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Robust Stabilizability of Switched Linear Time-Delay Systems with Polytopic Uncertainties ⋆

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Abstract: This paper is devoted to robust stability analysis via state feedback of linear systems with state delay that are composed of polytopic uncertain subsystems. By state feedback, we mean that the switchings among subsystems are dependent on system states. For continuous-time switched linear systems, we show that if there exists a common positive definite matrix for stability of all convex combinations of the extreme points which belong to different subsystem matrices, then the switched system is robustly stabilizable via state feedback. The stability conditions of both delay-independent and delay-dependent are analyzed using Lyapunov- Razumikhin functional approach. Furthermore, we propose the switching rules by using the obtained common positive definite matrix.

Keywords: Switched systems; Robust Stabilizability; Lyapunov-Razumikhin function; Switching rule; State feedback.

1. INTRODUCTION

In recent years, the study of switched systems has received growing attention. By a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time or discrete-time subsystems and a rule orchestrating the switching between the subsystems (see Branicky [1998] and Ye et al [1998]). The standard model for such systems is given in Branicky et al [1998]. These systems arise as models for phenomena which cannot be described by exclusively continuous or exclusively discrete processes. Examples include the control of manufacturing systems (Pepyne et al [2000], Song et al [2000]), communication networks, traffic control (Horowitz et al [2000], Livadas et al [2000], Varaiva [1993]), chemical processing (Engell [2000]) and automotive engine control and aircraft control (Antsaklis [2000]).

In the last two decades, there has been increasing interest in stability analysis and control design for switched systems (for example Branicky [1998],Ye et al [1998],Liberzon et al [1999]). A survey of basic problems in stability and design of switched systems has been proposed recently in Liberzon et al [1999]. The motivation for studying switched systems is from the fact that many practical systems are inherently multi-model in the sense that several dynamical subsystems are required to describe their behavior which may depend on various environmental factors, and that the methods of intelligent control design are based on the idea of switching between different controllers. In Zhai et al [2003], the authors only consider the quadratic stabilizability via state feedback for both continuous-time and discrete-time switched linear uncertain systems without time-delay.

Systems with time-delay constitute basic mathematical models of real phenomena, for instance, in circuit theory, economics and mechanics. The analysis and synthesis of controllers for switched systems with time-delay have been attracting increasingly more attention. Some results have been obtained. In Lu et al [2006], the asymptotic stability of switched systems whose subsystems are time delay systems is studied via LMI (linear matrix inequalities) approach; in Sehjeong et al [2006], they consider a switching system composed of a finite number of linear delay differential equations. There are a few existing results concerning quadratic stabilization of switched linear systems that are composed of several unstable linear time-invariant subsystems. However, model uncertainty is often encountered in control systems. At the knowledge of the

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authors, there is little research focused on dealing with such systems. Motivated by the above results, here we will consider switched linear systems with both polytopic uncertainties and time-delay. Specifically, we use Lyapunov-Razumikhin function approach to deal with time-delay.

Notations: The following notations are used throughout this paper. R is the set of real numbers and R^n, $R^{m\times n}$ are sets of real vectors with dimension n and real matrices with dimension $m \times n$, respectively. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). I and 0 denote the identity matrix and zero matrix with compatible dimensions. $*$ denotes the symmetry elements in the symmetric matrices, that is

$$\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$$

2. PROBLEM STATEMENT

In this section, we consider the linear time-delay system with polytopic uncertainties

$$\dot{x}(t) = A_{\sigma(x,t)}x(t) + A_{d_{\sigma(x,t)}}x(t-\tau). \tag{1}$$

where $x(t) \in R^n$ is the state, $\sigma(x,t)$ is a switching rule defined by $\sigma(x,t) : R^n \times R^+ \to \{1,2\}$, and R^+ denotes nonnegative real numbers. $\tau \geq 0$ is assumed to be a constant time delay. Therefore, the switched system is composed of two continuous-time subsystems with time-delay

$$S_1: \quad \dot{x}(t) = A_1x(t) + A_{d_1}x(t-\tau). \tag{2}$$

$$S_2: \quad \dot{x}(t) = A_2x(t) + A_{d_2}x(t-\tau). \tag{3}$$

Here, we assume that both S_1 and S_2 are uncertain systems of polytopic type described as

$$A_i = \sum_{j=1}^{N_i} \mu_{ij}A_{ij} \quad A_{d_i} = \sum_{j=1}^{N_i} \mu_{ij}A_{d_{ij}}. \tag{4}$$

where $\mu_i = (\mu_{i1}, \mu_{i2}, \cdots, \mu_{iN_i})$ belongs to

$$\left\{ \mu_i : \sum_{j=1}^{N_i} \mu_{ij} = 1, \mu_{ij} \geq 0 \right\} \tag{5}$$

Here $i \in \{1,2\}$ denotes the i'th subsystem. And $A_{ij}, A_{d_{ij}}$, $j = 1, 2, \cdots, N_i$ are constant matrices denoting the extreme points of the polytope A_i, A_{d_i}, and N_i is the number of the extreme points.

The following lemma will be useful in the proof of our main results.

Lemma 1. (see Cao et al [1998]) For any $x, y \in R^n$ and a matrix $W > 0$ with compatible dimensions, the following inequality holds

$$2x^Ty \leq x^TWx + y^TW^{-1}y \tag{6}$$

3. MAIN RESULTS

In this section, we will first give a delay-independent robust stability condition for system (1). Then a delay-dependent one will be derived.

3.1 DELAY-INDEPENDENT ANALYSIS

If S_1 or S_2 is robustly stable, we can always activate the stable subsystem so that the entire switched system is robustly stable. Therefore, to make the switching problem nontrivial, we make the following assumptions.

