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Stability of TCP/AQM Networks Based on A Switched Time-Delay System Modeling ⋆

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Abstract: We study the modeling and stability of TCP/AQM systems. The control-theoretic framework used in most of the previous work is linear system theory. Based on the linearization of nonlinear congestion control systems, classical linear techniques, such as the Nyquist or Bode criteria, are applied for the analysis of stability. The success of the linearization method depends on the assumption that the equilibrium is far away from the zero queue length point so that the linearization is well-defined. In this paper, the nonlinearity of the queue part is taken into consideration and TCP/AQM systems with proportional control are modeled as a class of switched time-delay systems. For such systems, we employ a Lyapunov approach and establish stability results. Simulations are presented to demonstrate the effectiveness of the stability analysis.

1. INTRODUCTION

Today's Internet has developed into a large-scale, heterogeneous, distributed system with unparalleled complexity. The Internet becomes one of the largest artificial feedback systems. As the Internet continues to evolve in scale and diversity, it is increasingly important to have a solid understanding on how this feedback system works.

Internet congestion occurs when the demand for some resources exceeds the available capacity. For example, if several users transfer files over a single bottleneck link at a rate that exceeds the capacity of the link, then some packets have to be dropped out. In turn, packet losses lead to the retransmission of lost packets and to the consequent ineffective utilization of network bandwidth.

Congestion control regulates the rate at which traffic sources inject packets into networks to ensure high-bandwidth utilization while avoiding network congestion. End-point congestion control, such as the Transmission Control Protocol (TCP), can be helped by Active Queue Management (AQM), whereby intermediate routers mark or drop packets with the objective of obtaining low packet losses, short queueing delay and high bandwidth utilization (see Kelly (1997), Floyd (1998), Hollot et al. (2001b), Srikant (2003), Paganini et al. (2001), Papachristodoulou et al. (2004)). AQM methods measure the router's queue length and attempt to throttle the sender's rate accordingly. AQM realizes the feedback by marking packets randomly with a probability determined by the queue length. Packets are then routed to the receiver. The receiver acknowledges receipt to the sender. Furthermore, the acknowledgment states whether the received packet was marked or not. When the sender received an acknowledgment, it can roughly infer the router queue length from the presence or absence of marking. The sender should then moderate its sending rate in the presence of long router queues and increase its sending rate in the absence of queues. As such, the marking probability effectively serves as a control signal to regulate the sender's sending rate. In practice, the configuration of TCP/AQM system is shown in Fig. 1. Notice that the feedback signal depends on the queue length, which can only take values between zero and the router buffer size. Moreover, AQM allows for arbitrary control laws to be instantiated at the router (see Fig. 1), and previous work has considered proportional and proportional-integral controllers in Hollot et al. (2001b), as well as nonlinear control laws in Floyd (1998).

The work on TCP model development and analysis mostly is based upon fluid analogies (Misra et al. (2000) etc.), hybrid systems (Bohacek et al. (2003) etc.) and linear systems with self-tuning parameters (Naitou et al. (2002)). In Misra et al. (2000), an ordinary differential equation model was developed for TCP/AQM systems based on fluid analogies. The model has been extensively used to study the stability and dynamic behavior of TCP/AQM systems (see Hollot et al. (2001a), Al-Hammouri et al. (2006), Hollot et al. (2002) and Hollot et al. (2001b)). The basic ideas of these papers are to linearize the differential equation model at the equilibrium and then use classical control system tools, such as the Nyquist or Bode criteria, to determine the stability of the TCP/AQM system. The linear analysis provides local stability results as long as the equilibrium is far way from the physical limits of the system states, such as the minimum or maximum value of queue lengths.

