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Infimal Feedback Capacity for a Class of Additive Coloured Gaussian Noise Channels

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Abstract: In this paper we consider the infimal signal-to-noise ratio (SNR) required for stabilisation of a linear time invariant (LTI) scalar unstable plant over a class of additive coloured Gaussian noise channels. We apply recent results in the literature to obtain the feedback capacity of such a class of channels. We prove that the infimal SNR constrained LTI solution, when dealing with a scalar unstable plant, does achieve a channel feedback capacity equal to the infimal rate of transmission required for stability. The optimality of such channel feedback capacity is a non trivial result since we consider additive 1st order moving average (MA) and autoregressive moving average (ARMA) coloured noise.

1. INTRODUCTION

The study of control over networks has been a growing area of research in recent years; see for example Antsaklis and Baillieul (2004); Nair et al. (2007) and references therein. Communication channels can impose additional limitations to feedback, such as constraints in data-rate and bandwidth, and effects of noise and time-delay. A recent line of research has studied stabilisability under a signal to noise ratio (SNR) constraint Braslavsky et al. (2007). These papers obtained the infimal SNR required to stabilise an unstable plant over a memoryless additive white Gaussian noise (AWGN) channel, whilst in Rojas et al. (2006) we addressed the case of additive coloured Gaussian noise channels with memory, see Figure 1.[1] As in Rojas et al. (2006) in the present paper we consider the channel to be located in the measurement path. The noise process $n(k)$ in Figure 1 is a zero-mean i.i.d. Gaussian white noise process with variance given by σ^2 and the channel input satisfies a power constraint $E\{s^2\}<P$. The capacity of a communication channel,

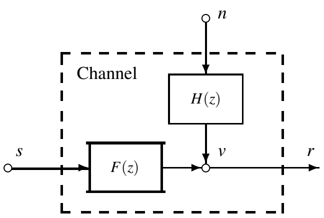

Fig. 1. Additive coloured Gaussian noise (ACGN) channel with memory.

defined as the maximum of the mutual information between the channel input and output (see (Cover and Thomas, 1991, p. 241)), is also a useful quantity describing a communication channel. For an AWGN channel this is given by

$$C_{channel}=\frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right) \text{ bits/trans.,} \tag{1}$$

and is thus completely determined by its SNR. As observed by Shannon (1956), the presence of feedback does not increase the capacity of a memoryless channel. On the other hand, when the channel model has memory, the expression for the channel capacity, (1), does not apply. Furthermore for a channel with memory, the capacity with feedback, C_{fb}, is greater or equal to that without feedback (see Cover and Pombra (1989)).

In order to address the issue of channel capacity when the additive noise is coloured and feedback is present, we consider a recent result in Kim (2006b) and a conjecture from Kim (2005). The main result in Kim (2006b) yields a procedure to obtain the channel feedback capacity for $F(z)=1$ and the channel additive noise $n(k)$ coloured by an order 1 moving average (MA(1)) filter

$$v(k)=n(k)+\alpha\, n(k-1), \tag{2}$$

with α in $[-1,\,1]$. The noise variance, without loss of generality Kim (2006b), is $\sigma^2=1$.

A conjecture for a similar result yields a procedure to obtain the channel feedback capacity for $F(z)=1$ and the channel additive noise $n(k)$ coloured by an order 1 autoregressive moving average (ARMA(1)) filter

$$v(k)+\eta v(k-1)=n(k)+\alpha\, n(k-1), \tag{3}$$

with α in $[-1,\,1]$, η in $(-1,\,1)$ and unitary noise variance $\sigma^2=1$.

We show in the present paper that for a scalar unstable plant, the infimal SNR for stabilisability (or equivalently the infimal channel input power constraint for stabilisability, since $\sigma^2=1$), achieves an infimal channel feedback capacity equal in value to the infimal channel rate of transmission required for stabilisation, as in Freudenberg et al. (2006).

The rest of the paper is organized as follows: Section 2 introduces some preliminary concepts. Section 3 presents a *water-*

[1] The use of the term *memory* as terminology in the present paper is restricted to the case of $F(z)\neq 1$ in Figure 1. Thus for example: an ACGN channel with memory presumes $F(z)\neq 1$ and $H(z)\neq 1$; an ACGN channel presumes $F(z)=1$ and $H(z)\neq 1$; an AWGN channel with memory presumes $F(z)\neq 1$ and $H(z)=1$ and a memoryless AWGN channel presumes $F(z)=1$ and $H(z)=1$.

filling argument to obtain a non-tight lower bound for the ACGN channel feedback capacity. Section 4 analyses the feedback capacity of an ACGN channel when the noise is coloured by a MA(1) filter with $\sigma^2=1$. In this case, the infimal SNR for stabilisability by LTI feedback imposes a demand on the power in the transmitted signal. The additive Gaussian MA(1) noise channel feedback capacity corresponding to this power demand is equal to the infimal feedback capacity required for stabilisation by any causal feedback. Section 5 focuses on the conjecture for a similar result involving the additive Gaussian ARMA (1) noise channel. Section 6 presents the conclusions and final remarks for this work.