Assumption 1: Both S_1 and S_2 are delay-independent robustly unstable.

Now, we need the definition of robust stabilizability via state feedback for the switched system (1).

Definition 1: The system (1) is said to be robustly stabilizable via state feedback if there exist positive-definite function $V(x) = x^TPx$, a positive number ϵ and a switching rule $\sigma(x,t)$ depending on x such that

$$\frac{d}{dt}V(x) < -\epsilon x^Tx \tag{7}$$

holds for all trajectories of the system (1).

Our aim in this section is to find a state feedback (state-dependent switching rule) $\sigma(x,t)$ such that the switched system (1) is robustly stable. We state and prove the following main result.

Theorem 2. The switched system (1) is delay-independent robustly stabilizable via state feedback if there exist constant scalars λ_{ij}'s ($i = 1,2; j = 1,2,\cdots,N_i$) satisfying $0 \leq \lambda_{ij} \leq 1$ and $P > 0$, $W > 0$ such that

$$P \geq W^{-1} \tag{8}$$

$$\begin{aligned}&[\lambda_{ij}A_{1i} + (1-\lambda_{ij})A_{2j}]^TP + P[\lambda_{ij}A_{1i} + (1-\lambda_{ij})A_{2j}] \\ &+P[\lambda_{ij}A_{d_{1i}}WA_{d_{1i}}^T + (1-\lambda_{ij})A_{d_{2j}}WA_{d_{2j}}^T]P + P < 0\end{aligned} \tag{9}$$

holds for all $i = 1,2; j = 1,2,\cdots,N_i$.

Proof. For the benefit of notation simplicity, we only give the proof in the case of $N_1 = N_2 = 2$. The extension from $N_1 = N_2 = 2$ to general case is very obvious.

From (9), we know that there always exists a positive scalar ϵ such that

$$\begin{aligned}&[\lambda_{ij}A_{1i} + (1-\lambda_{ij})A_{2j}]^TP + P[\lambda_{ij}A_{1i}+ \\ &(1-\lambda_{ij})A_{2j}] + P[\lambda_{ij}A_{d_{1i}}WA_{d_{1i}}^T+ \\ &(1-\lambda_{ij})A_{d_{2j}}WA_{d_{2j}}^T]P + P < -\epsilon I\end{aligned} \tag{10}$$

Then, for any $x \neq 0$, we obtain

$$\begin{aligned}&x^T[\lambda_{11}A_{11} + (1-\lambda_{11})A_{21}]^TPx + x^TP[\lambda_{11}A_{11} \\ &+(1-\lambda_{11})A_{21}]x + x^TP[\lambda_{11}A_{d_{11}}WA_{d_{11}}^T+ \\ &(1-\lambda_{11})A_{d_{21}}WA_{d_{21}}^T]Px + x^TPx < -\epsilon x^Tx\end{aligned} \tag{11}$$

$$\begin{aligned}&x^T[\lambda_{12}A_{11} + (1-\lambda_{12})A_{22}]^TPx + x^TP[\lambda_{12}A_{11} \\ &+(1-\lambda_{12})A_{22}]x + x^TP[\lambda_{12}A_{d_{11}}WA_{d_{11}}^T+ \\ &(1-\lambda_{12})A_{d_{22}}WA_{d_{22}}^T]Px + x^TPx < -\epsilon x^Tx\end{aligned} \tag{12}$$

$$\begin{aligned}&x^T[\lambda_{21}A_{12} + (1-\lambda_{21})A_{21}]^TPx + x^TP[\lambda_{21}A_{12} \\ &+(1-\lambda_{21})A_{21}]x + x^TP[\lambda_{21}A_{d_{12}}WA_{d_{12}}^T+ \\ &(1-\lambda_{21})A_{d_{21}}WA_{d_{21}}^T]Px + x^TPx < -\epsilon x^Tx\end{aligned} \tag{13}$$

$$\begin{aligned}&x^T[\lambda_{22}A_{12} + (1-\lambda_{22})A_{22}]^TPx + x^TP[\lambda_{22}A_{12} \\ &+(1-\lambda_{22})A_{22}]x + x^TP[\lambda_{22}A_{d_{12}}WA_{d_{12}}^T+ \\ &(1-\lambda_{22})A_{d_{22}}WA_{d_{22}}^T]Px + x^TPx < -\epsilon x^Tx\end{aligned} \tag{14}$$

which can be rewritten as

$$\lambda_{11}x^T\Xi_{11}x + (1-\lambda_{11})x^T\Xi_{21}x < -\epsilon x^Tx \tag{15}$$

$$\lambda_{12}x^T\Xi_{11}x + (1-\lambda_{12})x^T\Xi_{22}x < -\epsilon x^Tx \tag{16}$$

$$\lambda_{21}x^T\Xi_{12}x + (1-\lambda_{21})x^T\Xi_{21}x < -\epsilon x^T x \quad (17)$$
$$\lambda_{22}x^T\Xi_{12}x + (1-\lambda_{22})x^T\Xi_{22}x < -\epsilon x^T x \quad (18)$$

Here $\Xi_{ij} := A_{ij}^T P + PA_{ij} + PA_{d_{ij}} W A_{d_{ij}}^T P + P$

From the above inequalities, it is obvious to verify that either

$$x^T\Xi_{11}x < -\epsilon x^T x \qquad x^T\Xi_{12}x < -\epsilon x^T x \quad (19)$$

or

$$x^T\Xi_{21}x < -\epsilon x^T x \qquad x^T\Xi_{22}x < -\epsilon x^T x \quad (20)$$

is true. For example, if (19) is not true with $x^T\Xi_{11}x > -\epsilon x^T x$, then we get $x^T\Xi_{21}x < -\epsilon x^T x$ from (15) and $x^T\Xi_{22}x < -\epsilon x^T x$ from (16). The same is true for other cases.