The success of the linearization technique relies on how well the dynamics of the system is approximated by its linearization at the equilibrium. If the dynamical equations of the system are continuous and differentiable everywhere, the characteristics of the system at the equilibrium can

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be well represented by the linearized system. However, if the equilibrium lies on or is close to a discontinuous point of the system dynamics, then linearizing the system at the equilibrium could lead to wrong conclusions, even for local stability results (see Branicky (1998) and Liberzon (2003)). These discontinuities exist in most of the congestion control systems and arise from physical features (for example, the queue length of congestion control systems cannot be negative), resulting in the behavior switching between the cases of positive and zero queue lengths. As a result, congestion control systems can be viewed as switched systems with several different dynamics that switch from one to another dependent on the queue length.

In this paper, we take into account the aforementioned physical constraints of congestion control systems. The linearized TCP/AQM systems with a proportional control are modeled as a class of switched time-delay systems. Due to the presence of the queue, the system is nonlinear and hence the classical frequency domain approach cannot be applied. Instead, a Lyapunov approach is employed to deal with the resulting switching systems with time delay. We then present a computational technique of determining the proportional control gains for stability.

In the next section, we introduce the dynamics of TCP/AQM systems with proportional control and show how they can be modeled as a switched delay system. Stability analysis and simulations are conducted in Section III and IV. Concluding remarks are given in Section V.

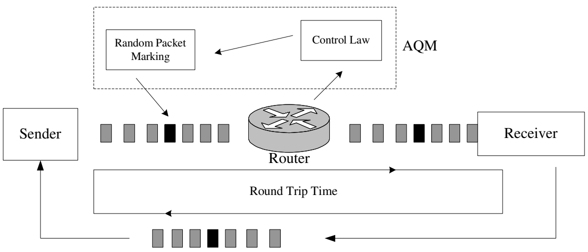

Fig. 1. A schematic of AQM marking packets, sender-receiver connection as a feedback system.

2. DYNAMIC MODEL

2.1 Dynamics of TCP and proportional AQM schemes

In Misra et al. (2000), a dynamic model of the TCP behavior was developed based on fluid flow and the analysis of stochastic differential equations. By ignoring the TCP timeout mechanism, the following simplified model was obtained in Hollot et al. (2002):

$$\dot{W}(t) = \frac{1}{R(t)} - \frac{W(t)}{2}\frac{W(t-R(t))}{R(t-R(t))}p(t-R(t)) \tag{1}$$

$$\dot{q}(t) = \begin{cases} -c + \dfrac{N(t)}{R(t)}W(t), & q > 0 \\ \max\{0, -c + \dfrac{N(t)}{R(t)}W(t)\}, & q = 0 \end{cases} \tag{2}$$

where

$W \doteq$ average TCP window size in packets;

$q \doteq$ queue length in packets;

$R(t) \doteq \dfrac{q(t)}{c} + T_p$, round-trip time in seconds;

$c \doteq$ link capacity in packets per second;

$T_p \doteq$ propagation delay in seconds;

$N \doteq$ load factor in number of TCP sessions;

$p \doteq$ probability of packet mark.

While (1) models the TCP window control dynamics, (2) describes the bottleneck queue length. The marking probability p takes values in $[0,1]$ and is a function of the queue length q, which in turn closes the loop of the feedback system. This feedback control law is also referred as to the Active Queue Management (AQM) control law.

An Active Queue Management (AQM) algorithm sends congestion information derived from the queue length. AQM runs on routers, updates and feedbacks the congestion information to senders by packet marking. AQM is a core process where packets are marked as a function of queue lengths. One of current AQMs is proportional or proportional-integral control, in which the control law of Fig. 1 is a P or PI controller In other words, packets are marked depending on the difference between the queue length q and its reference q_0. In this work, we shall focus on the stability analysis of TCP/AQM systems with proportional feedback control.

