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Generalized Controller for Directed Triangle Formations ⋆

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Abstract: This paper proposes and analyzes a distributed control law which generalizes three different previously considered control laws for maintaining a triangular formation in the plane consisting of three point-modelled, mobile autonomous agents. It is shown that the control law under consideration can cause any initially non-collinear, positively-oriented {resp. negatively-oriented} triangular formation to converge exponentially fast to a desired positively-oriented {resp. negatively-oriented} triangular formation. These findings clarify and extend earlier results.

1. INTRODUCTION

Ever since the appearance of the work of Baillieul and Suri {Baillieul and Suri [2003]} which emphasizes the potential problem of controlling a group of mobile autonomous agents in a "directed" formation containing a cycle, interest has focused on understanding this issue in depth. A formation is *directed* if each agent i can sense only the relative position of its "co-leaders" where by an agent i's *co-leaders* are meant other designated agents in the formation whose distances from agent i it is the responsibility of agent i to maintain. Since a directed triangular formation in the plane is the simplest formation with asymmetric co-leader relations which is both rigid and contains a cycle, it is natural to consider the problem of trying to maintain a directed triangular formation. Prompted by this, we consider the problem of maintaining a directed formation of three agents in a triangle by having each agent locally control its own position so that the distance to its co-leader {or next agent in the triangle} is constant. This particular problem has recently been addressed in Smith et al. [2006], Anderson et al. [2007] and Cao et al. [2007]. A distinguishing feature of this paper is that it considers a class of control laws which encompasses those considered in these three earlier reference. Another distinguishing feature is that the paper explicitly takes into account in the analysis the fact that the control laws considered in Smith et al. [2006] and Anderson et al. [2007] output control signals which grow without bound as the points in the formation approach each other. To deal with the manifold $\mathcal{Z}$ on which the control laws are not well-defined, one must consider a dynamical system whose state space excludes $\mathcal{Z}$. We prove that unique solutions to the systems of nonlinear differential equations involved either approach $\mathcal{Z}$ or exist for all time. We explicitly characterize a closed manifold $\mathcal{N}$ on which agents are collinearly positioned. Our main result is to show that the controls we consider will cause any initially non-collinear, "positively-oriented" {resp. negatively-oriented} triangular formation to converge exponentially fast to a prescribed positively-oriented {resp. negatively-oriented} triangular formation and then come to rest. The analysis in this paper clarifies and more completely explains the results in Smith et al. [2006] and Anderson et al. [2007]. We refer the reader to these papers for additional background and references on controlling triangular formations.

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2. TRIANGLE FORMATION

As in Cao et al. [2007], we consider a formation in the plane consisting of three mobile autonomous agents labelled $1, 2, 3$ where agent 1 follows 2, 2 follows 3 and 3 follows 1. For $i \in \{1, 2, 3\}$, we write $[i]$ for the label of agent i's *co-leader* where $[1] = 2$, $[2] = 3$ and $[3] = 1$. We assume that the desired distance between agents i and $[i]$ is d_i; here the d_i are positive numbers which satisfy the triangle inequalities:

$$d_1 < d_2 + d_3 \qquad d_2 < d_1 + d_3 \qquad d_3 < d_1 + d_2 \qquad (1)$$

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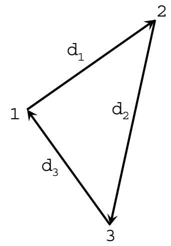


Fig. 1. Directed Point Formation

Note that there are two distinct triangular formations which satisfy the desired distance constraints. The first is as shown in Figure 1 and is referred to as a *clockwise-oriented triangle*. The second, called a *counterclockwise oriented triangle* is the triangle which results when the triangle shown in Figure 1 is flipped over.

In the sequel we write x_i for the Cartesian coordinate vector of agent i in some fixed global coordinate system in the plane, and y_{ij} for the position of agent j in some fixed coordinate system of agent i's choosing. Thus for $i \in \{1, 2, 3\}$, there is a rotation matrix R_i and a translation vector τ_i such that $y_{ij} = R_i x_j + \tau_i,\ j \in \{1, 2, 3\}$. We assume that agent i's motion is described by a simple kinematic point model of the form

$$\dot{y}_{ii} = u_i \quad i \in \{1, 2, 3\}$$

where u_i is agent i's control input. Thus in global coordinates,

$$\dot{x}_i = R_i^{-1} u_i, \quad i \in \{1, 2, 3\} \tag{2}$$

We assume that for $i \in \{1, 2, 3\}$, agent i can measure the relative position of agent $[i]$ in its own coordinate system. This means that for $i \in \{1, 2, 3\}$, agent i can measure the signal $R_i z_i$ where

$$z_i = x_i - x_{[i]}, \quad i \in \{1, 2, 3\} \tag{3}$$

Our aim is to define control laws of a sufficiently general form to encompass the control laws studied previously in Cao et al. [2007], Anderson et al. [2007], Smith et al. [2006]. Towards this end we consider controls of the form

$$u_i = -R_i z_i e_i, \quad i \in \{1, 2, 3\} \tag{4}$$

where

$$e_i = g_i(||R_i z_i||^2 - d_i^2), \quad i \in \{1, 2, 3\}$$

and g_i is a strictly monotone increasing function which is defined and continuously differentiable on the *open* interval $(-d_i^2, \infty)$ and satisfies $g_i(0) = 0$. Thus u_i is a well-defined, continuously differentiable control law on open space