Now, we define the switching rule as

$$\sigma(x,t) \in \{i \mid x^T\Xi_{ij}x < -\epsilon x^T x,\ j = 1,2\} \quad (21)$$

Then, based on the above discussion, we get

$$x^T\Xi_{\sigma 1}x < -\epsilon x^T x \quad x^T\Xi_{\sigma 2}x < -\epsilon x^T x \quad (22)$$

Thus it follows that

$$x^T\Xi_{\sigma}x < -\epsilon x^T x \quad (23)$$

since A_σ is a linear convex combination of $A_{\sigma 1}$ and $A_{\sigma 2}$, A_{d_σ} is a linear convex combination of $A_{d_{\sigma 1}}$ and $A_{d_{\sigma 2}}$.

Given $P > 0$, here we choose the Lyapunov function as $V(x(t)) = x^T(t)Px(t)$. The derivative of $V(x(t))$ along the solution of (1) is

$$\dot V(x(t)) = 2x^T(t)PA_\sigma x(t) + 2x^T(t)PA_{d_\sigma}x(t-\tau) \quad (24)$$

From Lemma 1, it follows that

$$\begin{aligned}\dot V(x(t)) \le\ & x^T(t)(A_\sigma^T P + PA_\sigma + PA_{d_\sigma}WA_{d_\sigma}^T P)x(t)\\ & + x^T(t-\tau)W^{-1}x(t-\tau)\end{aligned} \quad (25)$$

From (8), we get

$$\begin{aligned}\dot V(x(t)) \le\ & x^T(t)(A_\sigma^T P + PA_\sigma + PA_{d_\sigma}WA_{d_\sigma}^T P)x(t)\\ & + V(x(t-\tau))\end{aligned} \quad (26)$$

By Razumikhin Theorem, to prove (1) is robustly stable, it suffices to show that there exist an $\eta > 1$ and a $\delta > 0$ such that

$$\dot V(x(t)) \le -\delta x^T(t)x(t)$$

if

$$V(x(t+\theta)) < \eta V(x(t)),\ \forall\, \theta \in [-\tau, 0]$$

In the remainder of the proof, we will construct such η and δ and show that they satisfy the above condition.

From (23), we can see that there exists a $\delta_1 > 0$ such that

$$A_\sigma^T P + PA_\sigma + PA_{d_\sigma}WA_{d_\sigma}^T P + (1+2\delta_1)P < -\varepsilon I$$

Let $\eta = 1 + \delta_1$. Now suppose that $V(x(t+\theta)) < \eta V(x(t)),\ \forall\, \theta \in [-\tau, 0]$. Then from (26), it follows that

$$\begin{aligned}\dot V(x(t)) \le\ & x^T(t)(A_\sigma^T P + PA_\sigma + PA_{d_\sigma}WA_{d_\sigma}^T P + \eta P)x(t)\\ & < -(\delta_1 + \epsilon)x^T(t)x(t) = -\delta x^T(t)x(t)\end{aligned} \quad (27)$$

is true for all trajectories of (1). Thus the switched system is robustly stable.

Remark 3. Multiplying P^{-1} for both sides of inequalities (8) and (9) and letting $Q = P^{-1}$, we see that the matrix inequalities (8) and (9) are equivalent to the following matrix inequalities

$$Q - W \le 0 \quad (28)$$

$$\begin{aligned}& Q[\lambda_{ij}A_{1i} + (1-\lambda_{ij})A_{2j}]^T + [\lambda_{ij}A_{1i} + (1-\lambda_{ij})A_{2j}]\\ & Q + \lambda_{ij}A_{d_{1i}}WA_{d_{1i}}^T + (1-\lambda_{ij})A_{d_{2j}}WA_{d_{2j}}^T + Q < 0\end{aligned} \quad (29)$$

If the values of $\lambda_{ij}, i,j = 1,2$ are fixed, with respect to parameters Q and W,then inequalities (28) and (29) can be converted to linear matrix inequalities which can be solved efficiently using the interior-point method.

Remark 4. It is easy to extend Theorem 2 to the case of more than two subsystems. But the number of conditions to be checked grows very fast when the number of subsystems and the extreme points of each A_i are large.

3.2 DELAY-DEPENDENT ANALYSIS

Here to make the switching problem nontrivial, we also make the following assumption.

Assumption 2: Both S_1 and S_2 are delay-dependent robustly unstable.

Theorem 5. The switched system (1) is delay-dependent robustly stabilizable via state feedback if there exist constant scalars $\lambda'_{ij}s(i = 1,2; j = 1,2,\cdots,N_i)$ satisfying $0 \le \lambda_{ij} \le 1$, $\tau > 0$ and $P > 0$, $P_1 > 0$, $P_2 > 0$ such that

$$A_{ij}^T P_1^{-1} A_{ij} \le P \quad (30)$$

$$A_{d_{ij}}^T P_2^{-1} A_{d_{ij}} \le P \quad (31)$$

$$\begin{aligned}& [\lambda_{ij}\hat A_{1i} + (1-\lambda_{ij})\hat A_{2j}]^T P + P[\lambda_{ij}\hat A_{1i} + (1-\lambda_{ij})\\ & \hat A_{2j}] + \tau P[\lambda_{ij}A_{d_{1i}}(P_1+P_2)A_{d_{1i}}^T + (1-\lambda_{ij})A_{d_{2j}}\\ & (P_1+P_2)A_{d_{2j}}^T]P + 2\tau P < 0\end{aligned} \quad (32)$$

holds for all $i = 1,2;\ j = 1,2,\cdots,N_i$. Here $\hat A_{ij} = A_{ij} + A_{d_{ij}}$

Proof. For the benefit of notation simplicity, here we also give the proof in the case of $N_1 = N_2 = 2$.