2.2 Switched Time-delay Systems

As shown in Hollot et al. (2002), ignoring the second condition of (2), the linearized system of the TCP plant at the equilibrium (W_0, q_0, p_0) is expressed by

$$P(s) = \frac{b}{(s+\alpha)(s+\beta)}$$

where $\alpha = 2N/(d^2c)$, $\beta = 1/d$, $b = c^2/(2N)$, d is the round-trip time delay in seconds at the equilibrium. For simplicity, d is considered as a constant. The equilibrium is obtained by $\dot{W} = 0$ and $\dot{q} = 0$ so that $W_0^2 p_0 = 2$, $W_0 = dc/N$ and $d = q_0/c + T_p$. The perturbed variables about the equilibrium are represented as $\delta W \doteq W - W_0$, $\delta q \doteq q - q_0$ and $\delta p \doteq p - p_0$.

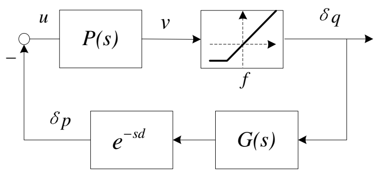

Fig. 2. The closed-loop system of TCP/AQM linearized model with a proportional controller and nonlinear queue.

The nonlinearity that lies in the queue part (2) is its boundary condition. Taking this into account, the closed-loop TCP/AQM system with proportional feedback control is shown in Fig. 2. A proportional feedback controller and nonlinearity f in the feedback system are introduced. The proportional AQM control law $G(s)$ and the nonlinear function f are given by

$$G(s) = k$$
$$\delta q = f(v) = \max\{v + q_0, 0\} - q_0$$

where f models the nonlinearity of (2), which is to prevent the occurrence of negative queue lengths. If the equilibrium is around the zero queue length point, the second equation of the queue part in (2) cannot be ignored any more, i.e., the nonlinearity must be taken into consideration. In previous work, the nonlinear function f was discarded in analysis because its nonlinearity makes the system intractable (Al-Hammouri et al. (2006)). Due to the presence of the queue, by which the nonlinearity of the routing algorithm is characterized, the conventional frequency domain approach cannot be used to establish stability of the nonlinear closed-loop system. Instead, a time domain approach, e.g., a Lyapunov approach, must be employed. In this work, we examine the nonlinear TCP/AQM system with a proportional controller, i.e., $G(s) = k_p = k$. In what follows, we first transform the nonlinear congestion system into an equivalent state space model, which will facilitate the stability analysis.

Note that the state space model of the plant $P(s) = \frac{b}{(s+\alpha)(s+\beta)}$ is given by

$$\dot{x} = Ax + Bu \tag{3}$$
$$v = Cx \tag{4}$$

where $A = \begin{bmatrix} 0 & 1 \\ -\alpha\beta & -(\alpha+\beta) \end{bmatrix}$ is a Hurwitz matrix, $B = [\,0\ 1\,]^T$, $C = [\,b\ 0\,]$, the control $u(t) = -\delta p(t-d) = -k\delta q(t-d)$ and the output v is the perturbed queue length if the nonlinear queue part and feedback delay are ignored.

Consequently, the corresponding closed-loop system with the proportional controller can be expressed as

$$\dot{x}(t) = Ax(t) - Bk\delta q(t-d) \tag{5}$$

where $\delta q = f(v) = \max\{v + q_0,\ 0\} - q_0$, or, equivalently,

$$\delta q = f(Cx) = \max\{Cx + q_0,\ 0\} - q_0$$

Taking into account of the delay, the closed-loop system leads to

$$\dot{x}(t) = Ax(t) - Bkf(Cx(t-d)) \tag{6}$$

Note that $Cx = [b \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = bx_1$ and the switching surface is $bx_1 + q_0 = 0$, (6) can be written as

$$\dot{x}(t) = \begin{cases} Ax(t) + Bkq_0, & \text{if } x_1 \le -\dfrac{q_0}{b} \\ Ax(t) + A_d x(t-d), & \text{if } x_1 > -\dfrac{q_0}{b} \end{cases} \tag{7}$$

with $A_d = -BkC$.