$$\mathbb{R}^6 - \mathcal{Z}_i$$

where $\mathcal{Z}_i = \{x : z_i = 0\}$ and $\mathbb{R}^6 - \mathcal{Z}$ is the complement of $\mathcal{Z}$ in $\mathbb{R}^6$. Note that the rotation matrices do not affect the definition of the e_i in that

$$e_i = g_i(||z_i||^2 - d_i^2), \quad i \in \{1, 2, 3\} \tag{5}$$

Moreover R_i cancels out of the update equation

$$\dot{x}_i = -z_i e_i, \quad i \in \{1, 2, 3\} \tag{6}$$

Set

$$\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \tag{7}$$

The closed loop system of interest is thus the smooth, time-invariant, dynamical system on the state space

$$\mathcal{X} = \mathbb{R}^6 - \mathcal{Z}$$

described in global coordinates by the equations

$$\dot{x}_i = -(x_i - x_{[i]}) g_i(||x_i - x_{[i]}||^2 - d_i^2), \quad i \in \{1, 2, 3\} \tag{8}$$

In the sequel we shall refer this system as the *overall system.*

3. ANALYSIS

Our aim is to study the geometry of the overall system. Towards this end let

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{9}$$

Let us note at once that because of Lipschitz continuity, for any initial state $x(0) = y \in \mathcal{X}$ there must exist a largest interval $[0, T_y)$ on which a unique solution to (8) exists. Next note that as a consequence of the definitions of the z_i in (3),

$$z_1 + z_2 + z_3 = 0 \tag{10}$$

and

$$\dot{z}_i = -z_i e_i + z_{[i]} e_{[i]}, \quad i \in \{1, 2, 3\} \tag{11}$$

Observe that the equilibrium points of the overall system are those values of the x_i for which

$$z_i e_i = 0, \quad i \in \{1, 2, 3\} \tag{12}$$

Since $z_i \neq 0$ for $x \in \mathcal{X}$,

$$\mathcal{E} = \{x : e = 0, x \in \mathcal{X}\} \tag{13}$$

is the equilibrium set of the overall system. It is possible to show by example that this set is not globally attractive and thus that the overall system is not globally asymptotically stable. On the other hand it will be shown that there is a thin set in $\mathbb{R}^6$ outside of which all trajectories approach $\mathcal{E}$ exponentially fast. The set to which we are referring corresponds to those formations in $\mathcal{X}$ which are collinear. To explicitly characterize this set, we need the following fact.

Lemma 1. The points at x_1, x_2, x_3 in $\mathbb{R}^6$ are collinear if and only if

$$\text{rank } [z_1\ z_2\ z_3] < 2$$

The simple proof is omitted.

To proceed, let $\mathcal{N}$ denote the set of points in $\mathbb{R}^6$ corresponding to points in the plane which are collinear. In other words

$$\mathcal{N} = \{x : \text{rank } [z_1\ z_2\ z_3] < 2,\ z_1 + z_2 + z_3 = 0\} \tag{14}$$

Note that $\mathcal{N}$ is a closed manifold in $\mathbb{R}^6$. Note in addition that $\mathcal{N}$ contains $\mathcal{Z}$ and is small enough to not intersect $\mathcal{E}$:

Lemma 2. $\mathcal{N}$ and $\mathcal{E}$ are disjoint sets.

Proof: To show that $\mathcal{N} \cap \mathcal{E}$ is empty, we assume the contrary. Let $x \in \mathcal{N} \cap \mathcal{E}$ be fixed. Since $\mathcal{E}$ and $\mathcal{Z}$ are disjoint, $x \notin \mathcal{Z}$. Therefore $z_i \neq 0$ for some $i \in \{1, 2, 3\}$. Then there must be a number λ such that $z_{[i]} = \lambda z_i$. Hence $z_{[i]+1} = -(1 + \lambda) z_i$. But $x \in \mathcal{E}$ so $||z_i|| = d_i,\ i \in \{1, 2, 3\}$. Thus $|\lambda| d_i = d_{[i]}$ and $|1 + \lambda| d_i = d_{[i]+1}$. Then $d_i + d_{[i]} = d_{[i]+1}$ when $\lambda \geq 0$, $d_i + d_{[i]+1} = d_{[i]}$ when $\lambda \leq -1$, and $d_{[i]} + d_{[i]+1} = d_i$ when $-1 < \lambda < 0$. All of these equalities contradict (1). Therefore $\mathcal{N}$ and $\mathcal{E}$ are disjoint sets. ■

That $\mathcal{N} \cap \mathcal{X}$ might be the place where formation control will fail is further underscored by the fact that formations points in $\mathcal{X}$ which are initially collinear, remain collinear along all trajectories starting at such points. To understand why this is so, first note that for any two vectors $p, q \in \mathbb{R}^2$, $\det[p\ q] = p'Gq$ where

$$G = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

From this and (10) it follows that $\det[z_1\ z_2] = -\det[z_1\ z_3]$. This and the definition of $\mathcal{N}$ in (14) imply that

$$\mathcal{N} \cap \mathcal{X} = \{x : x \in \mathcal{X},\ \det[z_1\ z_2] = 0\} \quad (15)$$

Moreover (11) implies that

$$\frac{d}{dt}\det[z_1\ z_2] = -(e_1 + e_2 + e_3)\det[z_1\ z_2] \quad (16)$$

Thus if $\det[z_1\ z_2] = 0$ at $t = 0$, then $\det[z_1\ z_2] = 0$ along all the trajectory of the overall system starting at y.

