated) principal axis is an axis of inertial symmetry, i.e., $I_{22}=I_{33}$ [Brocket 1983].

3.2 Case 2: One Degree of Actuation $(d=1)$

The actuated body axis in this case is the one about which the spacecraft angular velocity component is ω_3, and the unactuated body axes are the ones about which the spacecraft angular velocity components are ω_1 and ω_2. Therefore, $\mathcal{S}_{11}\in\mathbb{R}^{2\times 2}$, $\mathcal{S}_{12}\in\mathbb{R}^{2\times 1}$, $\mathcal{S}_{21}\in\mathbb{R}^{1\times 2}$, $\mathcal{S}_{22}\in\mathbb{R}$, $\omega_u=[\omega_1\ \omega_2]^T$, $\omega_a=\omega_3$, and $\tau=[0\ 0\ u]^T$, where $u\in\mathbb{R}$. The function ϕ may be chosen to be

$$\phi(\omega_u)=\omega_1^2+\omega_2^2. \quad (27)$$

The zero actuated state Jacobian $\mathcal{J}_a(\omega_u)$ is obtained as

$$\begin{aligned}\mathcal{J}_a(\omega_u)=&\frac{4\left[I_{23}(I_{12}I_{33}+I_{13}I_{23})-I_{13}(I_{11}I_{33}-I_{13}^2)\right]\omega_2}{D}\\ &-\frac{4\left[I_{13}(I_{12}I_{33}+I_{13}I_{23})-I_{23}(I_{22}I_{33}-I_{23}^2)\right]\omega_1}{D}.\end{aligned} \quad (28)$$

Therefore, the condition given by (17) reduces to the following two conditions

$$I_{23}(I_{12}I_{33}+I_{13}I_{23})-I_{13}(I_{11}I_{33}-I_{13}^2)\neq 0 \quad (29)$$

$$I_{13}(I_{12}I_{33}+I_{13}I_{23})-I_{23}(I_{22}I_{33}-I_{23}^2)\neq 0. \quad (30)$$

In particular,

(1) The spacecraft is *not* asymptotically stabilizable by a single torque actuator that is mounted on a principal axis, i.e., if $I_{13}=I_{23}=0$ [Aeyels & Szafranski 1988].
(2) The spacecraft is *not* stabilizable by a single torque actuator that is mounted in a principal plane, i.e., if $I_{12}=I_{13}=0$ or if $I_{12}=I_{23}=0$.
(3) If the axis system is chosen such that $I_{13}=0$, then the spacecraft is *not* asymptotically stabilizable by a single torque actuator that is mounted such that $I_{23}^2=I_{22}I_{33}$.
(4) If the axis system is chosen such that $I_{23}=0$, then the spacecraft is *not* asymptotically stabilizable by a single torque actuator that is mounted such that $I_{13}^2=I_{11}I_{33}$.
(5) If the axis system is chosen such that $I_{12}=0$, then the spacecraft is *not* asymptotically stabilizable by a single torque actuator that is mounted such that $I_{33}=(I_{13}^2+I_{23}^2)/I_{11}$ or such that $I_{33}=(I_{13}^2+I_{23}^2)/I_{22}$.

4. PERTURBED NULLPROJECTION

A fundamental property of nullprojection matrices is that they are rank deficient. To facilitate introducing the present methodology, we construct the full rank *perturbed controls coefficient nullprojector* by perturbing the controls coefficient nullprojection matrix to disencumber its rank deficiency.

Definition 3. The perturbed CCNP $\widetilde{\mathcal{P}}(\omega,\delta)$ is defined as

$$\widetilde{\mathcal{P}}(\omega,\delta):=I_{d\times d}-h(\delta)\mathcal{A}^{+}(\omega)\mathcal{A}(\omega) \quad (31)$$

where $h(\delta):\mathbb{R}\to\mathbb{R}$ is any continuous function such that

$$h(\delta)=1 \quad \text{if and only if} \quad \delta=0. \quad (32)$$

Proposition 3. The perturbed CCNP $\widetilde{\mathcal{P}}(\omega,\delta)$ is of full rank for all $\delta\neq 0$.