From (32), we know that there always exists a positive scalar ϵ such that

$$\begin{aligned}& [\lambda_{ij}\hat A_{1i} + (1-\lambda_{ij})\hat A_{2j}]^T P + P[\lambda_{ij}\hat A_{1i} + (1-\lambda_{ij})\\ & \hat A_{2j}] + \tau P[\lambda_{ij}A_{d_{1i}}(P_1+P_2)A_{d_{1i}}^T + (1-\lambda_{ij})A_{d_{2j}}\\ & (P_1+P_2)A_{d_{2j}}^T]P + 2\tau P < -\epsilon I\end{aligned} \quad (33)$$

Then, for any $x \ne 0$, we have

$$\begin{aligned}& x^T[\lambda_{11}\hat A_{11} + (1-\lambda_{11})\hat A_{21}]^T Px + x^T P[\lambda_{11}\hat A_{11} +\\ & (1-\lambda_{11})\hat A_{21}]x + \tau x^T P[\lambda_{11}A_{d_{11}}(P_1+P_2)A_{d_{11}}^T +\\ & (1-\lambda_{11})A_{d_{21}}(P_1+P_2)A_{d_{21}}^T]Px + 2\tau x^T Px < -\epsilon x^T x\end{aligned} \quad (34)$$

$$\begin{aligned}& x^T[\lambda_{12}\hat A_{11} + (1-\lambda_{12})\hat A_{22}]^T Px + x^T P[\lambda_{12}\hat A_{11} +\\ & (1-\lambda_{12})\hat A_{22}]x + \tau x^T P[\lambda_{12}A_{d_{11}}(P_1+P_2)A_{d_{11}}^T +\\ & (1-\lambda_{12})A_{d_{22}}(P_1+P_2)A_{d_{22}}^T]Px + 2\tau x^T Px < -\epsilon x^T x\end{aligned} \quad (35)$$

$$\begin{aligned}& x^T[\lambda_{21}\hat A_{12} + (1-\lambda_{21})\hat A_{21}]^T Px + x^T P[\lambda_{21}\hat A_{12} +\\ & (1-\lambda_{21})\hat A_{21}]x + \tau x^T P[\lambda_{21}A_{d_{12}}(P_1+P_2)A_{d_{12}}^T +\\ & (1-\lambda_{21})A_{d_{21}}(P_1+P_2)A_{d_{21}}^T]Px + 2\tau x^T Px < -\epsilon x^T x\end{aligned} \quad (36)$$

$$\begin{aligned}& x^T[\lambda_{22}\hat A_{12} + (1-\lambda_{22})\hat A_{22}]^T Px + x^T P[\lambda_{22}\hat A_{12} +\\ & (1-\lambda_{22})\hat A_{22}]x + \tau x^T P[\lambda_{22}A_{d_{12}}(P_1+P_2)A_{d_{12}}^T +\\ & (1-\lambda_{22})A_{d_{22}}(P_1+P_2)A_{d_{22}}^T]Px + 2\tau x^T Px < -\epsilon x^T x\end{aligned} \quad (37)$$

which can be rewritten as

$$\lambda_{11}x^T\Omega_{11}x + (1-\lambda_{11})x^T\Omega_{21}x < -\epsilon x^T x \quad (38)$$

$$\lambda_{12}x^T\Omega_{11}x + (1-\lambda_{12})x^T\Omega_{22}x < -\epsilon x^T x \quad (39)$$

$$\lambda_{21}x^T\Omega_{12}x + (1-\lambda_{21})x^T\Omega_{21}x < -\epsilon x^T x \quad (40)$$

$$\lambda_{22}x^T\Omega_{12}x + (1-\lambda_{22})x^T\Omega_{22}x < -\epsilon x^T x \quad (41)$$

Here $\Omega_{ij} = \hat{A}_{ij}^T P + P\hat{A}_{ij} + \tau P A_{d_{ij}}(P_1+P_2)A_{d_{ij}}^T P + 2\tau P$

It is very easy to verify from the above inequalities that either

$$x^T\Omega_{11}x < -\epsilon x^T x \quad x^T\Omega_{12}x < -\epsilon x^T x \quad (42)$$

or

$$x^T\Omega_{21}x < -\epsilon x^T x \quad x^T\Omega_{22}x < -\epsilon x^T x \quad (43)$$

is true. For example, if (42) is not true with $x^T\Omega_{11}x > -\epsilon x^T x$,then we get $x^T\Omega_{21}x < -\epsilon x^T x$ from (38) and $x^T\Omega_{22}x < -\epsilon x^T x$ from (39). The same is true for other cases.

Now, we define the switching rule as

$$\sigma(x,t) \in \{i \mid x^T\Omega_{ij}x < -\epsilon x^T x,\ j = 1,2\} \quad (44)$$

Then, based on the above discussion, we get

$$x^T\Omega_{\sigma 1}x < -\epsilon x^T x \quad x^T\Omega_{\sigma 2}x < -\epsilon x^T x \quad (45)$$

And thus

$$x^T\Omega_{\sigma}x < -\epsilon x^T x \quad (46)$$

since A_σ is a linear convex combination of $A_{\sigma 1}$ and $A_{\sigma 2}$, A_{d_σ} is a linear convex combination of $A_{d_{\sigma 1}}$ and $A_{d_{\sigma 2}}$.