In this way, the TCP/AQM system shown as Fig. 2 has been represented as a switched time-delay system (7). Since both time delay and switching are involved in the system, investigating the stability of such system requires to construct a Lyapunov function that can take care of both effects simultaneously. The stability analysis will be given in the next section.

3. STABILITY ANALYSIS

In this section, we present the stability results on the TCP/AQM system. In physical network systems, the parameter $b = c^2/(2N)$ is usually a very large number in comparison with q_0 and the designed controller gain k is a very small number. In view of these facts, i.e., $Bkq_0 \approx 0$ and $q_0/b \approx 0$ are sufficiently close to zero (see, for instance, Example 1), the dynamic behavior and stability of the system (7) can be understood by investigating the following simplified model

$$\dot{x}(t) = \begin{cases} Ax(t), & \text{if } x_1 \le 0 \\ Ax(t) + A_d x(t-d), & \text{if } x_1 > 0 \end{cases} \tag{8}$$

with $A_d = -BkC$.

In what follows, we characterize the stability of the switched delay system (8) using a Lyapunov approach. The main result of this section is the following theorem.

Theorem 1. The equilibrium of the TCP/AQM system (8) with time delay d and proportional gain k is globally asymptotically stable if there exist positive definite matrices P, R, W such that

$$S_1 := -A^T P - PA - W - Rd > 0 \tag{9}$$
$$S_2 := W - (BkC)^T P S_1^{-1} P(BkC) > 0 \tag{10}$$

To prove Theorem 1, we introduce the following useful lemma.

Lemma 1. (See Gu et al. (2003)). For any constant symmetric matrix $M \in \mathbb{R}^{n\times n}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbb{R}^m$ such that the integration in the following is well-defined, we have

$$\gamma \int_0^\gamma \omega^T(\beta) M \omega(\beta) d\beta \ge \left(\int_0^\gamma \omega(\cdot) d\beta \right)^T M \left(\int_0^\gamma \omega(\cdot) d\beta \right)$$

Proof of Theorem 1. The basic idea of the proof is to construct a common Lyapunov function, thus yielding the asymptotic stability of the entire switched system (8).

Consider the Lyapunov function

$$V = V_1 + V_2 + V_3 \tag{11}$$

where

$$V_1 = x^T(t) P x(t),\ V_2 = \int_{t-d}^{t} x^T(s) W x(s) ds$$

$$\text{and } V_3 = \int_{t-d}^{t} (d - t + s) x^T(s) R x(s) ds$$

P, R and W are symmetric, positive definite matrices such that $A^T P + PA < 0$. The unique solution P to this Lyapunov equation is guaranteed since A is Hurwitz.

In the case when $x_1 \le 0$, the time derivative of V along the solution trajectories of the system (8) is given by $\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3$, where

$$\dot{V}_1 = x^T(t)(A^T P + PA)x(t)$$

$$\dot{V}_2 = x^T(t)Wx(t) - x^T(t-d)Wx(t-d)$$
$$\dot{V}_3 = dx^T(t)Rx(t) - \int_{t-d}^{t} x^T(s)Rx(s)ds$$

By Lemma 1, we have

$$\dot{V}_3 \leq dx^T(t)Rx(t) - d\left[\frac{1}{d}\int_{t-d}^{t} x(s)ds\right]^T R\left[\frac{1}{d}\int_{t-d}^{t} x(s)ds\right]$$

From the above equations and inequality, it follows that

$$\begin{aligned}\dot{V} \leq\ & x^T(t)(A^TP + PA)x(t) + x^T(t)Wx(t)\\ & -x^T(t-d)Wx(t-d) + dx^T(t)Rx(t)\\ & -d\left[\frac{1}{d}\int_{t-d}^{t} x(s)ds\right]^T R\left[\frac{1}{d}\int_{t-d}^{t} x(s)ds\right]\\ =\ & -X^T\Pi_1 X\end{aligned}$$

where $X^T = \left[x^T(t)\ x^T(t-d)\ \left(\frac{1}{d}\int_{t-d}^{t} x(s)ds\right)^T \right]$ and $\Pi_1 = \begin{bmatrix} -A^TP - PA - W - Rd & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & Rd \end{bmatrix}$.