It can be shown that there are initially collinear formations in $\mathcal{N} \cap \mathcal{X}$ which tend to the boundary of $\mathcal{X}$, which of course is a form of instability. Despite the fact that misbehavior can occur within $\mathcal{N} \cap \mathcal{X}$, the dimension of $\mathcal{N}$ is less than 6 which means that "almost every" initial formation in $\mathcal{X}$ will be non-collinear. The good news is that *all* such initially non-collinear formations will converge to the desired formation and come to rest, and moreover, the convergence will occur exponentially fast. This is in essence, the geometric interpretation of our main result on triangular formations.

Theorem 1. Each trajectory of the overall system (8) starting outside of $\mathcal{N}$, converges exponentially fast to a finite limit point in $\mathcal{E}$.

The set of points $\mathcal{X} - \mathcal{N} \cap \mathcal{X}$ consists of two disjoint point sets, one for which $\det[z_1\ z_2] > 0$ and the other for which $\det[z_1\ z_2] < 0$. Once the theorem has been proved, it is easy to verify that formations starting at points such that $\det[z_1\ z_2] < 0$, converge to the clockwise oriented triangular formation shown in Figure 1 whereas formations starting at points such that $\det[z_1\ z_2] > 0$, converge to the corresponding counterclockwise oriented triangular formation.

The proof of Theorem 1 involves several steps. The first is to show that all trajectories in $\mathcal{X}$ which do not tend to $\mathcal{Z}$, exist for all time. To accomplish this, let $\Omega : (-d_1^2, \infty) \times (-d_2^2, \infty) \times (-d_3^2, \infty) \to \mathbb{R}$ denote the function

$$\Omega(w_1, w_2, w_3) = \int_0^{w_1} g_1(s)ds + \int_0^{w_2} g_2(s)ds + \int_0^{w_3} g_3(s)ds$$

Observe that the constraints on the g_i imply that Ω is continuously differentiable and positive definite; moreover Ω is an unbounded function of w where $w = [w_1\ w_2\ w_3]'$. Let

$$V = \Omega(||z_1||^2 - d_1^2, ||z_2||^2 - d_2^2, ||z_3||^2 - d_3^2)$$

Fix $y \in \mathcal{X}$ and let $[0, T_y)$ denote the maximal interval of existence of the overall system. Then for $t \in [0, T_y)$,

$$\dot{V} = -2\{(z_1'z_1e_1^2 - z_1'z_2e_1e_2) + (z_2'z_2e_2^2 - z_2'z_3e_2e_3) + (z_3'z_3e_3^2 - z_3'z_1e_3e_1)\}$$

or

$$\dot{V} = -||z_1e_1 - z_2e_2||^2 - ||z_2e_2 - z_3e_3||^2 - ||z_3e_3 - z_1e_1||^2 \quad (17)$$

Thus V is monotone non-increasing on $[0, T_y)$. Since V is also bounded below by 0, V must be bounded on $[0, T_y)$. In view of the fact that Ω is a continuous, unbounded function of w, each $||z_i||^2 - d_i^2$ is also bounded on $[0, T_y)$. Therefore each z_i is bounded on $[0, T_y)$. In view of (8), $\dot{x}$ is also bounded on $[0, T_y)$. At this point one of two things can happen: Either $x \to \mathcal{Z}$ as $t \to T_y$ or it does not. We are interested in the latter situation in which case either $T_y = \infty$ or $T_y < \infty$. If it were true that $T_y < \infty$, then x would approaches the finite limit

$$\bar{x} = \int_0^{T_y} \dot{x}dt$$

as $t \to T_y$. But if this were so, then there would have to be an interval $[T_y, T')$ of positive length on which a solution to the overall system starting at $\bar{x}$ exists. Uniqueness would then imply the existence on $[0, T')$ of a solution to the overall system starting at y. This contradicts the assumption that $[0, T_y)$ is the interval of maximal existence. Thus $T_y = \infty$. We summarize.

Proposition 1. Each trajectory of the overall system either tends to $\mathcal{Z}$ or exists on $[0, \infty)$. Moreover z is bounded along any trajectory of the overall system which does not approach $\mathcal{Z}$.

More can be said if $y \notin \mathcal{N}$. Suppose that this is so in which case $\det[z_1(0)\ z_2(0)] \neq 0$ because of (14). Suppose $T_y < \infty$. In view of Proposition 1, x would tend to $\mathcal{Z}$ as $t \to \infty$. Since $\mathcal{Z} \subset \mathcal{N}$, x would therefore tend to $\mathcal{N}$ as $t \to T_y$. In other words, $\det[z_1\ z_2] \to 0$ as $t \to T_y$, again because of (14) . But this is impossible because of (16). Therefore $T_y = \infty$. We've proved the following.

Corollary 1. Each trajectory of the overall system which starts outside of $\mathcal{N}$ exists on $[0, \infty)$ and remains outside of $\mathcal{N}$ for $t < \infty$.

