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Improved Delay-Dependent Stability for a Class of Linear Systems with Time-Varying Delay and Nonlinear Perturbations ⋆

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Abstract: This note deals with the robust stability analysis for time-delay systems with nonlinear perturbations. Firstly, a new class of Lyapunov functional candidate is proposed to develop some new criteria by considering the additional useful terms and introducing some free-weighting matrices. Then, an augmented Lyapunov functional is introduced to establish a novel improved stability condition. All results obtained are given in terms of linear matrix inequalities. A numerical example is given to illustrate the effectiveness of the proposed methods.

1. INTRODUCTION

Time-delay systems are frequently encountered in various areas, including engineering, biology, and economics and other areas (see Hale & Lunel (1993), Gu et al. (2003)). In practical systems, time delay is often a source of instability, oscillation, and poor performance (for example, Malek-Zavarei & Jamshidi (1987)). Therefore, the problem of stability of time delay systems has been the subject of considerable research efforts. Generally speaking, the stability analysis is mainly concerned with two categories: delay-independent stability criterion and delay-dependent stability criterion. The former do not include any information about the magnitude of the delay, while the latter do employ such information. It is well known that delay-dependent stability criterion is less conservative than the delay-independent ones, especially when the magnitude of the delay is small.

In recent years, the problem of robust stability for systems with nonlinear perturbations has also received considerable attention (see Siljak & Stipanovic (2000), Zuo et al. (2004) and the references therein). In Cao & Lam (2000), a model transformation technique was used to transform the system with a discrete delay to a system with a distributed delay, and delay-dependent stability criteria were obtained by using a Lyapunov functional approach. In Han (2004), based on the descriptor model transformation and the decomposition technique of a discrete-delay term matrix, the author gave a linear matrix inequalities (LMI)-based robust stability condition. An integral inequality was introduced to derive the above result, which may lead to considerable conservativeness. In Zuo & Wang (2006), a less conservative result was obtained by using some free matrices to express the relationship of the terms in the Leibniz-Newton formula. Although their results are superior than some existing ones, there still leaves room for further investigation since in the derivative of Lyapunov functional, some useful terms have been enlarged or ignored which have been discussed in He et al. (2007).

In the following of our paper, we will use the free-weighting-matrix (FWM) approach which has been widely used in He et al. (2004a), Wu et al. (2004a), He et al. (2004b), and Wu et al. (2004b). This approach provides great flexibility in solving LMIs, and yields less conservative results than the previous approaches. Furthermore, by employing the augmented Lyapunov functional (see He et al. (2007), Wu et al. (2004b), He et al. (2005)) and some useful lemmas (Zhang et al. (2005), Moon et al. (2001), we obtain a novel improved result.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following linear system with time-varying delay and nonlinear perturbations

$$\begin{aligned} \dot{x}(t) = & Ax(t) + Bx(t-d(t)) + f(x(t),t) \\ & + g(x(t-d(t)),t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state. $A \in R^{n\times n}$, $B \in R^{n\times n}$, are constant matrices. The time-varying vector-valued functions $f(x(t),t) \in R^n$ and $g(x(t-d(t)),t) \in R^n$

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are unknown and represent the parameter perturbations with respect to the current state $x(t)$ and delayed state $x(t-d(t))$ of the systems, respectively. They satisfy that $f(0,t)=0$, $g(0,t)=0$. The function $d(t)$ is a time-varying delay which satisfies

$$0 < d(t) \le h, \quad \dot{d}(t) \le \mu \tag{2}$$

where $h > 0$ is the upper bound of delay, and $0 \le \mu < 1$. The initial condition of system (1) is given by

$$x(\theta) = \phi(\theta) \quad \forall \theta \in [-h, 0], \tag{3}$$

where $\phi(\cdot)$ is a continuous vector valued initial function. In this note, we assume that $f(x(t),t)$, $g(x(t-d(t)),t)$ satisfy

$$\begin{aligned} \| f(x(t),t) \| &\le \alpha \| x(t) \| \\ \| g(x(t-d(t)),t) \| &\le \beta \| x(t-d(t)) \| \end{aligned} \tag{4}$$

where $\alpha \ge 0$, $\beta \ge 0$ are given constants. Note that constraint (4) can be rewritten as

$$\begin{aligned} & f^T(x(t),t) f(x(t),t) \le \alpha^2 x^T(t) x(t), \\ & g^T(x(t-d(t)),t) g(x(t-d(t)),t) \\ & \qquad \le \beta^2 x^T(t-d(t)) x(t-d(t)) \end{aligned} \tag{5}$$

In this paper, we will attempt to formulate some delay-dependent robust stability of the system described by (1)-(3). The following lemmas are useful in deriving the criteria.