Related results have been submitted for journal publication, see Middleton et al. (2007), where we discuss in detail the linear minimal SNR stabilisation for an additive Gaussian MA(1) noise channel, but omit the additive Gaussian ARMA (1) noise channel feedback capacity conjecture.

2. PRELIMINARIES

Consider the plant model, $G(z)$, the controller, $C(z)$, and filters $F(z)$ and $H(z)$ to be transfer functions. Furthermore we assume, unless stated otherwise, that the transfer functions $F(z)$ and $H(z)$ composing the ACGN channel with memory model are both stable, biproper and minimum phase. We also assume that $C(z)$ is such that the closed-loop system is stable in the sense that, for any distribution of initial conditions, the distribution of all signals in the loop will converge exponentially rapidly to a stationary distribution.

As mentioned in the introduction we consider the channel to be located in the measurement path, thus the plant output $y(k)$ is equal to the channel input $s(k)$.

The channel input power is then defined by $\|s\|^2_{Pow} \triangleq \mathscr{E}\{y^2\}$, where $\mathscr{E}$ denotes expectation, is required to satisfy an imposed power constraint

$$P > \mathscr{E}\{y^2\}, \tag{4}$$

for some predetermined power level P. Under reasonable stationarity assumptions (Åström, 1970, §4.4), the power in the channel input may be computed, in the disturbance free case, as

$$\mathscr{E}\{y^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |T_{yn}(e^{j\omega})|^2 \sigma^2 d\omega,$$

where

$$T_{yn}(z) = -\frac{C(z)G(z)H(z)}{1+C(z)G(z)F(z)}, \tag{5}$$

is the transfer function that relates $y(k)$ with $n(k)$. Since the feedback system is stable, we have that the power constraint (4) at the channel input translates into the SNR bound on the H_2 norm of $T_{yn}(z)$

$$\frac{P}{\sigma^2} > \|T_{yn}(z)\|^2_{H_2}. \tag{6}$$

From (6) we observe that a fundamental limitation in the SNR of the ACGN channel with memory will be given then by the minimum of its RHS. Thus

$$\frac{P}{\sigma^2} > \min_{C(z)\ stab.} \|T_{yn}\|^2_{H_2}, \tag{7}$$

we have the basis for stating the SNR infimisation problem for stabilisability.

Problem 1. **(LTI Stabilisation with infimal SNR).** *Find a proper rational stabilising controller $C(z)$ such that the feedback control loop is stable and the transfer function in (5) achieves the least restrictive constraint* (7) *imposed on the admissible channel SNR.*

A well known result on mean square stabilisability for finite-dimension linear systems obtained in Nair and Evans (2004) calls for the rate of transmission of the communication channel, R, to satisfy

$$R > \sum_{i=1}^{m} \log_2 |\rho_i| \text{ bits/trans.} \tag{8}$$

For the memoryless AWGN channel the presence of feedback does not increase the channel capacity, Shannon (1956). From Braslavsky et al. (2007) we have that, also for a memoryless AWGN channel, the infimal SNR for stabilisability of a plant with m unstable poles $|\rho_i|>1$, $\forall i=1,\cdots,m$, minimum phase and with relative degree one, is given by

$$\frac{P}{\sigma^2} > \left(\prod_{i=1}^{m} |\rho_i|^2\right) - 1. \tag{9}$$

By replacing (9) directly into (1) we regain the lower bound on the rate of transmission of the communication channel required for stabilisation as in Nair and Evans (2004).

$$C_{channel} > \sum_{i=1}^{m} \log_2 |\rho_i| \text{ bits/trans.} \tag{10}$$

The main interpretation of (10) is that an LTI controller can be optimal in terms of data rate transmission requirement for stabilisability when using a memoryless AWGN channel. In the next sections we wish to find conditions under which an LTI controller is optimal, in the sense just described, when dealing with an ACGN channel.