Our next goal is to show that there is an open set of points in $\mathcal{X}$ from which all solutions to the overall system tend to $\mathcal{E}$ exponentially fast. For this we will need the following.

Lemma 3. Let ρ be a positive number. There exist positive numbers μ and δ such that

$$\Omega(w_1, w_2, w_3) \geq \frac{\mu}{2}||w||^2,\ \ ||w||^2 \leq \rho \quad (18)$$

$$\Omega(w_1, w_2, w_3) \leq \frac{\delta}{2}||w||^2,\ \ ||w||^2 \leq \rho \quad (19)$$

Proof: Let

$$\mu_i = \inf_{|s| \leq \sqrt{\rho}} \frac{dg_i(s)}{ds},\ i \in \{1, 2, 3\}$$

Each μ_i is positive because each g_i is continuously differentiable and strictly increasing. From this and the assumption that $g_i(0) = 0,\ i \in \{1, 2, 3\}$ it follows that

$$|g_i(s)| \geq \mu_i|s|, |s| \leq \sqrt{\rho},\ i \in \{1, 2, 3\} \quad (20)$$

Therefore

$$\int_0^{w_i} g_i(s)ds \geq \frac{\mu_i}{2}w_i^2,\quad |w_i| \leq \sqrt{\rho}$$

Set $\mu = \min\{\mu_1, \mu_2, \mu_3\}$. It follows that (18) is true.

Since each g_i is Lipschitz continuous and satisfies $g_i(0) = 0$, there are positive constants $\delta_i,\ i \in \{1, 2, 3\}$ such that $|g_i(s)| \leq \delta_i|s|,\ |s| \leq \sqrt{\rho}$. It follows from this that

$$\int_0^{w_i} g_i(s)ds \leq \frac{\delta_i}{2}w_i^2,\quad |w_i| \leq \sqrt{\rho}$$

Set $\delta = \max\{\delta_1, \delta_2, \delta_3\}$. It follows that (19) is true. ■

To proceed, observe from (17) that

$$\dot{V} = -e'Q'Qe$$

where

$$Q = \begin{bmatrix} -z_1 & z_2 & 0 \\ 0 & -z_2 & z_3 \\ z_1 & 0 & -z_3 \end{bmatrix}$$

Note that Q is also the transpose of the rigidity matrix {Eren et al. [2002]} of the point formation shown in Figure 1. By inspection it is clear that the rank of Q is less than three just in case, for at least one distinct pair of integers $i, j \in \{1, 2, 3\}$, z_i is a scalar multiple of z_j; moreover, because of (10) for such i and j, z_k would also have to be a scalar multiple of z_j where k is the remaining integer in $\{1, 2, 3\}$. In other words,

$$\text{rank } Q < 3 \Longleftrightarrow \text{rank } [z_1 \ z_2 \ z_3] < 2$$

In the light of this and the definition of $\mathcal{N}$, it is clear that $Q'Q$ is positive definite if and only if $x \notin \mathcal{N}$. Let ρ be a positive number and define

$$\mathcal{S}(\rho) = \{x : \sum_{i=1}^{3}(||z_i||^2 - d_i^2)^2 < \rho, \quad z_1 + z_2 + z_3 = 0, \ x \in \mathcal{X}\}$$

Note that $\mathcal{E} \subset \mathcal{S}(\rho)$ and that $\mathcal{S}(\rho) \to \mathcal{E}$ as $\rho \to 0$. In view of Lemma 2 it is possible to choose ρ so small that $\mathcal{N}$ and $\mathcal{S}(\rho)$ are disjoint. Pick ρ so that this is so and also so that $\rho < \min\{d_1^2, d_2^2, d_3^2\}$. This last inequality ensures that the closure of $\mathcal{S}(\rho)$ and $\mathcal{Z}$ are also disjoint.

To proceed, let μ and δ be as in Lemma 3 and note from the inequalities therein that $\mu \leq \delta$. Pick any positive number $\rho^* < \frac{\mu}{\delta}\rho$. Since $\frac{\mu}{\delta} \leq 1$, $\mathcal{S}(\rho^*)$ is a strictly proper subset of $\mathcal{S}(\rho)$. It will now be shown that any trajectory starting in $\mathcal{S}(\rho^*)$ remains in $\mathcal{S}(\rho)$ and converges to $\mathcal{E}$ exponentially fast. Towards this end fix $y \in \mathcal{S}(\rho^*)$. Since $\mathcal{S}(\rho^*)$ and $\mathcal{N}$ are disjoint, $T_y = \infty$ because of Corollary 1. Let $\{x(t) : t \in [0, \infty)\}$ denote the corresponding trajectory of the overall system. Since $x(0) \in \mathcal{S}(\rho)$ and $\mathcal{S}(\rho^*)$ is a strictly proper subset of $\mathcal{S}(\rho)$, there must be a positive time T' such that $x(t) \in \mathcal{S}(\rho), \ t \in [0, T')$. Let T^* be the largest such time. In view of (19), and the definition of V,

$$V \leq \frac{\delta}{2}\sum_{i=1}^{3}(||z_i||^2 - d_i^2)^2, \ t \in [0, T^*) \tag{21}$$