Proof: The singular value decomposition of $\mathcal{A}(\omega)$ is given by

$$\mathcal{A}(\omega)=\mathbf{\Sigma}(\omega)\mathcal{V}^T(\omega) \quad (33)$$

where

$$\mathbf{\Sigma}(\omega)=\left[\ \|\mathcal{A}(\omega)\|_2 \quad \mathbf{0}_{1\times(d-1)}\right] \quad (34)$$

and $\mathcal{V}(\omega) \in \mathbb{R}^{d\times d}$ is orthonormal. By inspecting the four conditions identifying the MPGI, it can be easily verified that it is given for $\mathcal{A}(\omega)$ by

$$\mathcal{A}^{+}(\omega) = \mathcal{V}(\omega)\mathbf{\Sigma}^{+}(\omega) \tag{35}$$

where $\mathbf{\Sigma}^{+}(\omega)$ is the MPGI of $\mathbf{\Sigma}(\omega)$

$$\mathbf{\Sigma}^{+}(\omega) = \begin{bmatrix} \dfrac{1}{\| \mathcal{A}(\omega) \|_2} & \mathbf{0}_{1\times(d-1)} \end{bmatrix}^T \tag{36}$$

Therefore,

$$\mathcal{A}^{+}(\omega)\mathcal{A}(\omega) = \mathcal{V}(\omega)\mathbf{\Sigma}^{+}(\omega)\mathbf{\Sigma}(\omega)\mathcal{V}^T(\omega) \tag{37}$$

The right hand side of (37) is a singular value decomposition of $\mathcal{A}^{+}(\omega)\mathcal{A}(\omega)$, where the diagonal matrix $\mathbf{\Sigma}^{+}(\omega)\mathbf{\Sigma}(\omega)$ contains the singular values of $\mathcal{A}^{+}(\omega)\mathcal{A}(\omega)$ as its diagonal elements

$$\mathbf{\Sigma}^{+}(\omega)\mathbf{\Sigma}(\omega) = \begin{bmatrix} 1 & \mathbf{0}_{1\times(d-1)} \\ \mathbf{0}_{(d-1)\times 1} & \mathbf{0}_{(d-1)\times(d-1)} \end{bmatrix} \tag{38}$$

Consequently,

$$\widetilde{\mathcal{P}}(\omega,\delta) = I_{d\times d} - h(\delta)\mathcal{A}^{+}(\omega)\mathcal{A}(\omega) \tag{39}$$

$$= I_{d\times d} - h(\delta)\mathcal{V}(\omega)\mathbf{\Sigma}^{+}(\omega)\mathbf{\Sigma}(\omega)\mathcal{V}^T(\omega) \tag{40}$$

$$= \mathcal{V}(\omega)[I_{d\times d} - h(\delta)\mathbf{\Sigma}^{+}(\omega)\mathbf{\Sigma}(\omega)]\mathcal{V}^T(\omega) \tag{41}$$

$$= \mathcal{V}(\omega)\begin{bmatrix} 1-h(\delta) & \mathbf{0}_{1\times(d-1)} \\ \mathbf{0}_{(d-1)\times 1} & I_{(d-1)\times(d-1)} \end{bmatrix}\mathcal{V}^T(\omega) \tag{42}$$

which is of full rank for all $\delta \neq 0$.