3. NON-TIGHT LOWER BOUND FOR THE COLOURED NOISE FEEDBACK CAPACITY

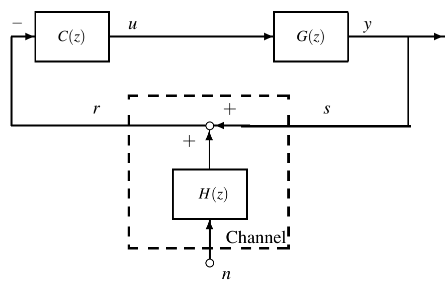

Fig. 2. ACGN channel in feedback configuration.

To compute the capacity of an ACGN channel with feedback, see Figure 2, we consider in a first approach a *water-filling* argument (Cover and Thomas (1991), Gallager (1968)).

We start by reformulating the ACGN channel as an equivalent (in terms of SNR required for stabilisability) AWGN channel with memory, see Figure 3. We obtain this by invoking the LTI single input single output (SISO) condition of all the systems involved in the feedback loop. Since the difference between $\tilde{G}(z)$ and the original $G(z)$ does not introduce any unstable pole, NMP zero nor different relative degree, the infimal SNR required for stabilisability will be the same for both plants $\tilde{G}(z)$ and $G(z)$.

As initially proposed, we can now deal with the memory element in the AWGN channel without feedback by means of a *water-filling* argument. The channel power constraint must satisfy

$$P = \int_{\omega\in W_{\mathscr{B}}} \left[\mathscr{B} - \frac{1}{|H^{-1}(e^{j\omega})|^2}\right] \frac{d\omega}{2\pi}, \tag{11}$$

Given P and $H(z)$ known, it is possible, from (11), to obtain $W_{\mathscr{B}}$ and as a consequence $\mathscr{B}$ (where $W_{\mathscr{B}}$ is the range of frequencies for which $1/|H^{-1}(e^{j\omega})|^2 \le \mathscr{B}$). The capacity of the channel is then given by

$$C_{channel} = \int_{\omega\in W_{\mathscr{B}}} \frac{1}{2} \log_2\left[|H^{-1}(e^{j\omega})|^2 \mathscr{B}\right] \frac{d\omega}{2\pi} \text{ bits/trans.}, \tag{12}$$

and the power spectral density for the input signal that achieves $C_{channel}$ is given by

$$S_s(\omega)=\begin{cases}\mathscr{B}-\frac{1}{|H^{-1}(e^{j\omega})|^2}, & \omega\in W_{\mathscr{B}},\\ 0, & \omega\notin W_{\mathscr{B}}.\end{cases} \tag{13}$$

For more details on *water-filling* see, for example, (Gallager, 1968, pp. 388-389). Notice, although, that the *water-filling* argument just presented is in open loop and that the channel feedback capacity, C_{fb}, is greater or equal to the channel capacity without feedback, $C_{channel}$ (see Cover and Pombra (1989)). Thus, what we obtained in this section is ultimately a non-tight lower bound on C_{fb}

$$C_{fb} > C_{channel} \tag{14}$$

with $C_{channel}$ as in (12).

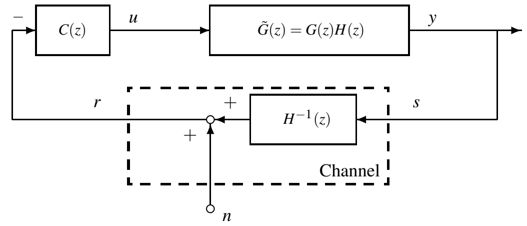

Fig. 3. Equivalent AWGN channel with memory configuration.

4. TIGHT LOWER BOUND FOR THE COLOURED NOISE FEEDBACK CAPACITY: FIRST ORDER MOVING AVERAGE CASE

Consider the case of an ACGN channel and a scalar unstable plant defined as $G(z)=1/(z-\rho)$ with one unstable pole ρ satisfying $\rho>1$.

We recall from Cover and Pombra (1989) that the channel capacity in the presence of feedback , C_{fb}, (see for example (Cover and Thomas, 1991, §8.12)) can be determined from the limiting solution (in the length of the message) of an optimisation problem. On the other hand, as noted in Kim (2006b) before, it can be very difficult in general to find a closed form solution for C_{fb}. Nonetheless an expression for C_{fb} in closed form is presented in Kim (2006b), but for a specific class of channels. The particular class of channels considered are additive white Gaussian noise channels, coloured by a MA(1) filter and, without loss of generality, unity channel noise variance. The result from Kim (2006b) is stated next for convenience.

