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Simulation and experimental tools for fractional order control education

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Abstract: The paper presents the simulation toolkit in MATLAB/Simulink® for the fractional order discrete, state-space system education. The toolkit has been written as a set of C-MEX S-functions which simulate several fractional order blocks - e.g. fractional order difference, fractional order discrete state-space representation, fractional order Kalman filter. This way the C code generation capability (using Real Time Workshop) has been provided. This allowed to easily implement the control algorithms in the experimental setup. For analysis of fractional order systems the ultracapacitor has been chosen. It represents the real-life system which is inherently of fractional order. Altogether the simulation toolkit and the experimental testbed form the basis for the advanced fractional order control lab at Institute of Control and Industrial Electronics, Warsaw University of Technology.

Keywords: control education, fractional systems

1. INTRODUCTION

The fractional order calculus and its applications at first glance seem to be a very difficult subject. This might seem especially so for graduate students who have gone through traditional (integer order) calculus and basic control during the undergraduate education e.g. in Control and Computer Engineering. However, teaching the fractional order calculus and its application in control does not have to be prohibitive for graduate students with training in calculus and control. What is needed is the set of tools to make the process easier and more productive.

There are very few tools for simulation and design of continuous time fractional order dynamic systems that may be considered as a support in teaching activities. One good example is the NINTEGER toolbox made by Duarte Valério, and described in Valério [2005]. It contains a set of MATLAB® functions to design and simulate fractional order systems generally in frequency domain. Another tool for simulation of fractional order systems in time domain written by Ivo Petraš implements Digital Fractional Order Differentiator/integrator - FIR type and IIR type, see Petraš [2003a,b].

In our advanced control systems lab we have prepared a set of simulations and experiments for graduate students who want to study the properties of fractional order systems. This is a combination of MATLAB/Simulink® toolkit and an experimental setup. The students entering the lab should have the basic knowledge of fractional order calculus and fractional order systems. This is the subject of one of the lectures on Advanced Control. First part of the lab is devoted to simulation of the fractional order systems. Students start with the simulational analysis of fractional difference, and they get acquainted with the discrete fractional order state-space (DFOSS) model in deterministic and stochastic case. Then, they are ready for designing the simple control algorithms for this model. Every student is required to design a pole placement regulator. Depending on the group the system can be deterministic or stochastic and the state variables can be fully accessible or not. In stochastic case a design of fractional order Kalman filter may be required.

The paper is organised as follows. Basic notions of discrete, fractional order systems are recalled in Section 2. General features of the toolkit are described in Sections 3, the characterstics of its components are given in Sections 3.1, 3.2, 3.3, and 3.4. The examples of students simulations are given also there. After having completed the simulation part the students can desing a real-life experiment with the fractional order plant. This is the system with the ultracapacitor described in Section 4. Student's task is to design a feedback pole placement controller for this plant and to check if its performance meets the design specifications. Examples of student's design is given in Section 4 too.

2. DISCRETE FRACTIONAL ORDER STATE-SPACE SYSTEM

Let us start our exposition with some basics of the fractional calculus used throughout the paper. In this paper the following definition of the fractional order difference will be used.

Definition 1. Fractional order difference is giving as follows:

$$\Delta^n x_k = \sum_{j=0}^{k} (-1)^j \binom{n}{j} x_{k-j}$$

where, $n \in \mathbb{R}$, is a fractional degree, $\mathbb{R}$, is the set of real numbers and $k \in \mathbb{N}$ ($\mathbb{N}$, is the set of natural numbers) is a number of sample for which the approximation of the derivative is calculated.