Proposition 4. The controls coefficient nullprojector $\mathcal{P}(\omega)$ commutes with its inverted perturbation $\widetilde{\mathcal{P}}^{-1}(\omega,\delta)$ for all $\delta \neq 0$. Furthermore, their matrix multiplication equals to the controls coefficient nullprojector itself, i.e.,

$$\mathcal{P}(\omega)\widetilde{\mathcal{P}}^{-1}(\omega,\delta) = \widetilde{\mathcal{P}}^{-1}(\omega,\delta)\mathcal{P}(\omega) = \mathcal{P}(\omega). \tag{43}$$

Proof: The first part of the identities follows from the Morrison-Sherman-Woodbery matrix inversion lemma

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1} \tag{44}$$

with $A = I_{d\times d}$, $B = -h(\delta)I_{d\times d}$, $C = I_{d\times d}$, and $D = \mathcal{A}^{+}(\omega)\mathcal{A}(\omega)$. The second part of the identities (43) is obtained by interchanging the definitions of B and D in the lemma and proceeding in the same manner, see [Bajodah 2008] for a detailed proof.

5. DYNAMICALLY SCALED GENERALIZED INVERSION

5.1 Singularity Analysis

Let the function $\phi(\omega_u) : \mathbb{R}^{(3-d)\times 1} \to \mathbb{R}$ be globally twice continuously differentiable and satisfies the condition given by (9). Proposition 1 implies that if the dynamics given by (10) is globally realizable by the underactuated Euler's system of equations, then

$$\lim_{\omega\to\mathbf{0}_{3\times 1}} \mathcal{A}(\omega) = \mathbf{0}_{1\times d}. \tag{45}$$

Accordingly, the definition of the $\mathcal{A}^{+}(\omega)$ given by (19) implies that for a nonzero initial condition ω_0, $\mathcal{A}^{+}(\omega)$ goes unbounded as the spacecraft detumbles. This is a source of instability for the closed loop system because it causes the control law expression given by (18) to become unbounded. A solution to this problem is made by switching the value of the CCGI according to (19) to $\mathcal{A}^{+}(\omega) = \mathbf{0}_{d\times 1}$ when the controls coefficient $\mathcal{A}(\omega)$ approaches singularity, which is equivalent to deactivating the particular part of the control law as the closed loop system reaches steady state [Bajodah 2007]. To avoid such a discontinuity in the control law, the growth-controlled *dynamically scaled generalized inverse* is introduced next.

5.2 Dynamically Scaled Generalized Inverse

Definition 4. The DSGI $\mathcal{A}_s^{+}(\omega) \in \mathbb{R}^{d\times 1}$ is given by

$$\mathcal{A}_s^{+}(\omega) = \frac{\mathcal{A}^T(\omega)}{\mathcal{A}(\omega)\mathcal{A}^T(\omega) + \|\omega_a\|_p^p} \tag{46}$$

where $\|\omega_a\|_p$ is the vector p norm of ω_a for some positive dynamic scaling power integer p.

5.3 Properties of the Dynamically Scaled Generalized Inverse

The following properties can be verified by direct evaluation of the CCGI $\mathcal{A}^{+}(\omega)$ given by (19) and its dynamic scaling $\mathcal{A}_s^{+}(\omega)$ given by (46).

(1) $\mathcal{A}_s^{+}(\omega)\mathcal{A}(\omega)\mathcal{A}^{+}(\omega) = \mathcal{A}^{+}(\omega)\mathcal{A}(\omega)\mathcal{A}_s^{+}(\omega) = \mathcal{A}_s^{+}(\omega)$
(2) $(\mathcal{A}_s^{+}(\omega)\mathcal{A}(\omega))^T = \mathcal{A}_s^{+}(\omega)\mathcal{A}(\omega)$
(3) $\lim_{\|\omega_a\|_p\to 0} \mathcal{A}_s^{+}(\omega) = \mathcal{A}^{+}(\omega)$.