Theorem 2. Consider the additive Gaussian MA(1) noise channel in (2). The channel feedback capacity C_{fb}, under a power constraint P and a zero-mean i.i.d. Gaussian white noise $n(k)$ with variance $\sigma^2=1$, is then given by $C_{fb}=-\log_2 x_o$ bits/trans., where x_o is the unique positive root of the fourth-order polynomial

$$Px^2=\left(1-x^2\right)(1-|\alpha|x)^2. \tag{15}$$

Proof. See Kim (2006b).

The following theorem applies the above result to show that for a minimum phase relative degree one plant with one unstable pole ρ, we can again regain a channel capacity equal to the data rate required for stabilisation, as in Nair and Evans (2004). We can observe in Figure 4 the proposed setting to achieve this. The element $\gamma_\alpha(k)$ is defined as

$$\gamma_\alpha(k)=\begin{cases}1 & \text{, if sign}(\rho)\neq\text{sign}(\alpha)\text{ ,}\\ (-1)^k & \text{, if sign}(\rho)=\text{sign}(\alpha),\end{cases} \tag{16}$$

and plays the role of encoder and decoder.

The present result is more restrictive than that for the memoryless AWGN channel in (10), since at present we can only state it for the case of scalar unstable plant dynamics.

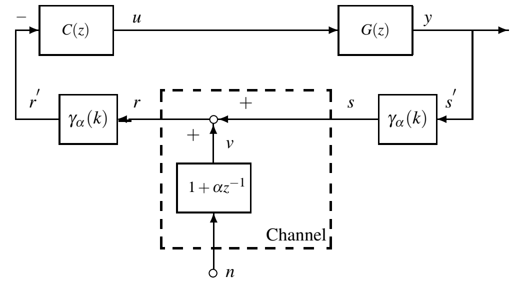

Fig. 4. Additive Gaussian MA(1) noise channel with feedback and explicit consideration of encoder and decoder.

Theorem 3. Suppose we restrict $\mathscr{E}\{s_k^2\}<P$. Let C_{fb} be the feedback capacity of the channel as given in Theorem 2. Assume $G(z)$ to be minimum phase, with relative degree 1 and a single unstable pole at $z=\rho$. Then a decoder and encoder that stabilise the feedback system subject to the power constraint exist if and only if $C_{fb}>\log_2|\rho|$ bits/trans.. Furthermore, the SNR limited stabilisation can be achieved by a linear time invariant encoder and decoder.

Proof.

(i) Necessity: from Proposition III.1 in Freudenberg et al. (2006) or equivalently Theorem 2.1 in Nair and Evans (2004).
(ii) Sufficiency: see the Appendix.

An example follows next in which we consider α to span the all range $[-1,1]$. We expect the infimal SNR solution to achieve a channel feedback capacity equal to the infimal rate of transmission required for stabilisation, as in Freudenberg et al. (2006).

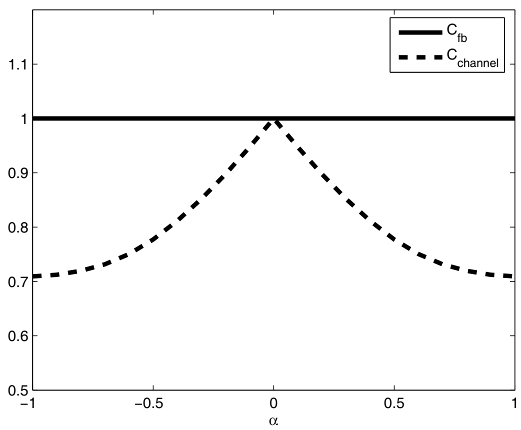

Fig. 5. Unstable pole at 2, minimum phase and relative degree 1. Channel feedback capacity obtained with infimal SNR , solid line. Channel capacity obtained by *water-filling* argument, dashed line.

Example 4. Assume that $G(z)$ is minimum phase, has an unstable pole $\rho=2$ and relative degree 1. Also consider an additive

Gaussian MA(1) noise channel with $\alpha\in[-1,1]$ and $\sigma^2=1$. The channel input power constraint accordingly to Theorem 3 is given by $P=3(1-0.5|\alpha|)^2$ (see the Appendix for details). In Figure 5 we have the channel feedback capacity as a function of α. It can be seen that the channel feedback capacity is 1 bit/trans. and constant over all the range of α. Therefore, it is possible to perceive the optimality of the LTI SNR constrained solution, in terms of Freudenberg et al. (2006), for a first order unstable plant and an additive Gaussian MA(1) noise channel with $\sigma^2=1$.