Lemma 1. For any vector a, b and matrices N, U, X, Y, where U, X are two symmetric matrices, such that

$$-2a^T N b \le a^T U a + b^T X b \tag{6}$$

where

$$\begin{bmatrix} U & N \\ N^T & X \end{bmatrix} \ge 0 \tag{7}$$

Proof. Applying the lemma in Moon et al. (2001) we can easily to obtain

$$-2a^T N b \le \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} U & Y-N \\ \star & X \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

where

$$\begin{bmatrix} U & Y \\ Y^T & X \end{bmatrix} \ge 0,$$

Let $Y = N$, (6) is obtained immediately.

Lemma 2. (Zhang et al. (2005)) Let $x(t) \in R^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices $X, M_1, M_2 \in R^{n\times n}$ and $Z \in R^{2n\times 2n}$, and a scalar function $h := h(t) \ge 0$:

$$-\int_{t-h}^{t} \dot{x}^T(s) X \dot{x}(s) ds \le \eta^T(t) \Upsilon \eta(t) + h\eta^T(t) Z \eta(t)$$

where

$$\Upsilon := \begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ \star & -M_2^T - M_2 \end{bmatrix}, \ \eta(t) := \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \ge 0,$$

with $Y := [M_1 \ M_2]$

3. NEW STABILITY CRITERIA

In the previous work such as He et al. (2004a), Wu et al. (2004a), and Fridman & Shaked (2003), the following well-known Lyapunov functional is used

$$\begin{aligned} V_1(x_t) &= x^T(t) P x(t) + \int_{t-d(t)}^{t} x^T(s) Q x(s) ds \\ &\quad + \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Z \dot{x}(s) ds d\theta \end{aligned} \tag{8}$$

However, we know that, in the derivative of $V_1(x_t)$ some useful terms have been enlarged or ignored which may bring conservativeness. Recently, in order to overcome this conservatism, in He et al. (2007), the authors used a new class of Lyapunov functional candidate,

$$\begin{aligned} V_2(x_t) &= x^T(t) P x(t) + \int_{t-d(t)}^{t} x^T(s) Q x(s) ds \\ &+ \int_{t-h}^{t} x^T(s) R x(s) ds + \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) (Z_1 + Z_2) \dot{x}(s) ds d\theta \end{aligned} \tag{9}$$

and in the derivative of $V_2(x_t)$, the term $-\int_{t-h}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds$ was separated to $-\int_{t-d(t)}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds - \int_{t-h}^{t-d(t)} \dot{x}^T(s) Z_1 \dot{x}(s) ds$. But when dealing with the term $-\int_{t-h}^{t} \dot{x}^T(s) Z_2 \dot{x}(s) ds$, the way they used has also conservativeness. In the following, we will employ Lemma 2 presented in Zhang et al. (2005) to handle this term.

Theorem 3. For given scalar $h > 0$, and $0 < \mu < 1$, the system described by (1)-(4) is asymptotically stable if there exist matrices $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, $Z_i = Z_i^T > 0$, $i = 1, 2$,

$$\bar{G} = \begin{bmatrix} G_1 \\ \vdots \\ G_6 \end{bmatrix}, \ \bar{H} = \begin{bmatrix} H_1 \\ \vdots \\ H_6 \end{bmatrix}, \ \bar{K} = \begin{bmatrix} K_1 \\ \vdots \\ K_6 \end{bmatrix}, \ \bar{L} = \begin{bmatrix} L_1 \\ \vdots \\ L_6 \end{bmatrix}$$

and scalars $\varepsilon_1 \ge 0$, $\varepsilon_2 \ge 0$ such that

$$\begin{bmatrix} \Xi & h\bar{G} & h\bar{H} & h\bar{K} \\ \star & -hZ_1 & 0 & 0 \\ \star & \star & -hZ_1 & 0 \\ \star & \star & \star & -hZ_2 \end{bmatrix} < 0, \tag{10}$$

where

$$\Xi = \Xi_1 + \Xi_2 + \Xi_2^T$$

$$\Xi_1 = \begin{bmatrix} \Xi_{11} & 0 & 0 & P & 0 & 0 \\ \star & \Xi_{22} & 0 & 0 & 0 & 0 \\ \star & \star & -R & 0 & 0 & 0 \\ \star & \star & \star & h(Z_1 + Z_2) & 0 & 0 \\ \star & \star & \star & \star & -\varepsilon_1 I & 0 \\ \star & \star & \star & \star & \star & -\varepsilon_2 I \end{bmatrix}$$

$$\begin{aligned} \Xi_{11} &= R + Q + \varepsilon_1 \alpha^2 I \\ \Xi_{22} &= -(1-\mu) Q + \varepsilon_2 \beta^2 I \end{aligned}$$

$$\Xi_2 = \left[\, \bar{G} + \bar{K} \ \ \bar{H} - \bar{G} \ \ -\bar{H} - \bar{K} \ 0 \ 0 \ 0 \,\right] + \bar{L}[\, A \ \ B \ \ 0 \ \ -I \ \ I \ \ I \,]$$