6. ASYMPTOTIC PERTURBED FEEDBACK LINEARIZATION

Theorem 2. Let $\phi(\omega_u)$ be globally twice continuously differentiable and satisfying the conditions given by (9) and (17), and let $\mathcal{A}(\omega)$ be the controls coefficient of the desired linear unactuated dynamics given by (10) relative to $\phi(\omega_u)$ along the trajectories of Euler's underactuated equations of motion (1), and let $\mathcal{B}(\omega)$ be the corresponding controls load. Also, let $\mathcal{P}(\omega)$ be the projection matrix to the nullspace of the controls coefficient $\mathcal{A}(\omega)$, given by (20). If the zero actuated state Jacobian of $\mathcal{A}(\omega)$ satisfies (17), then for any strictly stable $K \in \mathbb{R}^{d\times d}$, the control law

$$u = \mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) + \mathcal{P}(\omega)y \tag{47}$$

yields the origin of the underactuated Euler's system given by (1) globally asymptotically stable, where

$$y = K\omega_a - \mathcal{S}_{21}(\omega)\omega_u - \mathcal{S}_{22}(\omega)\omega_a \tag{48}$$

Proof: Consider the control law

$$u = \mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) + \widetilde{\mathcal{P}}(\omega,\delta)\eta \tag{49}$$

obtained by replacing $\mathcal{P}(\omega)$ in the control law given by (47) by the perturbed CCNP $\widetilde{\mathcal{P}}(\omega,\delta)$ given by (31), and

$$\eta = -\widetilde{\mathcal{P}}^{-1}(\omega,\delta)\Big[\mathcal{S}_{21}(\omega)\omega_u + \mathcal{S}_{22}(\omega)\omega_a - K\omega_a + \mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega)\Big]. \tag{50}$$

Applying the feedback linearizing control law given by (49) in the actuated subsystem given by (8) yields the globally asymptotically stable closed loop actuated subsystem

$$\dot{\omega}_a = K\omega_a. \tag{51}$$

Nevertheless, continuity of $\widetilde{\mathcal{P}}(\omega,\delta)$ in δ implies that if the magnitude of δ is small enough then the control law given by

$$u = \mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) + \mathcal{P}(\omega)\eta \tag{52}$$

yields globally asymptotically stable closed loop actuated subsystem also, which implies that

$$\lim_{t\to 0} \|\omega_a\| = 0 \tag{53}$$

Therefore, the third property of the DSGI implies that the control vector given by (52) uniformly converges to one choice of the control vectors given by (18), made by setting $y = \eta$. Hence, satisfying the condition given by (17) implies from proposition 2 that applying the control law given by (52) yields global asymptotic realization of the unactuated dynamics given by (10), and hence implies global asymptotic stabilization of the unactuated subsystem given by (7). Let the last term of the null-control vector η be denoted by h. Hence, Proposition (4) and the first DSGI property imply that

$$\mathcal{P}(\omega)h = -\mathcal{P}(\omega)\widetilde{\mathcal{P}}^{-1}(\omega,\delta)\mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) \quad (54)$$

$$= -\mathcal{P}(\omega)\mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) \quad (55)$$

$$= \mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) - \mathcal{A}^{+}(\omega)\mathcal{A}(\omega)\mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) \quad (56)$$

$$= \mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) - \mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) = \mathbf{0}_{d\times 1} \quad (57)$$

and that the expression given by (52) for u becomes

$$u = \mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) + \mathcal{P}(\omega)\left[K\omega_a - \mathcal{S}_{21}(\omega)\omega_u - \mathcal{S}_{22}(\omega)\omega_a\right]. \quad (58)$$

Comparing (58) with (47) results in the expression of y given by (48).

With the feedback control law given by (58), the closed loop actuated subsystem given by (8) becomes

$$\dot{\omega}_a = \mathcal{S}_{21}(\omega)\omega_u + \mathcal{S}_{22}(\omega)\omega_a + \mathcal{A}_s^{+}(\omega)\mathcal{B}(\omega) + \mathcal{P}(\omega)\left[K\omega_a - \mathcal{S}_{21}(\omega)\omega_u - \mathcal{S}_{22}(\omega)\omega_a\right]. \quad (59)$$