When $x_1 > 0$, the time derivative of V along the solution trajectories of (8) yields

$$\begin{aligned}\dot{V} \leq\ & x^T(t)(A^TP + PA)x(t) + 2x^T(t)P(-BkC)x(t-d)\\ & +x^T(t)Wx(t) - x^T(t-d)Wx(t-d) + dx^T(t)Rx(t)\\ & -d\left[\frac{1}{d}\int_{t-d}^{t} x(s)ds\right]^T R\left[\frac{1}{d}\int_{t-d}^{t} x(s)ds\right]\\ =\ & -X^T\Pi_2 X\end{aligned}$$

with $\Pi_2 = \begin{bmatrix} -A^TP - PA - W - Rd & P(BkC) & 0 \\ (BkC)^TP & W & 0 \\ 0 & 0 & Rd \end{bmatrix}$.

In order to guarantee the negative definiteness of $\dot{V}$, both Π_1 and Π_2 must be positive definite simultaneously, i.e., the following conditions need to be satisfied

$$-A^TP - PA - W - Rd > 0$$
$$W - (BkC)^TPS_1^{-1}P(BkC) > 0$$

which guarantee the asymptotic stability of the system (8). This completes the proof. $\square$

For convenience of applications, we choose a set of specific matrices W and R, leading to the following corollary.

Corollary 1. The equilibrium of the TCP/AQM system (8) with time delay d and proportional gain k is globally asymptotically stable if there exist positive numbers γ, ε_1, ε_2 with $\gamma - \varepsilon_1 - d\varepsilon_2 > 0$, such that

$$0 < k < \frac{2\alpha\beta(\alpha+\beta)\sqrt{\varepsilon_1(\gamma - \varepsilon_1 - d\varepsilon_2)}}{\gamma b\sqrt{(\alpha+\beta)^2 + (\alpha\beta+1)^2}} \tag{12}$$

Proof. With the conditions (9) and (10) holding, set $W = \varepsilon_1 I$, $R = \varepsilon_2 I$ and P such that $A^TP + PA = -\gamma I$ where ε_1, ε_2, γ are positive numbers and I denotes the identity matrix. As a result, $S_1 = (\gamma - \varepsilon_1 - d\varepsilon_2)I$ and

$$S_2 = \begin{bmatrix} \varepsilon_1 - \dfrac{(p_2^2 + p_3^2)b^2k^2}{\gamma - \varepsilon_1 - d\varepsilon_2} & 0 \\ 0 & \varepsilon_1 \end{bmatrix} > 0 \tag{13}$$

with $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. Using $A^TP + PA = -\gamma I$, P can be solved with $p_1 = \frac{(\alpha+\beta)\gamma}{2\alpha\beta} + \frac{(\alpha\beta+1)\gamma}{2(\alpha+\beta)}$, $p_2 = \frac{\gamma}{2\alpha\beta}$ and $p_3 = \frac{(\alpha\beta+1)\gamma}{2\alpha\beta(\alpha+\beta)}$. Then, substituting p_1, p_2 and p_3 into (13), a straightforward calculation leads to the inequality (12) immediately. $\square$

Substituting $\alpha = 2N/(d^2c)$, $\beta = 1/d$ and $b = c^2/(2N)$ into the condition (12), we obtain

$$0 < k < \frac{8N^2(cd + 2N)\sqrt{\varepsilon_1(\gamma - \varepsilon_1 - d\varepsilon_2)}}{\gamma c^3d^2\sqrt{c^2d^6 + c^2d^4 + 8cNd^3 + 4N^2d^2 + 4N^2}} \tag{14}$$