Proof. Choose the Lyapunov functional $V_2(x_t)$ as in (9). By the Leibniz-Newton formula, the following equations hold for any matrices $\bar{G}$, $\bar{H}$, $\bar{K}$ with appropriate dimensions

$$\alpha_1(t) := 2\zeta_1^T(t)\bar{G}[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds] = 0$$

$$\alpha_2(t) := 2\zeta_1^T(t)\bar{H}[x(t-d(t)) - x(t-h) - \int_{t-h}^{t-d(t)} \dot{x}(s)ds] = 0$$

$$\alpha_3(t) := 2\zeta_1^T(t)\bar{K}[x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}(s)ds] = 0$$

where

$$\zeta_1^T(t) = [x^T(t) \quad x^T(t-d(t)) \quad x^T(t-h) \quad \dot{x}^T(t) \quad f^T(x(t),t) \quad g^T(x(t-d(t)),t)]$$

Observe that

$$-\int_{t-h}^{t} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds = -\int_{t-d(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-h}^{t-d(t)} \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-h}^{t} \dot{x}^T(s)Z_2\dot{x}(s)ds \tag{11}$$

and note that

$$\theta_1(t) := 2\zeta_1^T(t)\bar{L}[-\dot{x}(t) + Ax(t) + Bx(t-d(t)) + f(x(t),t) + g(x(t-d(t)),t)] = 0$$

Calculating the derivative of $V_2(x_t)$ and adding the left side of $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$, $\theta_1(t)$ into it and using (11) yield

$$\begin{aligned}
\dot{V}_2(x_t) \le\ & 2x^T(t)P\dot{x}(t) + x^T(t)(Q+R)x(t) \\
& -(1-\mu)x^T(t-d(t))Qx(t-d(t)) \\
& -x^T(t-h)Rx(t-h) + h\dot{x}^T(t)(Z_1+Z_2)\dot{x}(t) \\
& -\int_{t-h}^{t} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds \\
& +\alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \theta_1(t) \\
\le\ & \zeta_1^T(t)[\Phi + h\bar{G}Z_1^{-1}\bar{G}^T + h\bar{H}Z_1^{-1}\bar{H}^T + h\bar{K}Z_2^{-1}\bar{K}^T]\zeta_1(t) \\
& -\int_{t-d(t)}^{t} [\zeta_1^T(t)\bar{G} + \dot{x}^T(s)Z_1]Z_1^{-1}[\bar{G}^T\zeta_1(t) + Z_1\dot{x}(s)]ds \\
& -\int_{t-h}^{t-d(t)} [\zeta_1^T(t)\bar{H} + \dot{x}^T(s)Z_1]Z_1^{-1}[\bar{H}^T\zeta_1(t) + Z_1\dot{x}(s)]ds \\
& -\int_{t-h}^{t} [\zeta_1^T(t)\bar{K} + \dot{x}^T(s)Z_2]Z_2^{-1}[\bar{K}^T\zeta_1(t) + Z_1\dot{x}(s)]ds
\end{aligned} \tag{12}$$

where $\Phi = \Phi_1 + \Xi_2 + \Xi_2^T$ and

$$\Phi_1 = \begin{bmatrix} Q+R & 0 & 0 & P & 0 & 0 \\ \star & -(1-\mu)Q & 0 & 0 & 0 & 0 \\ \star & \star & -R & 0 & 0 & 0 \\ \star & \star & \star & h(Z_1+Z_2) & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \end{bmatrix}$$

Ξ_2 is defined in Theorem 3.

Since $Z_i > 0$, $i = 1, 2$, then the last three parts in (12) are all less than 0. Set $\Psi = \Phi + h\bar{G}Z_1^{-1}\bar{G}^T + h\bar{H}Z_1^{-1}\bar{H}^T + h\bar{K}Z_2^{-1}\bar{K}^T$. So if $\Psi < 0$, then $\dot{V}_2(x_t) \le \zeta_1^T(t)\Psi\zeta_1(t)$.