Remark: The resulting closed loop system is a perturbation from the non-realizable linear system

$$\ddot{\phi} + c_1\dot{\phi} + c_2\phi = 0, \qquad \dot{\omega}_a = K\omega_a \quad (60)$$

obtained as $\lim_{t\to\infty}\mathcal{A}_s(\omega) = \mathcal{A}(\omega)$ *via replacing* $\mathcal{P}(\omega)$ *by* $\widetilde{\mathcal{P}}(\omega,\delta)$ *in the control law given by (47).*

Example 1. The spacecraft that we consider has a principal plane of inertial symmetry, where the three principal moments of inertia (in $\mathrm{Kg}-\mathrm{m}^2$) are 50, 50, and 85. The above analysis implies that the spacecraft is not asymptotically stabilizable by two torque actuators that are mounted along the two principal axes residing in the principal plane of inertial symmetry, i.e., if $I_{11} = 85\mathrm{Kg}-\mathrm{m}^2$ and $I_{22} = I_{33} = 50\mathrm{Kg}-\mathrm{m}^2$. Nevertheless, the spacecraft is asymptotically stabilizable if one of the two actuators is mounted along the third principal axis instead, i.e., if $I_{11} = I_{22} = 50\mathrm{Kg}-\mathrm{m}^2$ and $I_{33} = 85\mathrm{Kg}-\mathrm{m}^2$. The controls coefficient $\mathcal{A}(\omega)$ relative to the function $\phi(\omega_u)$ given by (21) is

$$\mathcal{A}(\omega) = \left[\frac{k(I_{22}-I_{33})}{I_{11}}\omega_1^{k-1}\omega_3 \quad \frac{k(I_{22}-I_{33})}{I_{11}}\omega_1^{k-1}\omega_2\right] \quad (61)$$

and its zero actuated state Jacobian is

$$\mathcal{J}_a(\omega_1) = \left[\frac{\partial\mathcal{A}^T(\omega)}{\partial\omega_2} \quad \frac{\partial\mathcal{A}^T(\omega)}{\partial\omega_3}\right]_{\substack{\omega_2=0\\ \omega_3=0}} \quad (62)$$

$$= \begin{bmatrix} 0 & \frac{k(I_{22}-I_{33})}{I_{11}}\omega_1^{k-1} \\ \frac{k(I_{22}-I_{33})}{I_{11}}\omega_1^{k-1} & 0 \end{bmatrix} \quad (63)$$

which is nonsingular for all $\omega_1 \neq 0$, implying that (10) is globally realizable by the underactuated Euler's equations. Figures (1) and (2) show the resulting angular velocities about the three principal axes and the required control variables for $k = 2$, desired linear unactuated dynamics constants $c_1 = 3$ and $c_2 = 1$, where the matrix K is taken to be diagonal with elements -1 and -4. To speed up convergence, a dynamic scaling power $p = 6$ is chosen. The initial spacecraft body angular velocity vector is $\omega(0) = [2.0 \quad -3.0 \quad -1.0]^T$. In the case of one degree of actuation, the range space of the controls coefficient is a scalar, and the nullprojector given by (20) becomes

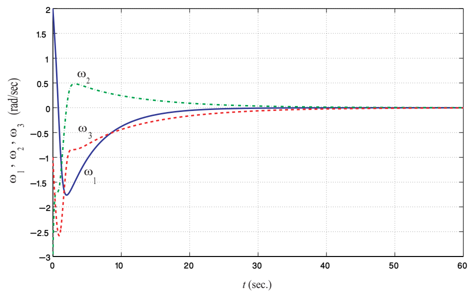

Fig. 1. $\omega_1, \omega_2, \omega_3$ vs. t: Two Degrees of Actuation

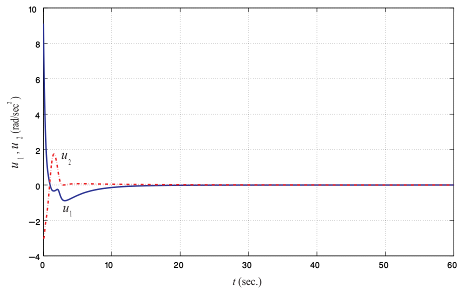

Fig. 2. u_1, u_2 vs. t: Two Degrees of Actuation

$$\mathcal{P}(\omega) = 1 - \frac{\mathcal{A}(\omega)}{\mathcal{A}(\omega)} = 0, \quad (64)$$