The stability region given in (14) depends on the choice of γ, ε_1 and ε_2. In what follows, we illustrate how to choose ε_1, ε_2 and γ so that the gain k can have a maximal range. For a given TCP/AQM system with some α, β, b and delay d, the inequality (14) can be rewritten as

$$0 < k < \frac{\sqrt{\varepsilon_1(\gamma - \varepsilon_1 - d\varepsilon_2)}}{\gamma}\cdot F \tag{15}$$

with F being a constant defined by

$$F = \frac{8N^2(cd + 2N)}{c^3d^2\sqrt{c^2d^6 + c^2d^4 + 8cNd^3 + 4N^2d^2 + 4N^2}}$$

In order to reach the maximal range of k, we only need to focus on the function $\frac{\sqrt{\varepsilon_1(\gamma-\varepsilon_1-d\varepsilon_2)}}{\gamma}$. Denote $g(\varepsilon_1, \varepsilon_2, \gamma) = \frac{\sqrt{\varepsilon_1(\gamma-\varepsilon_1-d\varepsilon_2)}}{\gamma}$. Consequently, the problem of finding the maximal range of k boils down to solving the following optimization problem

$$\begin{aligned} \max\ & g(\varepsilon_1, \varepsilon_2, \gamma) = \frac{\sqrt{\varepsilon_1(\gamma - \varepsilon_1 - d\varepsilon_2)}}{\gamma} \\ \text{s.t.}\ & \gamma - \varepsilon_1 - d\varepsilon_2 > 0,\ \ \varepsilon_1 > 0,\ \varepsilon_2 > 0,\ \gamma > 0 \end{aligned} \tag{16}$$

As for ε_2, the maximizer of $g(\cdot)$ is reached when $\varepsilon_2 \to 0$. It does not affect the maximum of $g(\cdot)$ much if ε_2 is ignored in $g(\varepsilon_1, \varepsilon_2, \gamma)$, as we can choose ε_2 arbitrarily small after ε_1 and γ are fixed. Let $m = \varepsilon_1 > 0$ and $n = \gamma - \varepsilon_1 > 0$, then $g(\varepsilon_1, \varepsilon_2, \gamma)$ reduces to $\overline{g}(m, n) = \sqrt{\frac{mn}{(m+n)^2}}$. It is easy to see that $\overline{g}(m, n)$ reaches its maximum $\frac{1}{2}$ at $m = n > 0$. Hence, $g(\varepsilon_1, \varepsilon_2, \gamma)$ obtained its maximum around $\frac{1}{2}$ when $\gamma = 2\varepsilon_1 > 0$ and $\varepsilon_2 \to 0$. By (15), the stability region of k ends up with

$$0 < k < \sqrt{\frac{1}{4} - \frac{d\varepsilon_2}{\gamma^2}}\cdot F \approx \frac{1}{2}\cdot F \tag{17}$$

Therefore, $0 < k < \frac{1}{2}\cdot F$ gives an estimation of the range of the controller gain k.

Remark 1. From the right hand side of the inequality (14), it is easy to see that the range of k can be enlarged to

$0 < k < \infty$ when the delay $d \to 0$. It means that any positive proportional gain k would stabilize the system (8) with sufficiently small delay. On the other hand, the stability region of k shrinks to null if the delay $d \to \infty$.

Remark 2. The system (8) is actually a particular case of a class of switched time-delay systems that take the form $\dot{x}(t) = A_i x(t) + A_{di} x(t-d)$, $i = 1, 2, \ldots$. For such systems, a continuous piecewise quadratic Lyapunov function is constructed as $V = x^T(t) P_i x(t) + \int_{t-d}^{t} x^T(s) W x(s) ds + \int_{t-d}^{t} (d - t + s) x^T(s) R x(s) ds$. Note that such Lyapunov functions are continuous everywhere in terms of x and t. The continuity of x on the switching surface is guaranteed by the term $x^T(t) P_i x(t)$, in which P_i is computed as in the case of switched linear systems in Johansson et al. (1998). The other two terms are introduced to take care of the delay effects of the systems. Since $A_1 = A_2 = A$ in (8), the continuous piecewise quadratic Lyapunov function reduces to a common Lyapunov function, i.e., $P_1 = P_2 = P$.