Given $P > 0$, here we choose definite Lyapunov function $V(x(t)) = x^T(t)Px(t)$.

Since $x(t)$ is continuously differentiable for $t \geq 0$, using the Leibniz-Newton formula, one can write

$$\begin{aligned} x(t-\tau) &= x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds \\ &= x(t) - \int_{t-\tau}^{t} [A_\sigma x(s) + A_{d_\sigma}x(s-\tau)]ds \end{aligned} \quad (47)$$

for $t \geq \tau$. Thus the system (1) can be written as

$$\begin{aligned} \dot{x}(t) &= (A_\sigma + A_{d_\sigma})x(t) \\ &\quad - A_{d_\sigma}\int_{t-\tau}^{t} [A_\sigma x(s) + A_{d_\sigma}x(s-\tau)]ds \end{aligned} \quad (48)$$

The derivative of $V(x(t))$ along the solution of (1) is

$$\begin{aligned} \dot{V}(x(t)) &= 2x^T(t)P(A_\sigma + A_{d_\sigma})x(t) \\ &\quad -2x^T(t)PA_{d_\sigma}\int_{t-\tau}^{t} [A_\sigma x(s) + A_{d_\sigma}x(s-\tau)]ds \end{aligned} \quad (49)$$

Now we translate the interval $[t-\tau, t]$ of $x(t)$ to $[-\tau, 0]$. And from the Lemma 1, we get

$$\begin{aligned} \dot{V}(x(t)) \leq\ & 2x^T(t)P(A_\sigma + A_{d_\sigma})x(t) \\ & +\tau x^T(t)PA_{d_\sigma}(P_1+P_2)A_{d_\sigma}^T Px(t) \\ & + \int_{-\tau}^{0} x^T(t+s)A_\sigma^T P_1^{-1}A_\sigma x(t+s)ds \\ & + \int_{-\tau}^{0} x^T(t-\tau+s)A_{d_\sigma}^T P_2^{-1}A_{d_\sigma}x(t-\tau+s)ds \end{aligned} \quad (50)$$

It follows from (30) and (31) that

$$\begin{aligned} x^T(t)A_\sigma^T P_1^{-1}A_\sigma x(t) \leq x^T(t)Px(t) \\ x^T(t)A_{d_\sigma}^T P_2^{-1}A_{d_\sigma}x(t) \leq x^T(t)Px(t) \end{aligned} \quad (51)$$

Let $\hat{A}_\sigma = A_\sigma + A_{d_\sigma}$. Hence

$$\begin{aligned} & \dot{V}(x(t)) \\ & \leq 2x^T(t)P\hat{A}_\sigma x(t) + \tau x^T(t)PA_{d_\sigma}(P_1+P_2)A_{d_\sigma}^T Px(t) \\ & \quad + \int_{-\tau}^{0} V(x(t+s))ds + \int_{-\tau}^{0} V(x(t-\tau+s))ds \end{aligned} \quad (52)$$

By Razumikhin Theorem, to guarantee the asymptotic stability of system, it is suffices to find two scalars $\eta > 1$ and $\delta > 0$ such that

$$\dot{V}x(t) < -\delta V(x(t))$$

if

$$V(x(t+\theta)) < \eta V(x(t)), \forall\, \theta \in [-2\tau, 0]$$

From (46), we know that there exists a $\delta_1 > 0$ such that

$$\begin{aligned} x^T\ & [\hat{A}_\sigma^T P + P\hat{A}_\sigma + \tau PA_{d_\sigma}(P_1+P_2)A_{d_\sigma}^T P \\ & +2\tau(1+2\delta_1)P]x < -\epsilon x^T x \end{aligned}$$

Let $\eta = 1+\delta_1$. Suppose that $V(x(t+\theta)) < \eta V(x(t)), \forall\, \theta \in [-2\tau, 0]$. Then from (52), we get

$$\begin{aligned} \dot{V}(x(t)) \leq\ & 2x^T(t)P\hat{A}_\sigma x(t) + 2\tau\eta x^T(t)Px(t) \\ & +\tau x^T(t)PA_{d_\sigma}(P_1+P_2)A_{d_\sigma}^T Px(t) \\ & = x^T(t)(\hat{A}_\sigma^T P + P\hat{A}_\sigma + \tau PA_{d_\sigma}(P_1+P_2)A_{d_\sigma}^T P \\ & +2\tau\eta P)x(t) < -2\tau\delta_1 x^T(t)Px(t) \end{aligned} \quad (53)$$

Thus the switched system is robustly stable.

Remark 6. If we multiple P^{-1} by the left and right to the inequalities (30)— (32) and let $Q = P^{-1}$, we can easily see that the matrix inequalities (30)— (32) are equivalent to the following matrix inequalities using Schur complement.

$$\begin{bmatrix} Q & QA_{ij}^T \\ * & P_1 \end{bmatrix} \geq 0 \quad (54)$$

$$\begin{bmatrix} Q & QA_{d_{ij}}^T \\ * & P_2 \end{bmatrix} \geq 0 \quad (55)$$

$$\begin{aligned} & Q[\lambda_{ij}\hat{A}_{1i} + (1-\lambda_{ij})\hat{A}_{2j}]^T + [\lambda_{ij}\hat{A}_{1i} + (1-\lambda_{ij}) \\ & \hat{A}_{2j}]Q + \tau[\lambda_{ij}A_{d_{1i}}(P_1+P_2)A_{d_{1i}}^T + (1-\lambda_{ij})A_{d_{2j}} \\ & (P_1+P_2)A_{d_{2j}}^T] + 2\tau Q < 0 \end{aligned} \quad (56)$$

Remark 7. The above three conditions involves bilinear matrix inequalities (BMI) with respect to λ_{ij}s. Although there is LMI Toolbox in Matlab which is convenient for us for solving LMI problem, it is not easy to solve BMI problem up till. We can use the branch and bound methods suggested or the homotopy-based algorithm to deal with the BMI problem.