Then using S-procedure, if there exist $\varepsilon_1 \ge 0$, and $\varepsilon_2 \ge 0$ such that

$$\begin{aligned}
&\zeta_1^T(t)\Psi\zeta_1(t) + \varepsilon_1(\alpha^2 x^T(t)x(t) - f^T(x(t),t)f(x(t),t)) \\
&+\varepsilon_2(\beta^2 x^T(t-d(t))x(t-d(t)) \\
&\qquad -g^T(x(t-d(t)),t)g(x(t-d(t)),t)) < 0
\end{aligned} \tag{13}$$

for all $\zeta_1(t) \neq 0$, which can be rewritten as

$$\zeta_1^T(t)\Gamma\zeta_1(t) < 0$$

where

$$\Gamma = \Psi + \begin{bmatrix} \varepsilon_1\alpha^2 I & 0 & 0 & 0 & 0 & 0 \\ \star & \varepsilon_2\beta^2 I & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & -\varepsilon_1 I & 0 \\ \star & \star & \star & \star & \star & -\varepsilon_2 I \end{bmatrix}$$

Then $\dot{V}_2(x_t) \le \zeta_1^T(t)\Gamma\zeta_1(t) < 0$, the system described by (1)-(4) is asymptotically stable. In view of Schur complement, $\Gamma < 0$ equivalent to (10).

In order to get more less conservative criteria, we will use the Lemma 2 to handle the term $-\int_{t-h}^{t} \dot{x}^T(s)Z_2\dot{x}(s)ds$ in the following.

Theorem 4. For given scalar $h > 0$, and $0 < \mu < 1$, the system described by (1)-(4) is asymptotically stable if there exist real matrices $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, $Z_i = Z_i^T > 0$, $i = 1, 2$,

$$\bar{G} = \begin{bmatrix} G_1 \\ \vdots \\ G_6 \end{bmatrix}, \ \bar{H} = \begin{bmatrix} H_1 \\ \vdots \\ H_6 \end{bmatrix}, \ \bar{L} = \begin{bmatrix} L_1 \\ \vdots \\ L_6 \end{bmatrix}$$

$$Y := [Y_1 \ Y_2], \ W := \begin{bmatrix} W_1 & W_2 \\ \star & W_3 \end{bmatrix} \tag{14}$$

and scalars $\varepsilon_1 \ge 0$, $\varepsilon_2 \ge 0$ such that

$$\begin{bmatrix} Z_2 & Y \\ \star & W \end{bmatrix} \ge 0, \tag{15}$$

$$\begin{bmatrix} \hat{\Xi} & h\bar{G} & h\bar{H} \\ \star & -hZ_1 & 0 \\ \star & \star & -hZ_1 \end{bmatrix} < 0, \tag{16}$$

where

$$\hat{\Xi} = \hat{\Xi}_1 + \hat{\Xi}_2 + \hat{\Xi}_2^T$$

$$\hat{\Xi}_1 = \begin{bmatrix} \hat{\Xi}_{11} & 0 & \hat{\Xi}_{13} & P & 0 & 0 \\ \star & \hat{\Xi}_{22} & 0 & 0 & 0 & 0 \\ \star & \star & \hat{\Xi}_{33} & 0 & 0 & 0 \\ \star & \star & \star & h(Z_1+Z_2) & 0 & 0 \\ \star & \star & \star & \star & -\varepsilon_1 I & 0 \\ \star & \star & \star & \star & \star & -\varepsilon_2 I \end{bmatrix}$$

$$\hat{\Xi}_{11} = Q + R + \varepsilon_1\alpha^2 I + Y_1 + Y_1^T + hW_1$$
$$\hat{\Xi}_{13} = -Y_1^T + Y_2 + hW_2$$
$$\hat{\Xi}_{22} = -(1-\mu)Q + \varepsilon_2\beta^2 I$$
$$\hat{\Xi}_{33} = -R - Y_2 - Y_2^T + hW_3$$

$$\hat{\Xi}_2 = \begin{bmatrix} \bar{G} & \bar{H}-\bar{G} & -\bar{H} & 0 & 0 & 0 \end{bmatrix} + \bar{L}\begin{bmatrix} A & B & 0 & -I & I & I \end{bmatrix}$$

Proof. Choose the same Lyapunov functional as in Theorem 4, and use the equations:

$$\alpha_1(t) := 2\zeta_1^T(t)\bar{G}[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds] = 0$$

$$\alpha_2(t) := 2\zeta_1^T(t)\bar{H}[x(t-d(t)) - x(t-h) - \int_{t-h}^{t-d(t)} \dot{x}(s)ds] = 0$$

and

$$\theta_1(t) := 2\zeta_1^T(t)\bar{L}[-\dot{x}(t) + Ax(t) + Bx(t-d(t)) + f(x(t),t) + g(x(t-d(t)),t)] = 0$$