But $x(0) \in \mathcal{S}(\rho^*)$ so

$$\sum_{i=1}^{3}(||z_i(0)||^2 - d_i^2)^2 < \rho^*$$

From this, (21) and the fact that V is non-decreasing on $[0, T^*)$, there follows $V(t) < \frac{\delta}{2}\rho^*, \ t \in [0, T^*)$. But in view of (18), and the definition of V,

$$V \geq \frac{\mu}{2}\sum_{i=1}^{3}(||z_i||^2 - d_i^2)^2, \ t \in [0, T^*) \tag{22}$$

Therefore

$$\sum_{i=1}^{3}(||z_i||^2 - d_i^2)^2 < \frac{\delta}{\mu}\rho^*, \ t \in [0, T^*) \tag{23}$$

Suppose $T^* < \infty$. Then (23) implies that

$$\sum_{i=1}^{3}(||z_i||^2 - d_i^2)^2 \leq \frac{\delta}{\mu}\rho^*, \ t \in [0, T^*] \tag{24}$$

But $\frac{\delta}{\mu}\rho^* < \rho$, so

$$\sum_{i=1}^{3}(||z_i||^2 - d_i^2)^2 < \rho, \ t \in [0, T^*]$$

Because of continuity, this means there is an interval $[0, T')$ larger than $[0, T^*)$ such that $x(t) \in \mathcal{S}(\rho), \ t \in [0, T')$. This is impossible because $[0, T^*)$ was defined to be the largest such interval. Therefore $T^* = \infty$. This proves that the trajectory remains in $\mathcal{S}(\rho)$ for all time.

It will now be shown that x tends to $\mathcal{E}$ exponentially fast. For this let $\widehat{\mathcal{S}}$ denote the closure of $\{z : x \in \mathcal{S}(\rho^*)\}$. It is clear that $\widehat{\mathcal{S}}$ is compact. In addition, since $\mathcal{S}(\rho^*)$ is a strictly proper subset of $\mathcal{S}(\rho)$ and $\mathcal{S}(\rho)$ and $\mathcal{N}$ are disjoint, $\pi(Q'Q) > 0, \ z \in \widehat{\mathcal{S}}$, where $\pi(Q'Q)$ is the smallest eigenvalue of $Q'Q$. Thus if we define

$$\lambda = \inf_{z \in \widehat{\mathcal{S}}} \pi(Q'Q)$$

then $\lambda > 0$ and for $t \in [0, \infty)$

$$\dot{V} \leq -\lambda||e||^2 \tag{25}$$

But $e_i = g(||z_i||^2 - d_i^2), \ i \in \{1, 2, 3\}$. In view of (20) and the fact that $x(t) \in \mathcal{S}(\rho)$ for all time, $|e_i| \geq \mu_i|||z_i||^2 - d_i^2|, \ i \in \{1, 2, 3\}$. This implies that

$$||e||^2 \geq \mu^2 \sum_{i=1}^{3}(||z_i||^2 - d_i^2)^2$$

where $\mu = \min\{\mu_1, \mu_2, \mu_3\}$. From this and (21) there follows $||e||^2 \geq \frac{2}{\delta}\mu^2 V$. Combining this with (25) one gets

$$\dot{V} \leq -\frac{2}{\delta}\lambda\mu^2 V$$

Therefore by the Bellman-Gronwall Lemma

$$V \leq V(0)e^{-\frac{2}{\delta}\lambda\mu^2 t}, \ t \geq 0$$

so V tends to 0 exponentially fast. In view of (22), each $(||z_i||^2 - d_i^2), \ i \in \{1, 2, 3\}$ also tends to zero exponentially fast. This proves that the trajectory under consideration approaches $\mathcal{E}$ exponentially fact. We summarize:

Proposition 2. There exists an open set of points in $\mathcal{X}$, namely $\mathcal{S}(\rho)$, within which all trajectories of the overall system converge to $\mathcal{E}$ exponentially fast.

Note that for a trajectory to converge to $\mathcal{E}$ means that the corresponding formation converges to the desired triangle. To show that the formation actually comes to rest is a simple matter of exploiting the fact that the $||\dot{x}_i||$ are bounded above by signals which are decaying to zero exponentially fast. A proof of this last observation will not be given here.

To show that *all* trajectories outside of $\mathcal{N}$ converge exponentially fast to $\mathcal{E}$ requires more work. In view of Proposition 2, we already know that any trajectory which enters $\mathcal{S}(\rho)$ in finite time must converge to $\mathcal{E}$ exponentially fast. The problem then is to show that any trajectory starting outside of $\mathcal{N}$ must enter $\mathcal{S}(\rho)$ in finite time. A key observation from (17) needed to prove this is that $\dot{V} < 0$ whenever the three velocity vectors $z_i e_i, \ i \in \{1, 2, 3\}$ are not all equal. Prompted by this, let

$$\mathcal{M} = \mathcal{Z}_0 \bigcup_{i=0}^{3} \mathcal{M}_i$$

where

$$\mathcal{Z}_0 = \mathcal{Z}_1 \cap \mathcal{Z}_2 \cap \mathcal{Z}_3$$
$$\mathcal{M}_0 = \{x : x \in \mathcal{N} \cap \mathcal{X},\ z_1e_1 = z_2e_2 = z_3e_3\}$$

and for $i \in \{1, 2, 3\}$

$$\mathcal{M}_i = \{x : z_i = 0,\ z_{[i]}e_{[i]} = z_{[[i]]}e_{[[i]]},\ x \in \mathbb{R}^6 - \mathcal{Z}_{[i]} \cup \mathcal{Z}_{[[i]]}\}$$

Note that $\mathcal{M} \subset \mathcal{N}$ and that $\mathcal{M}$ and $\mathcal{E}$ are disjoint because $\mathcal{N}$ and $\mathcal{E}$ are.