which implies that the nullspace of the controls coefficient $\mathcal{A}(\omega)$ has the dimension zero, and that the auxiliary part of the control law given by (18) vanishes. To maintain a nontrivial controls coefficient nullspace for which a null-control vector y can be designed for perturbed feedback linearization of the actuated dynamics, we consider an artificially actuated Euler's system of equations that has two degrees of actuation, and we create a dependency among the designed null-control vector y that accounts for the nonexisting control torque. Therefore, we let $\phi(\omega_u) = \omega_1^2$, and we form (10) and the corresponding Equation (12). The control law that is given by (47) can be rewritten in the form of the following two scalar equations

$$0 = \mathcal{A}_{s_{(1,1)}}^{+}(\omega)\mathcal{B}(\omega) + \mathcal{P}_{(1,1)}(\omega)y_1 + \mathcal{P}_{(1,2)}(\omega)y_2 \quad (65)$$

$$u = \mathcal{A}_{s_{(2,1)}}^{+}(\omega)\mathcal{B}(\omega) + \mathcal{P}_{(2,1)}(\omega)y_1 + \mathcal{P}_{(2,2)}(\omega)y_2. \quad (66)$$

Eq. (65) is a constraint on the null-control vector y, and it can further be written as

$$0 = \mathcal{A}_{s_{(1,1)}}^{+}(\omega)\mathcal{B}(\omega) + \kappa\mathcal{P}_{(1,1)}(\omega)y_1 + (1-\kappa)\mathcal{P}_{(1,1)}(\omega)y_1 + \mathcal{P}_{(1,2)}(\omega)y_2 \quad (67)$$

where the real number $\kappa \neq 0, 1$. Eq. (67) can be written as

$$y_1 = -\frac{1-\kappa}{\kappa}y_1 + \mathcal{C}_1(\omega)y_2 + \mathcal{D}_1(\omega) \quad (68)$$

where

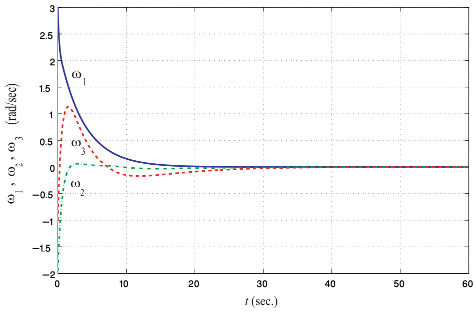

Fig. 3. $\omega_1, \omega_2, \omega_3$ vs. t: One Degree of Actuation

$$\mathcal{C}_1(\omega) = -\frac{\mathcal{P}_{(1,2)}(\omega)}{\kappa \mathcal{P}_{(1,1)}(\omega)}, \quad \mathcal{D}_1(\omega) = -\frac{\mathcal{A}^+_{s(1,1)}(\omega)}{\kappa \mathcal{P}_{(1,1)}(\omega)} \mathcal{B}(\omega). \tag{69}$$

Therefore, y can be written as

$$y = \mathcal{C}(\omega) y + \mathcal{D}(\omega) \tag{70}$$

where

$$\mathcal{C}(\omega) = \begin{bmatrix} -(1-\kappa)/\kappa & \mathcal{C}_1(\omega) \\ 0 & 1 \end{bmatrix} \quad \mathcal{D}(\omega) = \begin{bmatrix} \mathcal{D}_1(\omega) \\ 0 \end{bmatrix} \tag{71}$$