Calculating the derivative of $V_2(x_t)$ and adding the left side of $\alpha_1(t)$, $\alpha_2(t)$, $\theta_1(t)$ into it and using (11) yield

$$\begin{aligned} \dot{V}_2(x_t) \le & -\int_{t-h}^{t} \dot{x}^T(s)Z_2\dot{x}(s)ds \\ & +\zeta_1^T(t)[\Phi_1 + \hat{\Xi}_2 + \hat{\Xi}_2^T + h\bar{G}Z_1^{-1}\bar{G}^T + h\bar{H}Z_1^{-1}\bar{H}^T]\zeta_1(t) \\ & -\int_{t-d(t)}^{t} [\zeta_1^T(t)\bar{G} + \dot{x}^T(s)Z_1]Z_1^{-1}[\bar{G}^T\zeta_1(t) + Z_1\dot{x}(s)]ds \\ & -\int_{t-h}^{t-d(t)} [\zeta_1^T(t)\bar{H} + \dot{x}^T(s)Z_1]Z_1^{-1}[\bar{H}^T\zeta_1(t) + Z_1\dot{x}(s)]ds \end{aligned} \qquad (17)$$

where Φ_1 is defined in Theorem 3 and $\hat{\Xi}_2$ is defined in Theorem 4. Then using Lemma 2, we have

$$-\int_{t-h}^{t} \dot{x}^T(s)Z_2\dot{x}(s)ds \le$$

$$\begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \left\{ \begin{bmatrix} Y_1 + Y_1^T & -Y_1^T + Y_2 \\ \star & -Y_2^T - Y_2 \end{bmatrix} + hW \right\} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

where $\begin{bmatrix} Z_2 & Y \\ \star & W \end{bmatrix} \ge 0$, Y, W with structures given in (14). Then we can rewrite (17) as follows:

$$\begin{aligned} \dot{V}_2(x_t) \le & \zeta_1^T(t)[\hat{\Phi} + h\bar{G}Z_1^{-1}\bar{G}^T + h\bar{H}Z_1^{-1}\bar{H}^T]\zeta_1(t) \\ & -\int_{t-d(t)}^{t} [\zeta_1^T(t)\bar{G} + \dot{x}^T(s)Z_1]Z_1^{-1}[\bar{G}^T\zeta_1(t) + Z_1\dot{x}(s)]ds \\ & -\int_{t-h}^{t-d(t)} [\zeta_2^T(t)\bar{H} + \dot{x}^T(s)Z_1]Z_1^{-1}[\bar{H}^T\zeta_1(t) + Z_1\dot{x}(s)]ds \end{aligned} \qquad (18)$$

where $\hat{\Phi} = \hat{\Phi}_1 + \hat{\Xi}_2 + \hat{\Xi}_2^T$ and

$$\hat{\Phi}_1 = \begin{bmatrix} \hat{\Phi}_{11} & 0 & \hat{\Phi}_{13} & P & 0 & 0 \\ \star & -(1-\mu)Q & 0 & 0 & 0 & 0 \\ \star & \star & \hat{\Phi}_{33} & 0 & 0 & 0 \\ \star & \star & \star & h(Z_1+Z_2) & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \end{bmatrix}$$

$$\hat{\Phi}_{11} = Q + R + Y_1 + Y_1^T + hW_1$$
$$\hat{\Phi}_{13} = -Y_1^T + Y_2 + hW_2$$
$$\hat{\Phi}_{33} = -R - Y_2 - Y_2^T + hW_3$$

$\hat{\Xi}_2$ is defined in Theorem 4. Then using S-procedure and inequality (13), the proof follows a similar method in Theorem 3, then we can obtain Theorem 4.

4. NEW AUGMENTED STABILITY CONDITION

The augmented Laypunov functional introduced in He et al. (2007), Wu et al. (2004b), He et al. (2005), have shown less conservativeness. In He et al. (2007), He et al. used the augmented Lyapunov functional to deal with systems with a time-varying delay. Now, we will use this technique to handle our problem.