To show that trajectories starting outside of $\mathcal{N}$ must converge exponentially fast to $\mathcal{E}$, it is enough to show that all such trajectories are bounded away from $\mathcal{M}$, even in the limit as $t \to \infty$. In the sequel we explain why this is so.

Consider the function $\Phi : \mathcal{X} \to \mathbb{R}$ given by $\Phi(x) = -||z_1e_1 - z_2e_2||^2 - ||z_2e_2 - z_3e_3||^2 - ||z_3e_3 - z_1e_1||^2$ with the z_i and e_i as defined previously. Obviously $\Phi(x(t)) = \dot{V}$ when $x(t)$ is a solution to the overall system. We are interested in the following property of Φ when viewed as a function on $\mathcal{X}$.

Lemma 4. Let $\mathcal{T}$ be any subset of $\mathcal{X}$ whose closure is disjoint with $\mathcal{M} \cup \mathcal{E}$. Then

$$\sup_{x \in \mathcal{T}} \Phi(x) < 0 \qquad (26)$$

Proof: Observe that (26) will be true if

$$\Phi(x) < 0, \quad x \in \mathcal{T} \qquad (27)$$

and

$$\lim_{x \to \mathcal{B}} \Phi(x) < 0 \qquad (28)$$

where $\mathcal{B}$ is the boundary of $\mathcal{T}$. Note in addition that $\Phi(x) < 0$ whenever $\Phi(x) \neq 0$. Thus (27) and (28) are equivalent to

$$\Phi(x) \neq 0, \quad x \in \mathcal{T} \qquad (29)$$

and

$$\lim_{x \to \mathcal{B}} \Phi(x) \neq 0 \qquad (30)$$

respectively.

Since $\mathcal{T}$ and $\mathcal{M} \cup \mathcal{E}$ are disjoint, to prove (29) it is sufficient to prove that if $\Phi(x) = 0$ for some $x \in \mathcal{X}$, then $x \in \mathcal{M} \cup \mathcal{E}$. Therefore suppose $\Phi(x) = 0$ for some $x \in \mathcal{X}$. If $e = 0$, then it is clear that $x \in \mathcal{E} \cup \mathcal{M}$. Suppose $e \neq 0$ in which case at least one e_i is nonzero. Moreover, since Φ is well defined on $\mathcal{X}$, $z_1e_1 = z_2e_2 = z_3e_3$ because of the definition of Φ. Thus the three z_i are scalar multiples of one of them. Hence by Lemma 1, $x \in \mathcal{N}$. It follows that x is in $\mathcal{M}_0$ and consequently in $\mathcal{M} \cup \mathcal{E}$.

Since $\mathcal{B}$ and $\mathcal{M} \cup \mathcal{E}$ are disjoint, to prove (30) it is enough to show that if $\Phi(x) \to 0$ then $x \to \mathcal{M} \cup \mathcal{E}$. Suppose $\Phi(x) \to 0$ in which case either $x \to \mathcal{X}$ or $x \to \mathcal{Z}$ because $\mathbb{R}^6 = \mathcal{X} \cup \mathcal{Z}$. If the former true, then clearly $x \to \mathcal{M} \cup \mathcal{E}$ because as was just proved, the relations $\Phi(x) = 0$ and $x \in \mathcal{X}$ imply $x \in \mathcal{M} \cup \mathcal{E}$.

Suppose $x \to \mathcal{Z}$ in which case for at least one i, $z_i \to 0$. Without loss of generality, suppose $i = 1$. Note that if $x \to \mathcal{Z}_0$, then $x \to \mathcal{M}$ because of the definition of $\mathcal{M}$. Suppose therefore that $x \not\to \mathcal{Z}_0$. This means that for some i - say $i = 2$ - $z_2 \not\to 0$. By (10), $z_3 \not\to 0$. Thus $x \to \mathbb{R}^6 - \mathcal{Z}_2 \cup \mathcal{Z}_3$. Moreover $z_3e_3 - z_2e_2$ tends to zero because $\Phi(x) \to 0$. In view of the definition of $\mathcal{M}_2$, it is therefore clear that $x \to \mathcal{M}_2$. Therefore $x \to \mathcal{M} \cup \mathcal{E}$. ■

To proceed, suppose that $x(t)$ is a trajectory starting outside of $\mathcal{N}$ and that for all t, $x(t)$ is bounded away from $\mathcal{M}$. Then

$$\gamma = \inf_{t \to \infty} \delta(x(t))$$

must be a positive number where for $x \in \mathbb{R}^6$, $\delta(x)$ denotes the distance between x and $\mathcal{M}$. In view of the preceding, $x(t)$ will converge to $\mathcal{E}$ provided there is a finite time t_1 such that $x(t_1) \in \mathcal{S}(\rho)$. To prove that such a time must exist, we will assume the contrary and show that this leads to a contradiction.