The channel capacity obtained by the *water-filling* argument is always below the value of the channel feedback capacity (as stated by Pinsker (1969), Ebert (1970) and mentioned in Cover and Pombra (1989)), with the exception of $\alpha=0$.[2] Note that when $\alpha=0$, the additive Gaussian MA(1) noise channel becomes a memoryless AWGN channel for which the channel feedback capacity matches the channel capacity without feedback, (Shannon (1956)). Moreover, for this example, we can observe from Figure 5 that the feedback capacity satisfies

$$C_{channel}\leq C_{fb}\leq 2C_{channel} \text{ bits/trans.,} \tag{17}$$

and

$$C_{channel}\leq C_{fb}\leq C_{channel}+\tfrac{1}{2} \text{ bits/trans.,} \tag{18}$$

as proven in Cover and Pombra (1989).

5. TIGHT LOWER BOUND FOR THE COLOURED NOISE FEEDBACK CAPACITY: FIRST ORDER AUTOREGRESSIVE MOVING AVERAGE CASE

In the present section we analyse the case of channel feedback capacity when the channel noise is described by an additive Gaussian ARMA(1) noise as in (3). The present section is resting on a conjecture presented in Kim (2005) and attributed to Yang et al. (2004). The objective of the present section is not to prove the validity of such conjecture, but to test what feedback capacity can be achieved by the infimal LTI solution for the stabilisation SNR requirement. We reproduce next, from

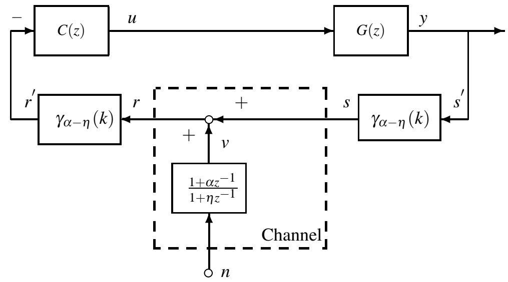

Fig. 6. Additive Gaussian ARMA(1) noise channel with feedback and explicit consideration of encoder and decoder.

(Kim, 2006a, p.37), the conjecture at the core of the present discussion.

Conjecture 5. Consider the additive Gaussian ARMA(1) noise channel in (3). The channel feedback capacity C_{fb}^{conj}, under a power constraint P and a zero-mean i.i.d. Gaussian white noise $n(k)$ with variance $\sigma^2=1$, is given by $C_{fb}^{conj}=-\log_2 x_o$ bits/trans., where x_o is the unique positive root of the fourth-order polynomial

$$Px^2=\frac{(1-x^2)(1+\upsilon\alpha x)^2}{(1+\upsilon\eta x)^2}, \tag{19}$$

[2] As can be observed from the Appendix the element $\gamma_\alpha(k)$ in Figure 4 imposes two different scenarios depending on the sign of α and ρ. Nonetheless, in both cases there is an equivalent transfer function $H(z)$ that can be examined as in Section 3.

and

$$\upsilon=\text{sgn}(\eta-\alpha)=\begin{cases}1, & \eta>\alpha,\\ 0, & \eta=\alpha,\\ -1, & \eta<\alpha.\end{cases} \tag{20}$$

Proof. In Kim (2005) it is argued that the above conjecture is due to Yang et al. (2004). A proof sketch of Conjecture 5 can be found in Kim (2005). A more detailed version can be found in Kim (2006a). To the knowledge of the authors of the present paper no peer-reviewed proof of Conjecture 5 is currently available.

In a similar fashion as in the previous section we follow with a theorem that applies Conjecture 5 to prove that for a minimum phase plant with relative degree one and one unstable pole ρ, we can again regain a channel capacity equal to the data rate required for stabilisation, as in Nair and Evans (2004). We can observe in Figure 6 the proposed setting to achieve this. The element $\gamma_{\alpha-\eta}(k)$ is defined as

$$\gamma_{\alpha-\eta}(k)=\begin{cases}1 & \text{, if sign}(\rho)\neq\text{sign}(\alpha-\eta)\text{ ,}\\ (-1)^k & \text{, if sign}(\rho)=\text{sign}(\alpha-\eta),\end{cases} \tag{21}$$

and plays the role of encoder and decoder. Notice that if $\eta=0$ we regain the situation discussed in the previous section.

Theorem 6. Suppose we restrict $\mathcal{E}\{s_k^2\}<P$. Let C_{fb}^{conj} be the feedback capacity of the channel as given in Theorem 5. Assume $G(z)$ to be minimum phase, with relative degree 1 and a single unstable pole at $z=\rho$. Then a decoder and encoder that stabilise the feedback system subject to the power constraint exist if and only if $C_{fb}^{conj}>log_2|\rho|$ bits/trans.. Furthermore, the SNR limited stabilisation can be achieved by a linear time invariant encoder and decoder.