Theorem 5. For given scalar $h > 0$, and $0 < \mu < 1$, the system described by (1)-(4) is asymptotically stable if there exist real matrices $Z_i = Z_i^T > 0,\ i = 1,\ 2$

$$P_a = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ \star & P_{22} & P_{23} \\ \star & \star & P_{33} \end{bmatrix} > 0, \qquad (19)$$

$$Q_a = \begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22} \end{bmatrix} \ge 0, \quad R_a = \begin{bmatrix} R_{11} & R_{12} \\ \star & R_{22} \end{bmatrix} \ge 0,$$

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ \star & X_{22} & X_{23} \\ \star & \star & X_{33} \end{bmatrix} > 0,\ U > 0, \qquad (20)$$

$$\tilde{G} = \begin{bmatrix} G_1 \\ \vdots \\ G_8 \end{bmatrix},\ \tilde{H} = \begin{bmatrix} H_1 \\ \vdots \\ H_8 \end{bmatrix},\ \tilde{L} = \begin{bmatrix} L_1 \\ \vdots \\ L_8 \end{bmatrix}$$

$$Y := [Y_1\ Y_2],\ W = \begin{bmatrix} W_1 & W_2 \\ \star & W_3 \end{bmatrix}$$

and scalars $\varepsilon_1 \ge 0$, $\varepsilon_2 \ge 0$ such that

$$\begin{bmatrix} U & \bar{P}_2 \\ \star & X \end{bmatrix} \ge 0, \quad \begin{bmatrix} Z_2 & Y \\ \star & W \end{bmatrix} \ge 0, \qquad (21)$$

$$\begin{bmatrix} \tilde{\Xi} & h\tilde{G} & h\tilde{H} \\ \star & -hZ_1 & 0 \\ \star & \star & -hZ_1 \end{bmatrix} < 0, \qquad (22)$$

where

$$\bar{P}_2 = [P_{12}^T \; P_{22} \; P_{23}], \;\; \tilde{\Xi} = \tilde{\Xi}_1 + \tilde{\Xi}_2 + \tilde{\Xi}_2^T$$

$$\tilde{\Xi}_1 = \begin{bmatrix} \tilde{\Xi}_{11} & \mu X_{12} & \tilde{\Xi}_{13} & \tilde{\Xi}_{14} & P_{12} & P_{13} & 0 & 0 \\ \star & \tilde{\Xi}_{22} & \mu X_{23} & P_{12}^T & \tilde{\Xi}_{25} & P_{23} & 0 & 0 \\ \star & \star & \tilde{\Xi}_{33} & P_{13}^T & P_{23}^T & P_{33} - Q_{12} & 0 & 0 \\ \star & \star & \star & \tilde{\Xi}_{44} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \tilde{\Xi}_{55} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -Q_{22} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\varepsilon_1 I & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\varepsilon_2 I \end{bmatrix}$$

$$\begin{aligned} \tilde{\Xi}_{11} &= R_{11} + Q_{11} + \mu X_{11} + Y_1 + Y_1^T + hW_1 + \varepsilon_1 \alpha^2 I \\ \tilde{\Xi}_{13} &= \mu X_{13} - Y_1^T + Y_2 + hW_2 \\ \tilde{\Xi}_{14} &= P_{11} + R_{12} + Q_{12} \\ \tilde{\Xi}_{22} &= -(1-\mu)R_{11} + \mu X_{22} + \varepsilon_2 \beta^2 I \\ \tilde{\Xi}_{25} &= -(1-\mu)R_{12} + P_{22} \\ \tilde{\Xi}_{33} &= -Q_{11} + \mu X_{33} - Y_2 - Y_2^T + hW_3 \\ \tilde{\Xi}_{44} &= h(Z_1 + Z_2) + R_{22} + Q_{22} \\ \tilde{\Xi}_{55} &= -(1-\mu)R_{22} + \mu U \end{aligned}$$

$$\begin{aligned} \tilde{\Xi}_2 = & \begin{bmatrix} \tilde{G} & \tilde{H} - \tilde{G} & -\tilde{H} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ & + \bar{L} \begin{bmatrix} A & B & 0 & -I & 0 & 0 & I & I \end{bmatrix} \end{aligned}$$

Proof. Construct the following augmented Lyapunov functional candidate as:

$$\begin{aligned} V_3(x_t) = & \xi^T(t) P_a \xi(t) + \int_{t-h}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T Q_a \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\ & + \int_{t-d(t)}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T R_a \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\ & + \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s) ds d\theta \end{aligned} \qquad (23)$$

where P_a, Q_a, R_a, Z_i, $i = 1, 2$, with structures given in (19), are matrices to be determined and

$$\xi^T(t) = \begin{bmatrix} x^T(t) & x^T(t-d(t)) & x^T(t-h) \end{bmatrix}$$

From the Leibniz-Newton formula, the following equations are true for matrices $\tilde{G}$, $\tilde{H}$ with appropriate dimensions

$$\tilde{\alpha}_1(t) := 2\zeta_2^T(t)\tilde{G}[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s) ds] = 0$$

$$\tilde{\alpha}_2(t) := 2\zeta_2^T(t)\tilde{H}[x(t-d(t)) - x(t-h) - \int_{t-h}^{t-d(t)} \dot{x}(s) ds] = 0$$

where

$$\begin{aligned} \zeta_2^T(t) = & \left[x^T(t) \; x^T(t-d(t)) \; x^T(t-h) \; \dot{x}^T(t) \right. \\ & \left. \dot{x}^T(t-d(t)) \; \dot{x}^T(t-h)) \; f^T(x(t),t) \; g^T(x(t-d(t)),t) \right] \end{aligned}$$