Suppose that for all t, $x(t)$ is in the complement of $\mathcal{S}(\rho)$ which we denote by $\bar{\mathcal{S}}(\rho)$. Thus for all t, $x(t)$ is in the closed set $\mathcal{V} = \{x : \delta(x) \geq \gamma,\ x \in \bar{\mathcal{S}}(\rho)\}$ which in turn is disjoint with $\mathcal{M} \cup \mathcal{E}$. Note that $\mathcal{V}$ contains the closure of the set $\mathcal{T} = \mathcal{X} \cap \mathcal{V}$; this means that $\mathcal{M} \cup \mathcal{E}$ and the closure of $\mathcal{T}$ are disjoint. Thus if we define

$$\sigma = \sup_{x \in \mathcal{T}} \Phi(x)$$

then $\sigma < 0$ because of Lemma 4. Since $\dot{V}(t) = \Phi(x(t))$, it must be true that $\dot{V}(t) \leq -\sigma, \quad t < \infty$. Thus $V \leq V(0) - \sigma t$ for all $t < \infty$. But this is impossible because V is nonnegative. Therefore x enters $\mathcal{S}(\rho)$ in finite time and consequently converges to $\mathcal{E}$.

We now turn to the problem of showing that all trajectories starting outside of $\mathcal{N}$ must be bounded away from $\mathcal{M}$, even in the limit as $t \to \infty$. As a first step toward this end, let us note that

$$\det[z_1(t)\ z_2(t)] = e^{-\int_\tau^t (e_1(s)+e_2(s)+e_3(s))ds} \det[z_1(\tau)\ z_2(\tau)] \qquad t \geq \tau \geq 0 \quad (31)$$

because of (16). In view of (15) it must therefore be true that any trajectory starting outside of $\mathcal{N}$ cannot enter $\mathcal{N}$ {and therefore $\mathcal{M}$} in finite time. It remains to be shown that any such trajectory can also not enter $\mathcal{M}$ even in the limit as $t \to \infty$. To prove that this is so we need the following facts. Let $\Theta : \mathcal{X} \to \mathbb{R}$ denote the function $\Theta(x) = e_1 + e_2 + e_3$ with the z_i and e_i as defined previously. Obviously $\Theta(x(t)) = e_1(t) + e_2(t) + e_3(t)$ when $x(t)$ is a solution to the overall system and the $e_i(t)$ are the values of the error signals along that solution. We are interested in the following property of Θ when viewed as a function on $\mathcal{X}$.

Lemma 5.

$$\lim_{x \to \mathcal{M}} \Theta(x) < 0 \qquad (32)$$

Proof: We first prove that $\Theta(x) < 0$ when $x \to \mathcal{M}_0$. Since the e_i, $i = 1, 2, 3$, are well defined on $\mathcal{M}_0$ and because of the continuity of e_i, it is enough to show $e_1(q) + e_2(q) + e_3(q) < 0$ for all $q \in \mathcal{M}_0$. Since $\mathcal{M}_0$ is a subset of $\mathcal{N}$, it is always true that $\|z_i(q)\| = \|z_{[i]}(q)\| + \|z_{[[i]]}(q)\|$ for some $i \in \{1, 2, 3\}$. Without loss of generality, suppose $\|z_1(q)\| = \|z_2(q)\| + \|z_3(q)\|$. This implies that $\|z_1(q)\| > \|z_2(q)\|$ because $z_3(q) \neq 0$. Observe that if $e_i(q) = 0$ for some $i \in \{1, 2, 3\}$, then $e_i(q) = 0$ for all $i \in \{1, 2, 3\}$ because $z_1(q)e_1(q) = z_2(q)e_2(q) = z_3(q)e_3(q)$ and $\|z_1(q)\|, \|z_2(q)\|, \|z_3(q)\| > 0$. However, $e_1(q)$, $e_2(q)$ and $e_3(q)$ cannot be zero at the same time because $\mathcal{M}_0$ and $\mathcal{E}$ are disjoint. Thus $e_i(q) \neq 0$ for all $i \in \{1, 2, 3\}$. Since

$e_1(q)z_1(q) = e_2(q)z_2(q)$ and $\|z_1(q)\| > \|z_2(q)\|$, it follows that $|e_1(q)| < |e_2(q)|$. Because of the equality $\|z_1(q)\| = \|z_2(q)\| + \|z_3(q)\|$, we know that $z_1(q)$ is pointing to the opposite direction with respect to that of $z_2(q)$ and $z_3(q)$, which implies that $e_1(q)e_2(q) < 0$ and $e_1(q)e_3(q) < 0$. Now suppose $e_1(q) < 0$. Then $e_2(q) > 0$ and $e_3(q) > 0$, which imply that $\|z_1(q)\| < d_1$, $\|z_2(q)\| > d_2$ and $\|z_3(q)\| > d_3$. Consequently $d_1 > \|z_1(q)\| = \|z_2(q)\| + \|z_3(q)\| > d_2 + d_3$ which contradicts the triangle inequality $d_1 < d_2 + d_3$. Hence, it must be true that $e_1(q) > 0$, $e_2(q) < 0$ and $e_3(q) < 0$. In view of the fact $|e_1(q)| < |e_2(q)|$, we know $e_1(q) + e_2(q) + e_3(q) < e_3(q) < 0$.