Proof.

(i) Necessity: from Proposition III.1 in Freudenberg et al. (2006) or equivalently Theorem 2.1 in Nair and Evans (2004).
(ii) Sufficiency: see the Appendix.

We conclude the section by exposing an example related to the result of Theorem 6.

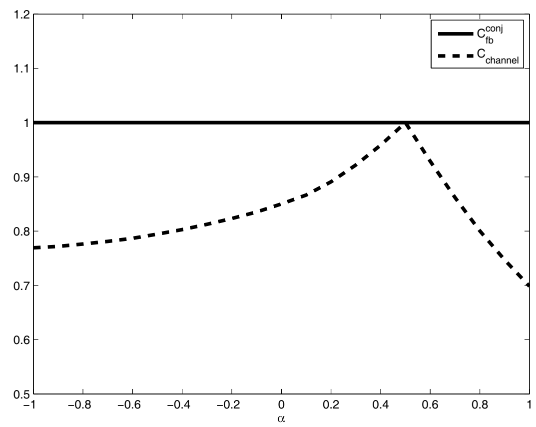

Fig. 7. Unstable pole at 2, minimum phase and relative degree 1. Channel feedback capacity obtained for an additive Gaussian ARMA(1) noise with infimal SNR, solid line. Channel capacity obtained by a *water-filling* argument, dashed line.

Example 7. Assume that $G(z)$ is minimum phase, has an unstable pole ρ at 2 and relative degree 1. Also consider an additive Gaussian ARMA(1) noise channel with $\alpha\in[-1,\,1]$, $\eta{=}0.5$ and $\sigma^2{=}1$. In Figure 5 we have the channel feedback capacity as a function of α. It can be seen that the channel feedback capacity is 1 bit/trans. and constant over all the range of α. Therefore, it is possible to perceive the optimality of the LTI SNR constrained solution, in terms of Freudenberg et al. (2006), for a first order unstable plant and an additive Gaussian ARMA(1) noise channel with $\sigma^2{=}1$.

6. CONCLUSION AND REMARKS

In this paper we analysed the channel capacity of a SNR constrained communication channel in feedback with an unstable plant and stabilising controller. We observe, as in Braslavsky et al. (2007), that the infimal SNR solution, for the case of a memoryless AWGN channel, imposes a capacity that matches the infimal data rate required for stabilisation as in Nair and Evans (2004). We conclude that the same is true for the case of a scalar unstable plant and an additive Gaussian MA(1) noise channel with unity variance noise and, depending on the validity of Conjecture 5, also for the case of an additive Gaussian ARMA(1) noise channel with unity variance. The conclusion is not trivial due to the presence of feedback and additive coloured channel noise. Future opportunities for research should consider filters other than the MA(1) and ARMA(1) for the additive Gaussian noise channel model and, perhaps, the lifting of the scalar plant condition.

APPENDIX

Proof of Theorem 3 (Sufficiency)

Assume the controller $C(z)$, in Figure 4, to be enforcing the infimal SNR required for stabilisability, and also that $\gamma_\alpha(k)$ is defined as in (16). Consider first the case of ρ and α having opposite sign, thus $\gamma_\alpha(k){=}1$. The MA(1) process $1{+}\alpha z^{-1}$ colouring the noise can be seen to play the role of $H(z)$ in our definition of a channel with memory in Figure 1. From Theorem 2 in Rojas et al. (2006) we have that for one unstable pole ρ, the infimal SNR required to guarantee stabilisability converts into a power constraint

$$P{=}\left(|\rho|^2{-}1\right)|1{-}|\alpha||\rho^{-1}||^2. \tag{22}$$

We therefore see that the choice $x_o{=}\frac{1}{|\rho|}$, satisfies (15), with P as in (22). We now turn to the issue of the channel input distribution and whether it achieves channel capacity or not. From Kim (2006b) we have that the channel feedback capacity C_{fb} is achieved by an asymptotically stationary ergodic input process satisfying $\mathscr{E}\{s^2\}{=}P$ (see also (Cover and Pombra, 1989, Section VIII)). We also have that the input distribution that achieves the channel capacity is obtained by a filtered version of the noise innovation ((Kim, 2006b, p. 3073))

$$\begin{aligned} &s_1{\sim}N(0,P),\\ &s(k){=}\beta s(k{-}1){+}\varsigma n(k{-}1), \qquad k{=}2,3,\cdots, \end{aligned} \tag{23}$$

where ς and β are given by

$$\begin{aligned} &\varsigma{=}sign(\alpha)\sqrt{P(1{-}\beta^2)}{=}sign(\alpha)|(\alpha{+}\beta^{-1})(\beta^2{-}1)|,\\ &\beta{=}{-}sign(\alpha)x_o, \end{aligned} \tag{24}$$

and

$$sign(\alpha){=}\begin{cases} 1 & \alpha{\geq}0,\\ -1 & \alpha{<}0. \end{cases}$$