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In most general case, when the orders of the system equation are not the same, the general fractional order states-space system can be defined as following (for more details see Sierociuk and Dzieliński [2006]):

Definition 2. The general linear discrete fractional order system in state-space representation is given as follows:

$$\Delta^{\Upsilon} x_{k+1} = A_d x_k + B u_k \tag{1}$$

$$x_{k+1} = \Delta^{\Upsilon} x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_j x_{k-j+1}. \tag{2}$$

$$y_k = C x_k \tag{3}$$

where:

$$\Upsilon_k = \operatorname{diag}\begin{bmatrix} \binom{n_1}{k} \\ \vdots \\ \binom{n_N}{k} \end{bmatrix} \quad \Delta^{\Upsilon} x_{k+1} = \begin{bmatrix} \Delta^{n_1} x_{1,k+1} \\ \vdots \\ \Delta^{n_N} x_{N,k+1} \end{bmatrix}$$

and $\mathcal{N} = [n_1, ..., n_N]^T$ is a matrix of the system equations orders.

Solution of such a defined system and its properties are given in Dzieliński and Sierociuk [2006, 2007b]

3. FRACTIONAL ORDER STATE SPACE TOOLKIT

The Fractional Order State Space Toolkit was written as a set Simulink blocks of C-MEX S-Functions which are written in C programming language. This technic was used especially for making the use of this toolkit with Real Time Workshop (RTW) possible. RTW is used in order to achieve the executable code generation (e.g. for dSPACE control cards). Functions also use work vectors that allow to use more than one block of the same type at one time. The toolkit is available on the Web at http://www.ee.pw.edu.pl/ ˜ dsieroci/fsst/fsst.htm and manual can be found in Sierociuk [2005].

The toolkit contains the following blocks :

- "Fractional Order Difference" described in Section 3.1
- "Fractional Order State-Space System" described in Section 3.2
- "Fractional Order Stochastic State-Space System" described in Section 3.3
- "Fractional Kalman Filter" described in Section 3.4

3.1 Fractional Order Difference Block

The code of the Fractional Order Difference Block is implemented in fodif.c file (Figure 1.

The S-function fodif has the following parameters: N, Ts, $Nbuf$, where

N is a matrix of orders
Ts is a sample time
$Nbuf$ is a width of a circular buffer of past states vectors (memory length L).

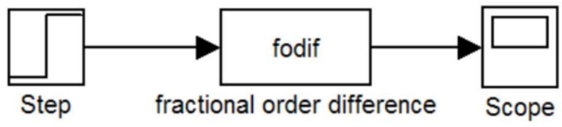

Fig. 1. Use of Fractional Order Difference Block

Example 1. Fractional order differences of step function

The results of obtaining fractional order differences of step function for different orders $n = 0.5, 0.7, 1.5$ are presented in Figure 2.

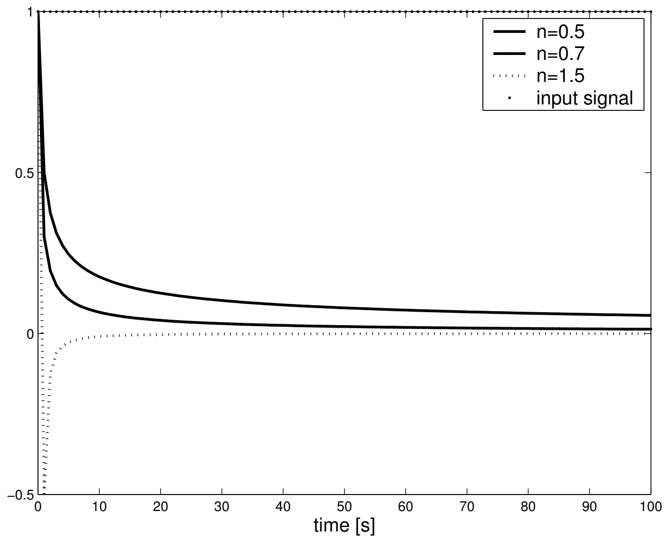

Fig. 2. Differences of step signal for $n = 0.5, 0.7, 1.5$

3.2 Discrete Fractional Order State-Space System Block

The code of the Discrete Fractional Order State-Space System Block is implemented in fsim_x0.c file.

The S-function fsim_x0 has the following parameters: $Ad, B, C, N, Ts, Nbuf, x0$.