Observe that

$$\begin{aligned} -\int_{t-h}^{t} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s) ds = & -\int_{t-d(t)}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds \\ - \int_{t-h}^{t-d(t)} \dot{x}^T(s) Z_1 \dot{x}(s) ds - & \int_{t-h}^{t} \dot{x}^T(s) Z_2 \dot{x}(s) ds \end{aligned} \qquad (24)$$

and note that

$$\begin{aligned} \tilde{\theta}_1(t) := & 2\zeta_2^T(t)\tilde{L}[-\dot{x}(t) + Ax(t) + Bx(t-d(t)) \\ & + f(x(t),t) + g(x(t-d(t)),t)] = 0 \end{aligned}$$

The derivative of $V_p(x_t) = \xi^T(t) P_a \xi(t)$ is expressed as

$$\begin{aligned} \dot{V}_p(x_t) & = 2\xi^T(t) P_a \begin{bmatrix} \dot{x}(t) \\ (1 - \dot{d}(t))\dot{x}(t-d(t)) \\ \dot{x}(t-h) \end{bmatrix} \\ & = 2\xi^T(t) P_a \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-d(t)) \\ \dot{x}(t-h) \end{bmatrix} - 2\dot{d}(t)\dot{x}(t-d(t))\bar{P}_2 \xi(t) \end{aligned}$$

Using Lemma 1, it follows that

$$\begin{aligned} & - 2\dot{d}(t)\dot{x}^T(t-d(t))\bar{P}_2\xi(t) \\ & \leq \mu \dot{x}^T(t-d(t)) U \dot{x}(t-d(t)) + \mu \xi^T(t) X \xi(t) \end{aligned}$$

where $\begin{bmatrix} U & \bar{P}_2 \\ \star & X \end{bmatrix} \geq 0$, $\bar{P}_2 = \begin{bmatrix} P_{12}^T & P_{22} & P_{23} \end{bmatrix}$, U, X with structures given in (20). Calculating the derivative of $V_3(x_t)$ and adding the left side of $\tilde{\alpha}_1(t)$, $\tilde{\alpha}_2(t)$, $\tilde{\theta}_1(t)$ into it and using (24) yield

$$\begin{aligned} \dot{V}_3(x_t) \leq & 2\xi^T(t) P_a \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-d(t)) \\ \dot{x}(t-h) \end{bmatrix} + \mu \xi^T(t) X \xi(t) \\ & + \mu \dot{x}^T(t-d(t)) U \dot{x}(t-d(t)) + \alpha_1(t) + \alpha_2(t) + \theta_1(t) \\ & + \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T (Q_a + R_a) \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\ & - \begin{bmatrix} x(t-h) \\ \dot{x}(t-h) \end{bmatrix}^T Q_a \begin{bmatrix} x(t-h) \\ \dot{x}(t-h) \end{bmatrix} \\ & -(1-\mu) \begin{bmatrix} x(t-d(t)) \\ \dot{x}(t-d(t)) \end{bmatrix}^T R_a \begin{bmatrix} x(t-d(t)) \\ \dot{x}(t-d(t)) \end{bmatrix} \\ & + h\dot{x}^T(t)(Z_1 + Z_2)\dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds \\ & - \int_{t-h}^{t-d(t)} \dot{x}^T(s) Z_1 \dot{x}(s) ds - \int_{t-h}^{t} \dot{x}^T(s) Z_2 \dot{x}(s) ds \end{aligned} \qquad (25)$$

Then using Lemma 2, we have

$$\begin{aligned} & - \int_{t-h}^{t} \dot{x}^T(s) Z_2 \dot{x}(s) ds \leq \begin{bmatrix} x(t) \\ x(t-h)) \end{bmatrix}^T \times \\ & \left\{ \begin{bmatrix} Y_1 + Y_1^T & -Y_1^T + Y_2 \\ \star & -Y_2^T - Y_2 \end{bmatrix} + hW \right\} \begin{bmatrix} x(t) \\ x(t-h)) \end{bmatrix} \end{aligned}$$

where $\begin{bmatrix} Z_2 & Y \\ \star & W \end{bmatrix} \geq 0$.