Now we prove if x approaches $\mathcal{M}_i$, $i = 1, 2, 3$, then $\Theta(x) < 0$. Suppose $x \to \mathcal{M}_1$. Then $z_1 \to 0$ and $z_2e_2 - z_3e_3 \to 0$. Thus $z_1 + z_2 \to 0$ and neither is zero so $e_2 + e_3 \to 0$. Meanwhile, e_1 gets negative because of the definition of g_1, so the sum of the e_i has a negative limit.

Finally, we prove that $\Theta(x) < 0$ when $x \to \mathcal{Z}_0$. Since the z_i tend to zero for $i = 1, 2, 3$, the e_i are negative in view of the definition of g_i. So the sum of the e_i has a negative limit.

Summarizing these three cases, we conclude that (32) is true. ■

We are now ready to show that any trajectory starting outside of $\mathcal{N}$, cannot approach $\mathcal{M}$ in the limit as $t \to \infty$. Suppose the opposite is true, namely that $x(t)$ is a trajectory starting outside of $\mathcal{N}$ which approaches $\mathcal{M}$ as $t \to \infty$. Then in view of (31), (15), and the fact that $\mathcal{M} \subset \mathcal{N}$,

$$\lim_{t\to\infty} |\det[z_1 \;\; z_2]| = 0 \tag{33}$$

We will now show that this is false.

In view of Lemma 5, there must be an open set $\mathcal{V}$ containing $\mathcal{M}$ on which the inequality in the lemma continues to hold. In view of Lemma 2 and the fact that $\mathcal{M} \subset \mathcal{N}$, it is possible to choose $\mathcal{V}$ small enough so that in addition to the preceding, $\mathcal{V}$ and $\mathcal{E}$ are disjoint. For $x(t)$ to approach $\mathcal{M}$ means that for some finite time T, $x(t) \in \mathcal{V}$, $t \in [T, \infty)$. This implies that $e_1 + e_2 + e_3 < 0$, $t \geq T$. In view of (31), $|\det[z_1 \;\; z_2]| \geq |\det[z_1(T) \;\; z_2(T)]|$, $t \geq T$. But

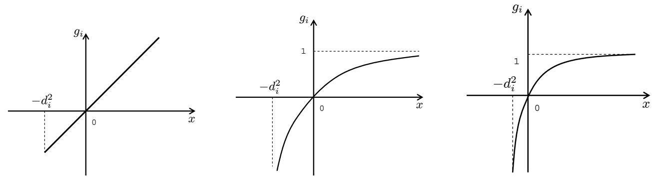
$$|\det[z_1(T) \;\; z_2(T)]| = e^{-\int_0^T (e_1(s)+e_2(s)+e_3(s))ds} |\det[z_1(0) \;\; z_2(0)]|$$

Moreover, $|\det[z_1(0) \;\; z_2(0)]| > 0$ because z starts outside of $\mathcal{N}$. Therefore $|\det[z_1 \;\; z_2]| > |\det[z_1(T) \;\; z_2(T)]| > 0$, $t \geq T$ which contradicts (33). This completes the proof of Theorem 1. ■

The preceding proves among other things that trajectories starting outside of $\mathcal{N}$ cannot approach $\mathcal{M}$. But $\mathcal{N} \cap \mathcal{X}$ is an invariant manifold. Moreover, we've already proved that all trajectories starting outside of $\mathcal{N}$ converge to $\mathcal{E}$. We can therefore conclude that any trajectory starting outside of $\mathcal{N}$ can never enter $\mathcal{N}$.

4. CONCLUDING REMARKS

The aim of this paper has been to analyze a control law which generalizes the controls proposed previously in Cao et al. [2007], Anderson et al. [2007] and Smith et al. [2006]. The control laws studied in Cao et al. [2007] are $u_i = -z_ie_i = -z_ig_i(\|z_i\|^2 - d_i^2) = -z_i(\|z_i\|^2 - d_i^2)$ where g_i take the form of linear functions



(a) $g_i(x) = x$, $x \geq -d_i^2$, in Cao et al. [2007]

(b) $g_i(x) = 1 - \frac{d_i}{(x+d_i^2)^{\frac{1}{2}}}$, $x > -d_i^2$, in Anderson et al. [2007]

(c) $g_i(x) = 1 - \frac{d_i^2}{x+d_i^2}$, $x > -d_i^2$, in Smith et al. [2006]

Fig. 2. Previously proposed controls

$$g_i(x) = x \tag{34}$$

for $x \geq -d_i^2$. The control laws proposed in Anderson et al. [2007] are $u_i = -z_ie_i = -z_ig_i(\|z_i\|^2 - d_i^2) = -z_i\frac{\|z_i\|-d_i}{\|z_i\|}$ where

$$g_i(x) = 1 - \frac{d_i}{(x + d_i^2)^{\frac{1}{2}}} \tag{35}$$

for $x > -d_i^2$. Finally, control laws considered in Smith et al. [2006] can be modified to control a directed triangular formation. Then $u_i = -z_ie_i = -z_ig_i(\|z_i\|^2 - d_i^2) = -z_i\frac{\|z_i\|^2-d_i^2}{\|z_i\|^2}$ where

$$g_i(x) = 1 - \frac{d_i^2}{x + d_i^2} \tag{36}$$

Additionally, extension of the ideas to agents with dynamics is currently in contemplation.

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