Ad, B, C, N are the system matrices, where $Ad \in \mathbb{R}^{N_x \times N_x}$, $B \in \mathbb{R}^{N_x \times N_u}$, $C \in \mathbb{R}^{N_y \times N_x}$, $N \in \mathbb{R}^{N_x}$.
Ts is the sampling time.
$Nbuf$ is the memory length.
$x0$ is the vector of initial conditions.

where N_x is the number of states, N_u is the number of inputs, N_y is the number of outputs (see Figure 3).

Example 2. Simulation of Discrete Fractional Order State-Space System for different orders $n = 0.5, 0.8, 1$

The system is given by the following matrices:

$$A_d = \begin{bmatrix} 0 & 0.1 \\ -0.1 & -0.2 \end{bmatrix}, \mathcal{N} = \begin{bmatrix} n \\ n \end{bmatrix} \tag{4}$$

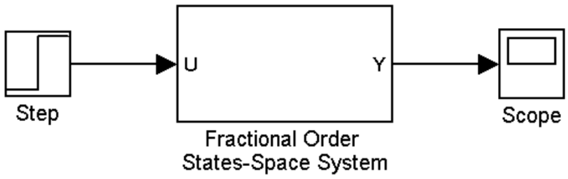

Fig. 3. Use of Discrete Fractional Order State-Space System Block

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [\,0.1\ \ 0.3\,] \tag{5}$$

The results of simulation are presented in Figure 4.

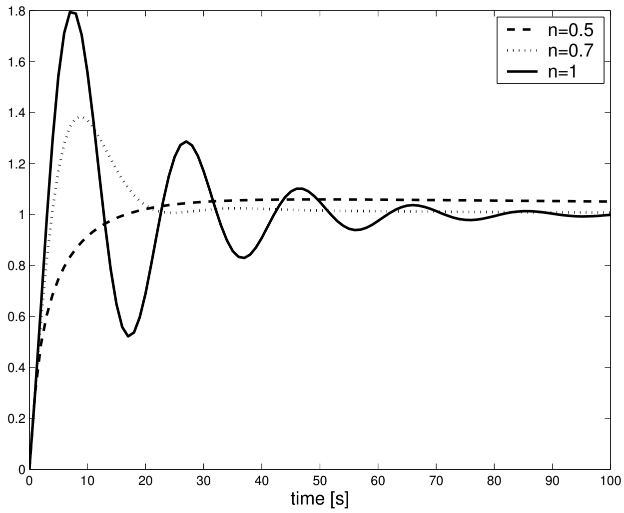

Fig. 4. Outputs of the Discrete Fractional Order State-Space System for $n = 0.5, 0.8, 1$

3.3 Discrete Stochastic Fractional Order System Block

Definition 3. The discrete linear fractional order stochastic system in state-space representation is given by the following set of equations

$$\Delta^n x_{k+1} = A_d x_k + B u_k + \omega_k \tag{6}$$

$$x_{k+1} = \Delta^n x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \binom{n}{j} x_{k+1-j} \tag{7}$$

$$y_k = C x_k + \nu_k \tag{8}$$

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The Discrete Stochastic Fractional Order State-Space System Block has the same properties as the Discrete Fractional Order State-Space System Block. It has only one additional input *omega* for the system noise ω_k. The output noise ν_k is easy to add outside of the block as it is presented in Figure 5.

Example 3. Simulation of Stochastic Fractional Order State-Space System for different values of noise variance

The system has the same matrices as in Example 2, however the orders matrix is equal to

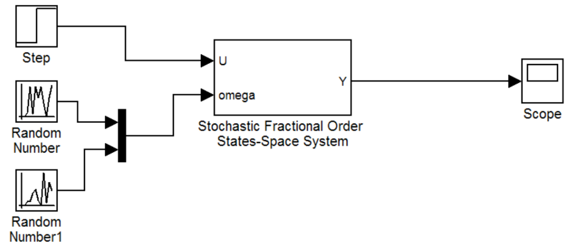

Fig. 5. Use of the Discrete Stochastic Fractional Order State-Space System Block

$$\mathcal{N} = \begin{bmatrix} 0.7 \\ 1.2 \end{bmatrix} \tag{9}$$

and the noise has the values:

in the first case:

$$E[\omega_k \omega_k^T] = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

in the second case:

$$E[\omega_k \omega_k^T] = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}$$

The results are presented and compared with deterministic system in Figure 6.