Now we can rewrite (25) as follows:

$$\dot{V}_3(x_t) \le \zeta_2^T(t)[\tilde{\Phi} + h\tilde{G}Z_1^{-1}\tilde{G}^T + h\tilde{H}Z_1^{-1}\tilde{H}^T]\zeta_2(t)$$
$$- \int_{t-d(t)}^{t} [\zeta_2^T(t)\tilde{G} + \dot{x}^T(s)Z_1]Z_1^{-1}[\tilde{G}^T\zeta_2(t) + Z_1\dot{x}(s)]ds \quad (26)$$
$$- \int_{t-h}^{t-d(t)} [\zeta_2^T(t)\tilde{H} + \dot{x}^T(s)Z_1]Z_1^{-1}[\tilde{H}^T\zeta_2(t) + Z_1\dot{x}(s)]ds$$

where $\tilde{\Phi} = \tilde{\Phi}_1 + \tilde{\Xi}_2 + \tilde{\Xi}_2^T$ and

$$\tilde{\Phi}_1 = \begin{bmatrix} \tilde{\Phi}_{11} & \mu X_{12} & \tilde{\Phi}_{13} & \tilde{\Phi}_{14} & P_{12} & P_{13} & 0 & 0 \\ \star & \tilde{\Phi}_{22} & \mu X_{23} & P_{12}^T & \tilde{\Phi}_{25} & P_{23} & 0 & 0 \\ \star & \star & \tilde{\Phi}_{33} & P_{13}^T & P_{23}^T & P_{33} - Q_{12} & 0 & 0 \\ \star & \star & \star & \tilde{\Phi}_{44} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \tilde{\Phi}_{55} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -Q_{22} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 & 0 \end{bmatrix}$$

$$\tilde{\Phi}_{11} = R_{11} + Q_{11} + \mu X_{11} + Y_1 + Y_1^T + hW_1$$
$$\tilde{\Phi}_{13} = \mu X_{13} - Y_1^T + Y_2 + hW_2$$
$$\tilde{\Phi}_{14} = P_{11} + R_{12} + Q_{12}$$
$$\tilde{\Phi}_{22} = -(1-\mu)R_{11} + \mu X_{22}$$
$$\tilde{\Phi}_{25} = -(1-\mu)R_{12} + P_{22}$$
$$\tilde{\Phi}_{33} = -Q_{11} + \mu X_{33} - Y_2 - Y_2^T + hW_3$$
$$\tilde{\Phi}_{44} = h(Z_1 + Z_2) + R_{22} + Q_{22}$$
$$\tilde{\Phi}_{55} = -(1-\mu)R_{22} + \mu U$$

$\tilde{\Xi}_2$ is defined in Theorem 5. Since $Z_i > 0$, $i = 1, 2$, then the last two parts in (26) are all less than 0. Set $\tilde{\Psi} = \tilde{\Phi} + h\tilde{G}Z_1^{-1}\tilde{G}^T + h\tilde{H}Z_1^{-1}\tilde{H}^T$. So if $\tilde{\Psi} < 0$, then $\dot{V}_3(x_t) \le \zeta_2^T(t)\tilde{\Psi}\zeta_2(t) < 0$. The remaining proof of Theorem 5 is similar to that of Theorem 3, thus omitted.

5. EXAMPLE

In order to demonstrate the effectiveness of the method we have presented, the same example (used in Cao & Lam (2000), Han (2004) and Zuo & Wang (2006)) is given in this section to compare with the results of the previous methods. Consider the system with

$$A = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix}$$

Applying the methods proposed in Cao & Lam (2000), Han (2004) and Zuo & Wang (2006) and Theorem 3, 4 and 5 in this paper, the maximum value of h for stability of system under different α and β, is listed in Table 1. It is obvious that the stability criteria presented in this paper give more less conservative results than the existing ones.

Table 1: Compared results for h by different methods

	$\alpha = 0$ $\mu = 0$	$\beta = 0.1$ $\mu = 0.5$	$\alpha = 0.1$ $\mu = 0$	$\beta = 0.1$ $\mu = 0.5$
Cao & Lam	0.6811	0.5467	0.6129	0.4950
Han(2004)	1.3279	0.6743	1.2503	0.5716
Zuo(2006)	2.0422	1.1424	1.8753	1.0097
Theorem 3	2.7419	1.1811	2.0000	1.0466
Theorem 4	3.0556	1.3360	2.1240	1.1883
Theorem 5	∞	1.6198	3.5254	1.4938

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