Substituting the expression of y given by (70) in the control law u given by (47) yields

$$u = \mathcal{A}^+_s(\omega)\mathcal{B}(\omega) + \mathcal{P}(\omega)\left[\mathcal{C}(\omega)y + \mathcal{D}(\omega)\right]. \tag{72}$$

In addition to globally realizing the asymptotically stable unactuated dynamics given by (10), the control law given by (72) accounts for the spacecraft single degree of actuation via a controls coefficient nullprojector of a higher dimension by constraining the freedom of the corresponding null-control vector y. Proceeding with the perturbed feedback linearizing control design, the null-control vector y given by (48) can easily be shown to render the closed loop fully-actuated subsystem of (8) globally asymptotically stable and a perturbation from the system given by

$$\dot{\omega}_a = K\omega_a. \tag{73}$$

Example 2. The control torque actuator is mounted along a spacecraft body-fixed axis about which the moment of inertia is $I_{33} = 80\mathrm{Kg}-\mathrm{m}^2$. The moments of inertia about two chosen orthogonal axes in a plane normal to the actuated axis are (in $\mathrm{Kg}-\mathrm{m}^2$) $I_{11} = 70$ and $I_{22} = 50$, and their product of inertia is $I_{12} = -90\mathrm{Kg}-\mathrm{m}^2$. The remaining two products of inertia are $I_{13} = 0$ and $I_{23} = 70\mathrm{Kg}-\mathrm{m}^2$. The function ϕ used to assess the spacecraft asymptotic stabilizability is chosen to be

$$\phi(\omega_u) = \omega_1^2 + \omega_2^2, \tag{74}$$

and the condition given by (29) and (30) implies that the desired second-order stable unactuated dynamics given by (10) is globally realizable by the underactuated Euler's model of the spacecraft. Nevertheless, to avoid the trivial nullprojection discussed above, the function ϕ is chosen to be

$$\phi(\omega_u) = \omega_1^2, \tag{75}$$

and the control law given by (72) is used to account for the nonexisting control torque. Figures (3) and (4) show the resulting angular velocity components about the three body-fixed axes and the required control variable for constants $c_1 = 8$ and $c_2 = 4$, $\kappa = 5$,where the matrix K is taken to be diagonal with elements -3 and -1, and a dynamic scaling power $p = 6$ is chosen. The initial spacecraft body angular velocity vector is $\omega(0) = [3.0 \quad -2.0 \quad -1.0]^T$.

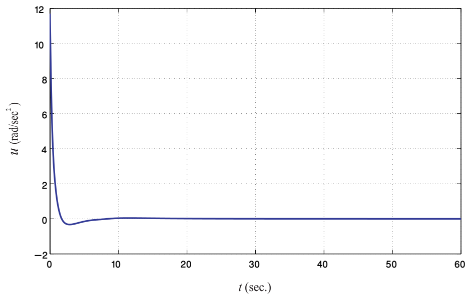

Fig. 4. u_1 vs. t: One Degree of Actuation

7. CONCLUSION

Based on the recently developed controls coefficient analysis, a new methodology is introduced for stabilizability assessment of underactuated spacecraft dynamics with arbitrary inertia distribution. Accordingly, necessary and sufficient stabilizability conditions are derived for spacecraft equipped with one and two degrees of actuation. The controls coefficient generalized inversion methodology is applied thereafter to design asymptotically stabilizing underactuated spacecraft control laws. The generalized inverse employed in the control laws is modified by a dynamic scaling factor, and uniformly converges to the standard Moore-Penrose generalized inverse, asymptotically realizing a prescribed unactuated dynamics. The null-control vector in the auxiliary part of the control law is designed for perturbed feedback linearization of the actuated dynamics. The dynamic scaling factor power substantially affects the scaled generalized inverse convergence properties, and low dynamic scaling factor power can destabilize the scaled generalized inverse. Studying the effect of the scaling factor on closed loop stability and performance is a future research work by the author.

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