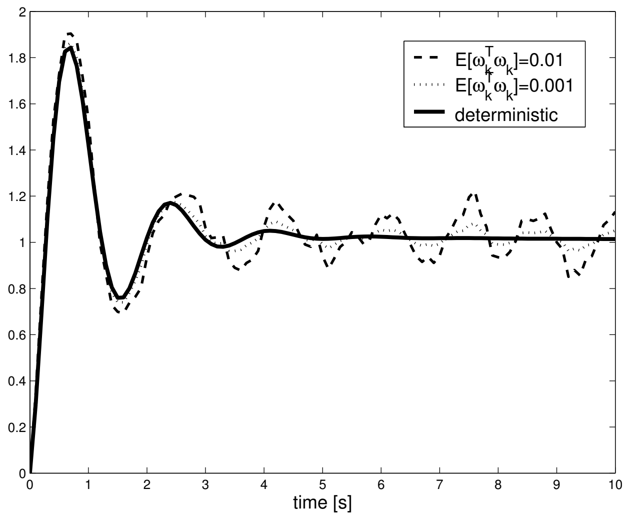

Fig. 6. Outputs of the Discrete Stochastic Fractional Order State-Space System for different values of noise variance

3.4 Fractional Kalman Filter Block

For the system defined in Section 3.2 the Fractional Kalman Filter is defined as follows (for more detail see Sierociuk and Dzieliński [2006]):

$$\begin{aligned}\Delta^{\Upsilon}\tilde{x}_{k+1} &= A_d\hat{x}_k + Bu_k\\ \tilde{x}_{k+1} &= \Delta^{\Upsilon}\tilde{x}_{k+1} + \sum_{j=1}^{k+1}(-1)^j\Upsilon_j\hat{x}_{k-j+1}\\ \tilde{P}_k &= (A_d+\Upsilon_1)\,P_{k-1}\,(A_d+\Upsilon_1)^T + \\ &\quad +Q_{k-1} + \sum_{j=2}^{k}\Upsilon_j P_{k-j}\Upsilon_j^T\\ K_k &= \tilde{P}_k C^T(C\tilde{P}_k C^T + R_k)^{-1}\\ \hat{x}_k &= \tilde{x}_k + K_k(y_k - C\tilde{x}_k)\\ P_k &= (I - K_k C)\tilde{P}_k\end{aligned}$$

where: R is a covariance matrix of output noise ν and Q is a covariance matrix of system noise ω. Both of this noises are assumed to be independent and with zero expected value.

The S-function fkf.c has the following parameters appropriately $A, B, C, N, P, Q, R, x_0, Ts, Nbuf$.

A, B, C, N are the system matrices.
P, Q, R, x_0 are the FKF matrices where $P \in \mathbb{R}^{N_x \times N_x}$, $Q \in \mathbb{R}^{N_x \times N_x}$, $R \in \mathbb{R}^{N_y \times N_y}$, $x_0 \in \mathbb{R}^{N_x}$
Ts is the sampling time
$Nbuf$ is the memory length

The number of system outputs (rows number of C matrix) is limited to one (in future versions it will not be limited).