Remark 3. Theorem 1 gives a conservative yet strong stability result in the sense that the stability follows independent of cell partition. Moreover, the inequalities (9) and (10) hold regardless of any switching schemes.

4. SIMULATIONS

Three examples are studied to simulate the set-point tracking responses of the system shown in Fig. 2. We apply the established stability results to the system (7) and demonstrate the effectiveness of the stability results.

Example 1. Consider a network with the following parameters: $N = 60$, $c = 3750$ pkt/sec, $d = 0.246$ sec, and $q_0 = 50$ (the same scenario as Example 4.1 in Al-Hammouri et al. (2006)). We have $W_0 = 15.375$ and $p_0 = 0.0085$. Choosing $\gamma = 11$, $\varepsilon_1 = 5.5$, $\varepsilon_2 = 0.01$, with $g(\varepsilon_1, \varepsilon_2, \gamma) = \sqrt{\varepsilon_1(\gamma - \varepsilon_1 - d\varepsilon_2)}/\gamma \approx 0.5$, gives that the TCP/AQM system is stable if $0 < k < 1.513 \cdot 10^{-5}$. Fig. 3 shows the response of the system (the queue length) when $k = 1.5 \cdot 10^{-5}$. If we keep increasing k, the system starts to oscillate at $k = 1.86 \cdot 10^{-4}$, which is shown in Fig. 4.

Example 2. Let $N = 60$, $c = 1250$ pkt/sec, $d = 0.22$ sec, and $q_0 = 50$ (as Example 4.2 in Al-Hammouri et al. (2006)). We have $W_0 = 4.58$ and $p_0 = 0.0952$. Choose $\gamma = 50$, $\varepsilon_1 = 25$, $\varepsilon_2 = 0.001$, then $g(\varepsilon_1, \varepsilon_2, \gamma) \approx 0.5$. The system is stable as $0 < k < 3.781 \cdot 10^{-4}$. The output response is shown in Fig. 5 using $k = 3.78 \cdot 10^{-4}$. If we increase k, the system will become unstable when $k = 2.83 \cdot 10^{-3}$, as shown in Fig. 6.

Example 3. Assume the network parameters are $N = 75$, $c = 1250$ pkt/sec, $d = 0.15$ sec, and $q_0 = 50$ (the same scenario as Example 4.3 in Al-Hammouri et al. (2006)). We have $W_0 = 2.5$ and $p_0 = 0.32$. Setting $\gamma = 100$, $\varepsilon_1 = 50$, $\varepsilon_2 = 0.01$ gives $g(\varepsilon_1, \varepsilon_2, \gamma) \approx 0.5$ and the stability region for k as $k \in (0, 0.0011)$. Choose $k = 0.001$ and the corresponding output response is shown in Fig. 7. Fig. 8 shows that the system becomes to oscillate at $k = 0.0097$.

Note that since the marking probability p takes values in $[0, 1]$, it is only physically realizable that the chosen k is not too large so that p is not larger than 1. In Example 3, the value of q is no more than 110 from Fig. 8 and thus δq is less than 60. Hence the range of the marking probability p can be calculated as $p = p_0 + k\delta q = 0.32 + 0.0097 \cdot 60 = 0.902$, which lies in $[0, 1]$.

The derived stability region of k is somewhat conservative from the simulation comparisons. The stability region criterion based on Corollary 1 gives an explicitly analytic condition instead of solving LMIs at the cost of being conservative. The explicit stability condition (17) is a trade off between the computational complexity and conservativeness in the analysis. The conservativeness might be reduced by a more meticulous examination of other Lyapunov function candidates or numerically solving the inequalities (9) and (10), which however are unlikely to result in explicit stability conditions.