4. NUMERICAL EXAMPLE

Example 1. (Delay-independent case) Consider the system (1) composed of two subsystems where

$$\begin{aligned} A_{11} &= \begin{bmatrix} 4 & -5 \\ 0 & -4 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 4 & -4.9 \\ 0 & -4 \end{bmatrix} \\ A_{d_{11}} &= \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \quad A_{d_{12}} = \begin{bmatrix} -1 & 0.1 \\ -1 & -1 \end{bmatrix} \end{aligned} \quad (57)$$

and

$$\begin{aligned} A_{21} &= \begin{bmatrix} -8 & 4 \\ 0 & 1 \end{bmatrix} \quad A_{22} = \begin{bmatrix} -8 & 4 \\ 0.1 & 1 \end{bmatrix} \\ A_{d_{21}} &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \quad A_{d_{22}} = \begin{bmatrix} -1 & 0.1 \\ 1 & -1 \end{bmatrix} \end{aligned} \quad (58)$$

Both $A_{11}, A_{d_{11}}$ and $A_{12}, A_{d_{12}}$ are delay-independent robustly unstable. Similarly, both $A_{21}, A_{d_{21}}$ and $A_{22}, A_{d_{22}}$ are delay-independent robustly unstable. Therefore, both S_1 and S_2 are robustly unstable.

Now, we let $\lambda_{11} = \lambda_{12} = \lambda_{21} = \lambda_{22} = 0.5$, in order to satisfy the conditions in Theorem 2, by using MATLAB tools we get

$$P = \begin{bmatrix} 1.7252 & 0.4435 \\ 0.4435 & 0.7159 \end{bmatrix} \quad W = \begin{bmatrix} 0.8582 & -0.2987 \\ -0.2987 & 1.8791 \end{bmatrix}$$

Here we let

$$\begin{aligned} \Gamma_{ij} := & [\lambda_{ij}A_{1i} + (1-\lambda_{ij})A_{2j}]^T P \\ & + P[\lambda_{ij}A_{1i} + (1-\lambda_{ij})A_{2j}] \\ & + P[\lambda_{ij}A_{d_{1i}}WA_{d_{1i}}^T + (1-\lambda_{ij})A_{d_{2j}}WA_{d_{2j}}^T]P + P \end{aligned}$$

Then we obtained

$$\Gamma_{11} = \begin{bmatrix} -2.5400 & -0.8733 \\ -0.8733 & -0.4932 \end{bmatrix} \quad \Gamma_{12} = \begin{bmatrix} -2.5454 & -0.9633 \\ -0.9633 & -0.5546 \end{bmatrix}$$
$$\Gamma_{21} = \begin{bmatrix} -2.5440 & -0.8701 \\ -0.8701 & -0.4913 \end{bmatrix} \quad \Gamma_{22} = \begin{bmatrix} -2.5494 & -0.9602 \\ -0.9602 & -0.5527 \end{bmatrix}$$

We can see they satisfy the conditions of (8) and (9). Therefore, this switched system is robustly stablilizable via state feedback.

Now, we will investigate the system state trajectory using two specific subsystems as follows

$$\begin{aligned} A_1 &= 0.5A_{11} + 0.5A_{12} = \begin{bmatrix} 4.0 & -4.95 \\ 0 & -4.0 \end{bmatrix} \\ A_{d_1} &= 0.5A_{d_{11}} + 0.5A_{d_{12}} = \begin{bmatrix} -1.0 & 0.05 \\ -1.0 & -1.0 \end{bmatrix} \end{aligned} \tag{59}$$

and

$$\begin{aligned} A_2 &= 0.4A_{21} + 0.6A_{22} = \begin{bmatrix} -8.0 & 4.0 \\ 0.06 & 1.0 \end{bmatrix} \\ A_{d_2} &= 0.4A_{d_{21}} + 0.6A_{d_{22}} = \begin{bmatrix} -1.0 & 0.06 \\ 1.0 & -1.0 \end{bmatrix} \end{aligned} \tag{60}$$

which belong to S_1 and S_2 and are both unstable.

Here, we suppose that the initial state is $x_0 = [2 \quad 5]^T$. Then, we get

$$\begin{aligned} x_0^T(A_{11}^T P + PA_{11} + PA_{d_{11}}WA_{d_{11}}^T P + P)x_0 &= -239.1073 \\ x_0^T(A_{12}^T P + PA_{12} + PA_{d_{12}}WA_{d_{12}}^T P + P)x_0 &= -238.9184 \\ x_0^T(A_{21}^T P + PA_{21} + PA_{d_{21}}WA_{d_{21}}^T P + P)x_0 &= 159.1965 \\ x_0^T(A_{22}^T P + PA_{22} + PA_{d_{22}}WA_{d_{22}}^T P + P)x_0 &= 152.4795 \end{aligned}$$

we choose $\sigma(x_0, 0) = 1$ according to the switching rule (21).

If we suppose the initial state is $x_0 = [2 \quad -4]^T$. Then, we get

$$\begin{aligned} x_0^T(A_{11}^T P + PA_{11} + PA_{d_{11}}WA_{d_{11}}^T P + P)x_0 &= 48.9492 \\ x_0^T(A_{12}^T P + PA_{12} + PA_{d_{12}}WA_{d_{12}}^T P + P)x_0 &= 48.8762 \\ x_0^T(A_{21}^T P + PA_{21} + PA_{d_{21}}WA_{d_{21}}^T P + P)x_0 &= -57.1063 \\ x_0^T(A_{22}^T P + PA_{22} + PA_{d_{22}}WA_{d_{22}}^T P + P)x_0 &= -56.2328 \end{aligned}$$

we choose $\sigma(x_0, 0) = 2$ according to the switching rule (21).