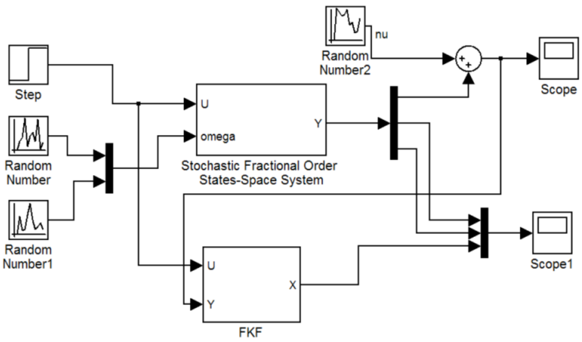

Fig. 7. Diagram of the state variables estimation using Fractional Kalman Filter

Example 4. State variables estimation using Fractional Kalman Filter

The system has the same matrices as in Example 3, the noise variances have the values

$$E[\nu_k\nu_k^T] = 0.01, E[\omega_k\omega_k^T] = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

Fractional Kalman Filter parameters used in the example are:

$$P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

$$R = [\,0.01\,], \quad \hat{x}_0 = [\,5\ 5\,]$$

The block diagram of the system is presented in Figure 7, and the results of simulations are given in Figure 8

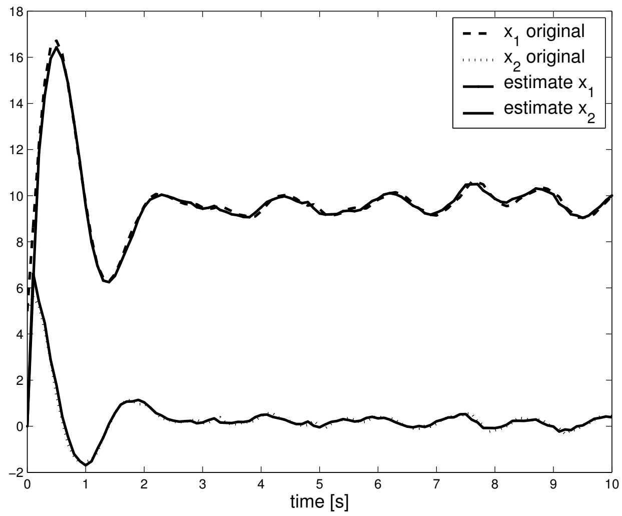

Fig. 8. Results of estimation with FKF compared with original state variables

3.5 State feedback control with Fractional Kalman Filter

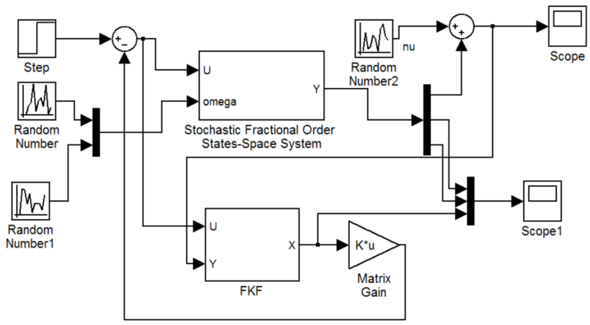

Fig. 9. Diagram of the state feedback control with FKF as an estimator

Example 5. State feedback control with Fractional Kalman Filter as a state variables estimator

The system has the same matrices as in Example 3, the noise variances have the values

$$E[\nu_k\nu_k^T] = 0.01, E[\omega_k\omega_k^T] = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

Fractional Kalman Filter parameters used in the example are:

$$P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

$$R = [\,0.01\,], \quad \hat{x}_0 = [\,0\ 0\,]$$

The controller matrix is given as

$$K = [\,-0.05\ \ -0.05\,] \tag{10}$$

The block diagram of the state feedback control system is given in Figure 9 The results are presented in Figure 10.

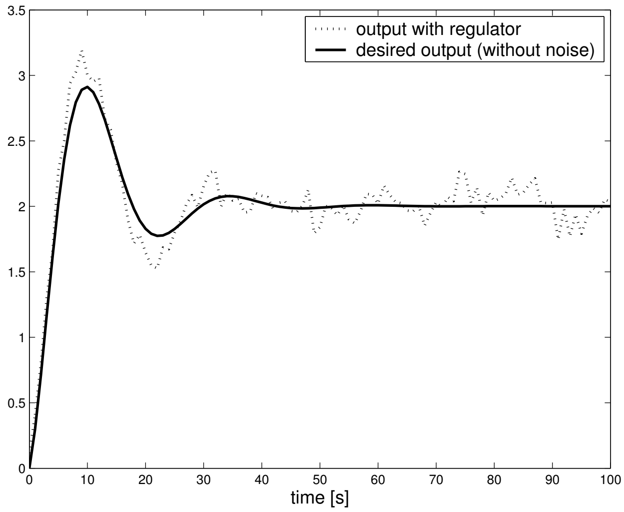

Fig. 10. Results of state feedback control with FKF compared with output of the desired system