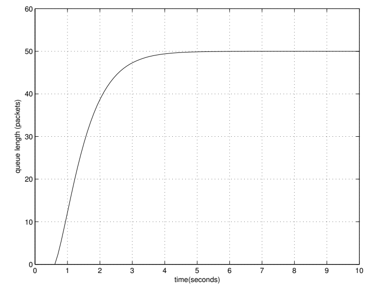

Fig. 3. The output response (queue length) of Example 1 using $k = 1.5 \cdot 10^{-5}$

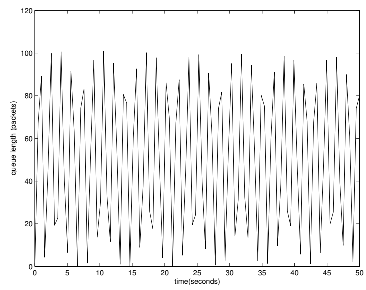

Fig. 4. The output response (queue length) of Example 1 using $k = 1.86 \cdot 10^{-4}$

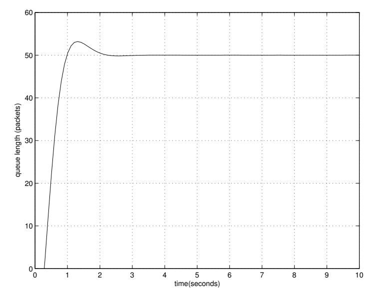

Fig. 5. The output response (queue length) of Example 2 using $k = 3.78 \cdot 10^{-4}$

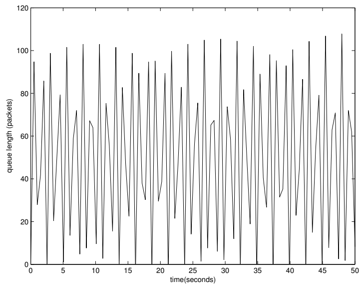

Fig. 6. The output response (queue length) of Example 2 using $k = 2.83 \cdot 10^{-3}$

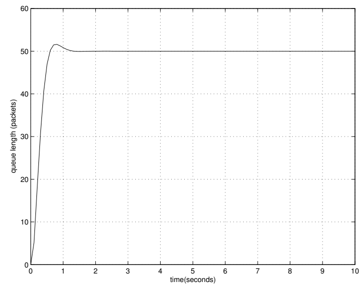

Fig. 7. The output response (queue length) of Example 3 using $k = 0.001$

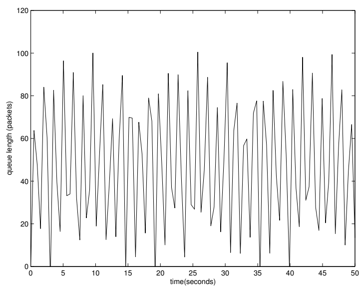

Fig. 8. The output response (queue length) of Example 3 using $k = 0.0097$

5. CONCLUSION

The work presented in this paper has provided an additional insight in understanding the stability of TCP/AQM networks under proportional control, in the presence of the queue nonlinearity. Our method is different from the existing ones in the literature, which do not take into account the involved nonlinearity. A class of common quadratic Lyapunov functions was developed for investigating the stability of TCP/AQM network that is modeled by a switched time-delay system. The range of the proportional gain was determined based on the Lyapunov analysis and the related LMI technique. With the help of such analysis tools, it is expected that the stability problem of TCP/AQM system under PI control can also be solved. This result will be reported in a future work.

On the other hand, most of the Internet congestion control systems involve nonlinear queue parts, which are usually discarded in the existing literature. Such nonlinearities cannot be ignored especially when the equilibria lie on or are closed to the physical limits. As demonstrated in this work, congestion control systems involving queue nonlinearities can be modeled as switched time-delay systems. The proposed Lyapunov methodology can deal with both time delay and switching simultaneously. It is believed that the developed tool might also be applied to other Internet congestion control systems with both time-delay and nonlinearity.

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