In the same way, we use the switching rule (21) to choose the subsystem mode for every time instant and evolve the system forth on. Although both A_1 and A_2 are unstable, but we see the system state converge to zero under the switching rule we proposed in (21) from Figure 1.

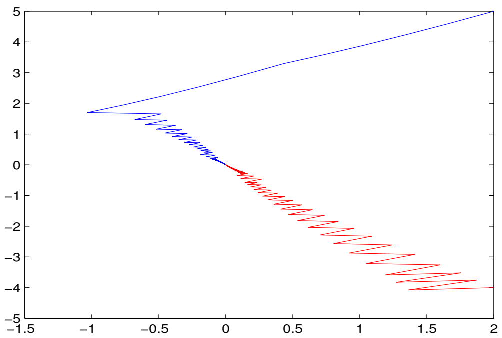

Fig. 1. The state of switched system with $\tau = 0.1$

Example 2. (Delay-dependent case) Consider the switched system (1) composed of two subsystems where

$$\begin{aligned} A_{11} &= \begin{bmatrix} 1 & 0 \\ 0.4 & -2.8 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 1 & 0.1 \\ 0.4 & -2.8 \end{bmatrix} \\ A_{d_{11}} &= \begin{bmatrix} 0.8 & 0 \\ 0 & -0.3 \end{bmatrix} \quad A_{d_{12}} = \begin{bmatrix} 0.8 & 0.1 \\ 0 & -0.3 \end{bmatrix} \end{aligned} \tag{61}$$

and

$$\begin{aligned} A_{21} &= \begin{bmatrix} -12.9 & 0.3 \\ 0 & 2 \end{bmatrix} \quad A_{22} = \begin{bmatrix} -12.9 & 0.3 \\ 0.1 & 2 \end{bmatrix} \\ A_{d_{21}} &= \begin{bmatrix} -0.6 & 0 \\ -0.2 & -0.5 \end{bmatrix} \quad A_{d_{22}} = \begin{bmatrix} -0.6 & 0.1 \\ -0.2 & -0.5 \end{bmatrix} \end{aligned} \tag{62}$$

Both $A_{11}, A_{d_{11}}$ and $A_{12}, A_{d_{12}}$ are delay-dependent robustly unstable. Similarly, both $A_{21}, A_{d_{21}}$ and $A_{22}, A_{d_{22}}$ are delay-dependent robustly unstable. Therefore, both S_1 and S_2 are robustly unstable.

Now, we let $\lambda_{11} = \lambda_{12} = \lambda_{21} = \lambda_{22} = 0.4$, $\tau = 0.172$ in order to satisfy the conditions in Theorem 5, by using MATLAB tools we get

$$P_1 = \begin{bmatrix} 29.5902 & -2.6474 \\ -2.6474 & 18.1402 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0.1914 & -0.0431 \\ -0.0431 & 0.7347 \end{bmatrix}$$
$$P = \begin{bmatrix} 5.7035 & -0.2924 \\ -0.2924 & 0.4472 \end{bmatrix}$$

Here we let

$$\begin{aligned} \Sigma_{ij} := & [\lambda_{ij}\hat{A}_{1i} + (1-\lambda_{ij})\hat{A}_{2j}]^T P \\ & + P[\lambda_{ij}\hat{A}_{1i} + (1-\lambda_{ij})\hat{A}_{2j}] \\ & + \tau P[\lambda_{ij}A_{d_{1i}}(P_1+P_2)A_{d_{1i}}^T + (1-\lambda_{ij})A_{d_{2j}} \\ & \times (P_1+P_2)A_{d_{2j}}^T]P + 2\tau P \end{aligned}$$

Then we obtained

$$\Sigma_{11} = \begin{bmatrix} -4.6381 & -0.0488 \\ -0.0488 & -0.0009 \end{bmatrix} \quad \Sigma_{12} = \begin{bmatrix} -2.6493 & -0.0099 \\ -0.0099 & -0.0074 \end{bmatrix}$$
$$\Sigma_{21} = \begin{bmatrix} -5.0492 & 0.3325 \\ 0.3325 & -0.0389 \end{bmatrix} \quad \Sigma_{22} = \begin{bmatrix} -3.0604 & 0.3714 \\ 0.3714 & -0.0454 \end{bmatrix}$$

We can see they satisfy the conditions of (30) to (32). Therefore, this switched system is robustly stablilizable via state feedback.

Now, we will investigate the system state trajectory using two specific subsystems as follows

$$\begin{aligned} A_1 &= 0.2A_{11} + 0.8A_{12} = \begin{bmatrix} 1.0 & 0.08 \\ 0.4 & -2.8 \end{bmatrix} \\ A_{d_1} &= 0.2A_{d_{11}} + 0.8A_{d_{12}} = \begin{bmatrix} 0.8 & 0.08 \\ 0 & -0.3 \end{bmatrix} \end{aligned} \tag{63}$$

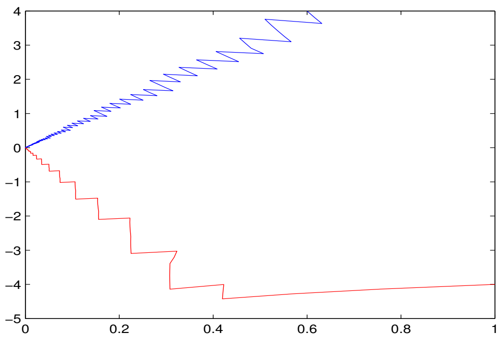

Fig. 2. The state of the switched system with $\tau = 0.172$

and

$$A_2 = 0.4A_{21} + 0.6A_{22} = \begin{bmatrix} -12.9 & 0.3 \\ 0.06 & 2.0 \end{bmatrix}$$
$$A_{d_2} = 0.4A_{d_{21}} + 0.6A_{d_{22}} = \begin{bmatrix} -0.6 & 0.06 \\ -0.2 & -0.5 \end{bmatrix} \quad (64)$$

which belong to S_1 and S_2 and are both unstable.