4. STATE FEEDBACK CONTROL OF THE SYSTEM WITH ULTRACAPACITOR

Ultracapacitors are electrical energy storage devices which offer high power density which was not possible to achieve in traditional capacitors. In the papers Quintana et al. [2006], Westerlund and Ekstam [1994] very efficient approach using fractional order calculus for continuous time modelling was presented. The papers Dzieliński and Sierociuk [2007c] and Dzieliński and Sierociuk [2007a] present results of ultracapacitors modeling using Discrete Fractional Order State Space and continuous frequency domain model.

The experimental setup, used in modeling, estimation and control experiments, contains the electronic circuit with ultracapacitor connected to the DS1103 PPC Control Card. The electronic circuit is presented in Figure 11 and is composed of the operational amplifier OPA544, a 180Ω resistor and an ultracapacitor of $0.22F$ at $5V$. OPA 544 is a high current operational amplifier and it works in voltage follower configuration.

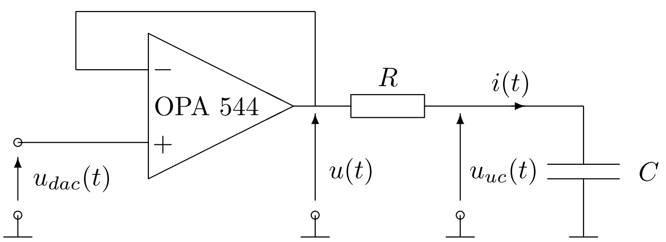

Fig. 11. Electric diagram of ultracapacitor system

Figure 12 presents Simulink scheme of the state feedback regulator for the system with ultracapacitor for which electronic circuit scheme is presented in Figure 11. Additional blocks representing A/D and D/C converters available in DS1103 PPC Control Card are shown, too. The saturation block is used only for the purpose of restraining the ultracapacitor's voltage.

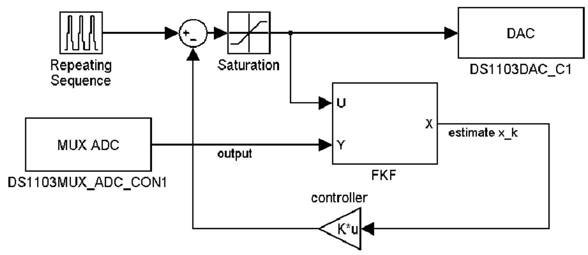

Fig. 12. State feedback regulator for the system with ultracapacitor

By output error minimalization the following discrete fractional order state-space system is obtained:

$$A_d = \begin{bmatrix} 0 & 1 \\ 0.035311 & 0.001815 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathcal{N} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$$
$$C = [\,-0.018624 \;\; 0.188432\,], D = [\,0\,], T_s = 0.1s$$

Results of discrete fractional order modelling of the system with ultracapacitor are presented in Figures 13 and 14.