After some algebra we have that the transfer function $\hat{T}_{sn}$ relating the channel additive white Gaussian noise $n(k)$ and the

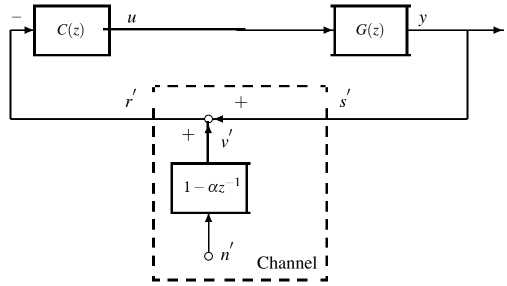

Fig. 8. Equivalent additive Gaussian MA(1) noise channel with feedback and implicit consideration of encoder and decoder.

channel input $s(k)$ is given by $\hat{T}_{sn}{=}\frac{\varsigma}{z-\beta}$, when the optimal controller $\hat{C}(z)$ achieving the infimal SNR required for stabilisability is in place. Thus, indeed the optimal SNR solution generates a channel input distribution that achieves the channel feedback capacity. The obtained channel feedback capacity, with $x_o{=}\frac{1}{|\rho|}$, is fixed at the value of $C_{fb}{=}\log_2|\rho|$ bits/trans. Consider now the opposite case in which ρ and α have the same sign. The sign involved in the unit gain for the encoder and decoder are synchronized, that is when the encoder value is -1 the decoder value is also -1. From the signal definitions in Figure 4 we can claim

$$r(k)'{=}({-}1)^k r(k), \text{ and } s(k)'{=}({-}1)^k s(k), \tag{25}$$

where it is assumed that encoder and decoder are 1 at $k{=}0$. Define at this point an alternative white noise sequence $n(k)'$ as

$$n(k)'{=}({-}1)^k n(k). \tag{26}$$

If we couple this alternative white noise sequence with the alternative colouring LTI filter $1{-}\alpha z^{-1}$, as in Figure 8, it is possible to verify that the obtained coloured noise is effectively $v(k)'{=}({-}1)^{-k}v(k)$, and that the additive Gaussian MA(1) noise channel input and output in Figure 8 are precisely $s(k)'$ and $r(k)'$ (with the encoder and decoder set back to be again time invariant gains of magnitude 1). The net gain in considering the proposed time varying scheme is that we have actually changed the LTI filter colouring the noise (now $1{-}\alpha z^{-1}$). As a consequence the infimal SNR required for stabilisability converts into the power constraint in (22). A similar close inspection of (15) for P defined as in (22) shows that $x_o{=}\frac{1}{|\rho|}$ is still the appropriate solution. Turning now to the issue of the channel input distribution and whether it achieves channel capacity or not, after some algebra we have that $\hat{T}_{s'n'}{=}\frac{-\varsigma}{z+\beta}$, from which we obtain the recursive expression for the channel input as

$$s(k)'{=}{-}\beta s(k{-}1)'{-}\varsigma\, n(k{-}1)'. \tag{27}$$

By recalling (25) and (26) and replacing in (27) we have $s(k){=}\beta s(k{-}1){+}\varsigma\, n(k{-}1)$, thus regaining (23). By the argument presented in (Kim, 2006b, p. 3073) we can claim that the obtained channel input distribution achieves channel feedback capacity.

Proof of Theorem 6 (Sufficiency)

Consider for the first half of this proof that $\rho{>}0$. As hinted by the definition of υ in (20) we need to consider three cases: $\alpha{<}\eta$, $\alpha{=}\eta$ and $\alpha{>}\eta$.