Here, we suppose that the initial state is $x_0 = [0.6 \quad 4]^T$. Then, we get

$$x_0^T \Upsilon_{11} x_0 = -13.4843 \qquad x_0^T \Upsilon_{12} x_0 = -10.7991$$
$$x_0^T \Upsilon_{21} x_0 = 5.7930 \qquad x_0^T \Upsilon_{22} x_0 = 7.1225$$

$$\Upsilon_{ij} := \hat{A}_{ij}^T P + P\hat{A}_{ij} + \tau P A_{d_{ij}} (P_1 + P_2) A_{d_{ij}}^T P + 2\tau P$$

we choose $\sigma(x_0, 0) = 1$ according to the switching rule (44).

If we suppose that the initial state is $x_0 = [1 \quad -4]^T$. Then, we get

$$x_0^T \Upsilon_{11} x_0 = 129.6460 \qquad x_0^T \Upsilon_{12} x_0 = 119.4713$$
$$x_0^T \Upsilon_{21} x_0 = -93.5325 \qquad x_0^T \Upsilon_{22} x_0 = -90.9114$$

we choose $\sigma(x_0, 0) = 2$ according to the switching rule (44).

In the same way, we use the switching rule (44) to choose the subsystem mode for every time instant and evolve the system forth on. Although both A_1 and A_2 are unstable, but we see the system state converge to zero under the switching rule we proposed in (44) from Figure 2.

5. CONCLUSION

In this paper, we have considered robust stabilizability via state feedback for continuous-time system that are composed of polytopic uncertain subsystems. We have shown that if there exists common positive definite matrices for stability of all convex combinations of the extreme points which belong to different subsystem matrices, then the switched system is robustly stabilizable via state feedback. We have analyzed the stability conditions for the case of delay-independent and delay-dependent and also given the switching rules σ. Numerical examples show the effectiveness of the proposed method.

REFERENCES

P. Antsaklis. Special issue on hybrid systems: Theory and applications–A brief introduction to the theory and applications of hybrid systems. *Proc IEEE*, vol.88, no.7, pages 887–897, Jul 2000.

M.S. Branicky. Multiple lyapunov functions and others analysis tools for switched and hybrid systems. *IEEE Trans. Autom. Contr*, vol.43, no.4, pages 475–482, Apr 1998.

M. Branicky, V. Borkar, and S. Mitter. A unified framework for hybrid control: Modal and optimal control theory. *IEEE Trans. Autom. Contr*, vol.43, no.1, pages 31–45, Jan 1998.

Y.-Y. Cao, Y.-X. Sun, and C. Cheng. Delay–dependent robust stabilization of uncertain systems with multiple state delays. *IEEE Trans. Autom. Control*, vol.43, pages 1608–1612, 1998.

S. Engell, S. Kowalewski, C. Schulz, and O. Strusberg. Continuous–discrete interactions in chemical processing plants. *Proc IEEE*, vol.88, no.7, pages 1050–1068, Jul 2000.

R. Horowitz, and P. Varaiya. Control design of an automated highway system. *Proc IEEE*, vol.88, no.7, pages 913–925, Jul 2000.

Lu Jian-ning, and ZHAO Guang-zhou. Stability analysis based on LMI for switched systems with time delay. *Journal of Southern Yangtze University (Natural Science Edition)*, vol.5, no.2, Apr 2006.

Sehjeong Kim, Sue Ann Campbell, and Xinzhi Liu. Stability of a class of linear switching system with time delay. *IEEE Transactions on circuits and systems*, vol.53, no.2, pages 384–393, Feb 2006.

D. Liberzon, and A.S. Morse. Basic problems in stability analysis of switched systems. *IEEE Control Systems Magazine*, vol.19, no.5, pages 59–70, Oct 1999.

C. Livadas, J. Lygeros, and N. A. Lynch. High-level modeling and analysis of the traffic alert and collision avoidance system (TCAS). *Proc IEEE*, vol.88, no.7, pages 926–948, Jul 2000.

D. Pepyne, and C. Cassandaras. Optimal control of hybrid systems in manufacturing. *Proc. IEEE*, vol.88, no.7, pages 1008–1122, Jul 2000.

M. Song, T. Tran, and N. Xi. Integration of task scheduling, action planning, and control in robotic manufacturing systems. *Proc IEEE*, vol.88, no.7, pages 1097–1107, Jul 2000.

P. Varaiva. Smart cars on smart roads: Problems of control. *IEEE Trans. Autom. Contr*, vol.38, no.2, pages 195–207, Feb 1993.

H. Ye, A.N. Michel, and L. Hou. Stability analysis of switched systems. *Presented at the Conf. Decision Control*, Tampa, Florida, 1998.

G. Zhai, H. Lin, and P. J. Antsaklis. Quadratic stabilizability of switched linear systems with polytopic uncertainties. *Int. J. Control*, vol.76, no.7, pages 747–753, 2003.