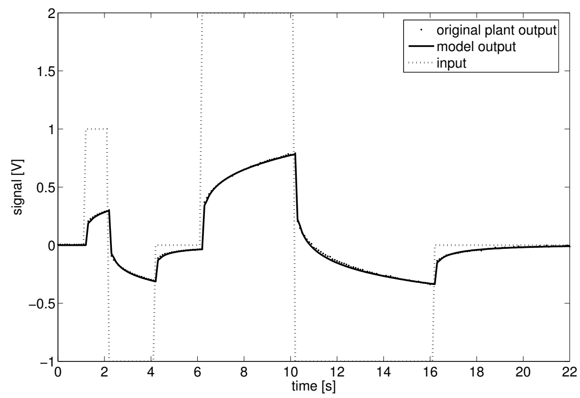

Fig. 13. Results of discrete modelling of system with ultracapacitor, outputs

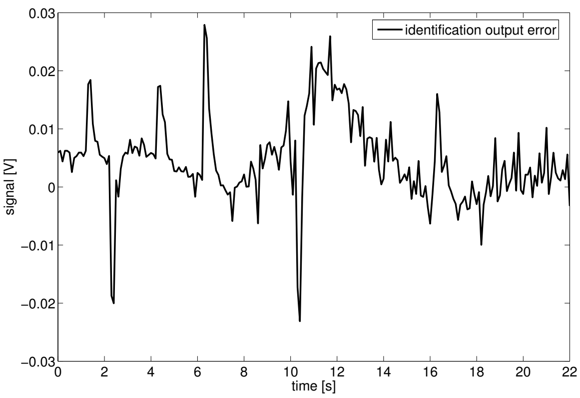

Fig. 14. Results of discrete modelling of system with ultracapacitor, error

For such an identified model the state feedback controller is used. The controller matrix has the form:

$$K = [0.05 \quad 0.05]$$

Figure 15 presents results of state feedback control of the system with ultracapacitor.

Parameters of the Fractional Kalman Filter used in the example are as follows:

$$P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

and

$$R = [\,0.01\,] .$$

the initial condtions are assumed to be

$$x_0 = [\,0 \;\, 0\,] .$$

Figure 15 shows the system output compared with the model given by system matrix $A-BK$. Figure 16 presents the system input and reference input.

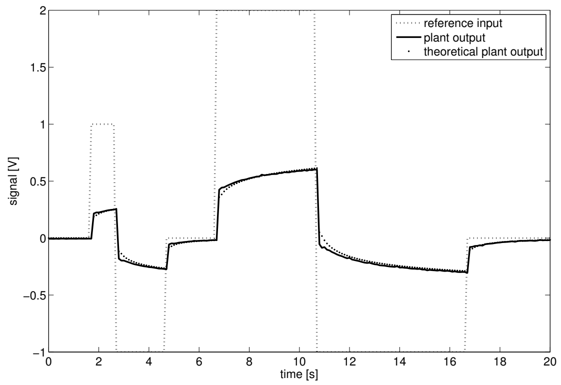

Fig. 15. State feedback control results, outputs

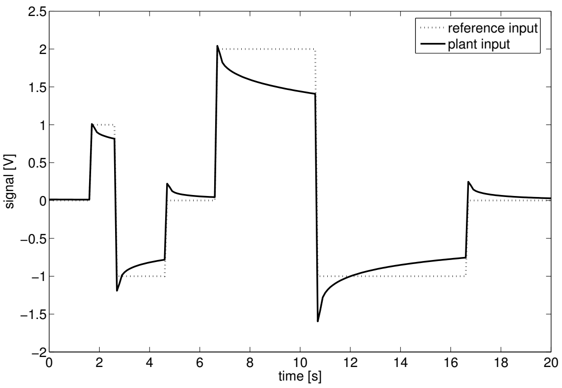

Fig. 16. State feedback control results, inputs

5. CONCLUSIONS

In the paper the simulation software toolkit and the experimental setup supporting the control education of fractional order systems has been presented. The toolkit provides basic insight into the performance and desing of fractional order systems in discrete state space representation. Several fractional order building blocks have already been implemented as C-MEX S-functions allowing the simulation of simple control systems. This in turn allows to build a real-life fractional order control system for a plant of fractional order - ultracapacitor.

Both tools have been used in Advanced Control lab for a selected group of students and the set of examples of students simulations and experiments is presented. Their first impressions after having gone through the lab were encouraging. The students were able to grasp the basic feeling of fractional order systems fairly quickly and they achieved the correct results in their simulation and real experiments without major difficulties. The students evaluation of the experiments was above the departamental average.

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