$\alpha{<}\eta$: recall that the proposed LTI filter colouring the noise is given by

$$H(z){=}\frac{z+\alpha}{z+\eta}, \tag{28}$$

with $\alpha{\in}[-1,1]$ and $\eta{\in}(-1,1)$. For the present case we have that $\gamma_{\alpha-\eta}(k)$ defined in Figure 6 is 1 for all k and the infimal input channel power constraint, from Theorem 2 in Rojas et al. (2006), is given by

$$P=(\rho^2-1)\left(\frac{\rho+\alpha}{\rho+\eta}\right)^2=(|\rho|^2-1)\left(\frac{|\rho|+\alpha}{|\rho|+\eta}\right)^2. \quad (29)$$

Replace the infimal channel input power presented in (29) into (19), recall that $\upsilon=1$ for this case, and observe that a suitable solution of the polynomial in x is given by $x_o=\frac{1}{|\rho|}$, thus the infimal C_{fb}^{conj} is given by $\log_2|\rho|$ bits/trans.

$\alpha=\eta$: in this case we have that the additive channel noise becomes effectively white. The channel input power constraint accordingly to Theorem 2 in Rojas et al. (2006) (see also Braslavsky et al. (2007)) is given by

$$P=\rho^2-1=|\rho|^2-1, \quad (30)$$

which when replaced in (19), recalling that $\upsilon=0$ for the present case, has as a solution $x_o=\frac{1}{|\rho|}$, and thus the infimal C_{fb}^{conj} is given by $\log_2|\rho|$ bits/trans.

$\alpha>\eta$: in order to prove the present case observe that the channel additive coloured noise $v(k)$, given $H(z)=(z+\alpha)/(z+\eta)$, can be alternatively described as

$$v(k)=n(k)+(\alpha-\eta)\sum_{l=1}^{k}(-\eta)^{l-1}n(k-l). \quad (31)$$

Consider now, as in the second half of the sufficiency proof of Theorem 3, that $n(k)$ is replaced by $n(k)'=(-1)^k n(k)$ therefore the coloured noise becomes

$$\begin{aligned} v(k)&=n(k)'+(\alpha-\eta)\sum_{l=1}^{k}(-\eta)^{l-1}n(k-l)' \\ &=(-1)^k n(k)+(\alpha-\eta)(-1)^{k-1}\sum_{l=1}^{k}(\eta)^{l-1}n(k-l) \\ &=(-1)^k \underbrace{\left[n(k)+(\eta-\alpha)\sum_{l=1}^{k}(\eta)^{l-1}n(k-l)\right]}_{v(k)'}=(-1)^k v(k)'. \end{aligned} \quad (32)$$

The last line in the above result is equivalent to have replaced the LTI filtering colouring the additive channel noise $(z+\alpha)/(z+\eta)$ by $(z-\alpha)/(z-\eta)$ instead. The new additive coloured channel noise is $v(k)'$, whilst the effect of the factor $(-1)^k$ is eliminated by the choice of $\gamma_{\alpha-\eta}$, which is also $(-1)^k$. Consider for example k even, thus we have $r(k)'=s(k)'+v(k)'$, and if k is odd, then

$$r(k)'=-r(k)=-(s(k)+v(k))=-(-s(k)'-v(k)')=s(k)'+v(k)', \quad (33)$$

from which we can conclude that, by the choice of $\gamma_{\alpha-\eta}$ and the use of $n(k)'$, the original scheme in Figure 6 is now effectively the one represented in Figure 9. The application of Theorem 2

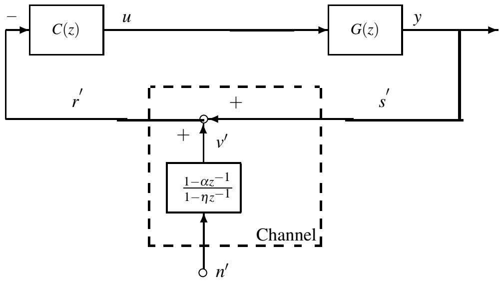

Fig. 9. Equivalent additive Gaussian ARMA(1) noise channel with feedback and implicit consideration of encoder and decoder.

in Rojas et al. (2006) for the loop in Figure 9 gives a channel input infimal power of $P=(\rho^2-1)\left(\frac{\rho-\alpha}{\rho-\eta}\right)^2=(|\rho|^2-1)\left(\frac{|\rho|-\alpha}{|\rho|-\eta}\right)^2$, which when replaced in (19), recalling that $\upsilon=-1$ for the present case, has as a solution $x_o=\frac{1}{|\rho|}$, and thus the infimal C_{fb}^{conj} is given by $\log_2|\rho|$ bits/trans. The proof ends by considering now $\rho<0$ and the three cases: $\alpha<\eta$, $\alpha=\eta$ and $\alpha>\eta$.

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