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Adaptive Tracking and Recursive Identification for a Class of Hammerstein Systems ⋆

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Abstract: In this work, a weighted least squares (WLS) based adaptive tracker is designed for a class of Hammerstein systems. Incorporating with the diminishing excitation technique, the proposed adaptive tracker leads to the minimality of the tracking errors and strong consistency of the estimates for the unknown system parameters. A numerical example is given and the simulation results are consistent with the theoretical analysis.

Keywords: Hammerstein system, adaptive tracking, optimality, weighted least squares, identification, strong consistency.

1. INTRODUCTION

The Hammerstein system consisting of a static function $f(\cdot)$ followed by a linear dynamics (see Fig.1) is a commonly used model in practice, such as chemical engineering Eskinat [1991], electronics circuits Kim&Konstantinou [2001], biological cybernetics Hunter&Korenberg [1986], and others. So, its identification and adaptive control issues naturally receive much attention from researchers. Though the most of existing theoretical results on identification of the Hammerstein systems, e.g., Eskinat [1991], Narendra&Gallman [1966], Stoica&Söderström [1982], Bai&Liu [2004], Chaoui et al [2005], Greblicki [2002], Chen [2004], Chen [2005], Zhao&Chen [2006] are for open-loop systems, there are also a few works Anbumani et al [1981], Agarwal&Seborg [1987], Wittenmark [1993], Samuelsson et al [2005], Chen [2007] on the adaptive control problems for the Hammerstein systems. The adaptive control problem is considered in Anbumani et al [1981], Agarwal&Seborg [1987], Wittenmark [1993] for the discrete time Hammerstein models, and in Samuelsson et al [2005] for the continuous time, but in these works, there are only numerical examples. Maybe, the first rigorous theoretical analysis for adaptive regulation of Hammerstein systems in noise environment is given in Chen [2007], where a direct adaptive regulator based on stochastic approximation (Chen [2002]), is designed such that the regulation error is asymptotically minimized.

In this paper, we consider adaptive tracking, for which the reference signal is normally allowed to be any given bounded signal, while for regulation it is restricted to be a constant.

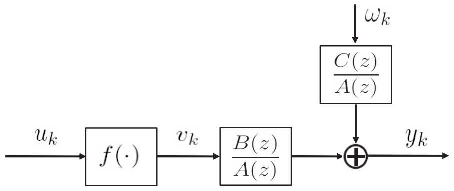

Fig. 1. Hammerstein system

Let us consider the single-input single-output (SISO) Hammerstein system with linear subsystem being an ARMAX model and the nonlinear function being a polynomial:

$$A(z)y_{k+1} = B(z)f(u_k) + C(z)\omega_{k+1}, \quad k \geq 0, \tag{1}$$

$$A(z) \triangleq 1 + a_1 z + \cdots + a_p z^p, \; B(z) \triangleq b_1 + b_2 z + \cdots + b_q z^{q-1},$$

$$C(z) \triangleq 1 + c_1 z + \cdots + c_r z^r, \quad f(x) = \sum_{i=1}^{s} f_j x^j,$$

where z is the backward-shift operator, u_k and y_k denote the system input and output, respectively, ω_k is the driven noise, $f(\cdot)$ is the unknown static function and $v_k \triangleq f(u_k)$ is the unavailable internal signal.

Let $\{y_k^*\}$ be a bounded reference signal. The problem of adaptive tracking consists in designing feedback control u_k depending on $\{u_0, \cdots, u_{k-1}, y_0, \cdots, y_k, y_0^*, \cdots, y_{k+1}^*\}$ in order to minimize

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} (y_k - y_k^*)^2 \tag{2}$$

Inspired by the adaptive tracker for linear systems (see Chen&Guo [1991], Chapter 5), in this work a weighted least squares (WLS) based adaptive tracker for system (1) is introduced and the tracking error (2) is shown to be minimized. Further, by the diminishing excitation technique used in Chen&Guo [1991], both the minimality of the control performance (2) and the strong consistency

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of the estimates for the Hammerstein system (1) are obtained simultaneously. The main results of this work are Theorem 3 and Theorem 6, which correspond to Theorem 5.4 and Theorem 6.2 in Chen&Guo [1991] for linear systems.

The rest of the paper is arranged as follows. The minimum of the tracking error and the adaptive control are shown in Section 2. The optimality of the WLS based adaptive tracker and the strong consistency of the WLS estimates are presented in Section 3. An illustrative example is given in Section 4. Some concluding remarks are addressed in Section 5.

2. STATEMENT OF PROBLEM

2.1 Optimal Tracking Control

In the following, for two sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$, by $a_k = O(b_k)$, we mean $|a_k| \le c|b_k|$ for some constant $c > 0$ and by $a_k = o(b_k)$, we mean $\frac{a_k}{b_k} \to 0$ as $k \to \infty$.

Prior to defining the adaptive control, let us first derive the lower bound of the tracking error and the optimal tracking control for the case where the system (1) is known. It is well-known that for the linear ARMAX system the optimal tracking control is determined by solving a *Diophantine* equation and the tracking error has the lower bound R_ω, where $R_\omega \triangleq \lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n}\omega_k^2$ (c.f. Chen&Guo [1991]). Similar results also hold for the Hammerstein system (1).

Lemma 1. Let $\{\mathcal{F}_k\}$ be a family of non-decreasing σ-algebras and $y_k \in \mathcal{F}_k$, $\forall\, k \ge 0$. Assume that **i)** $\{\omega_k, \mathcal{F}_k\}$ is a martingale difference sequence with $\sup\limits_k E(\omega_{k+1}^2|\mathcal{F}_k) < \infty$ a.s. and $R_\omega \triangleq \lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n}\omega_k^2 < \infty$; **ii)** the reference signal $\{y_k^*\}$ is bounded and y_k^* is $\mathcal{F}_{k-1}$-measurable, $\forall\, k \ge 1$. Then for any $\mathcal{F}_k$-measurable control u_k, the tracking error has the following lower bound,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n}(y_k - y_k^*)^2 \ge R_\omega. \tag{3}$$

Further, if u_k can be solved from

$$\begin{aligned} y_{k+1}^* = &-\big(A(z)-1\big)y_{k+1} + B(z)f(u_k) \\ &+ \big(C(z)-1\big)\big(y_{k+1} - y_{k+1}^*\big), \end{aligned} \tag{4}$$

or equivalently from

$$\begin{aligned} b_1\sum_{j=1}^{s} f_j u_k^j =& y_{k+1}^* + (A(z)-1)\,y_{k+1} - (B(z)-b_1)\,f(u_k) \\ &- (C(z)-1)\left(y_{k+1} - y_{k+1}^*\right), \end{aligned}$$

then u_k is the optimal tracking control.

Proof: The proof is similar to Theorem 3.6 in Chen&Guo [1991] for the optimal tracking control of linear systems. ■

Since $A(z)$, $B(z)$, $C(z)$ and $f(\cdot)$ are unknown, the optimal tracking control u_k defined by (4) cannot be used directly.

2.2 WLS Based Adaptive Control

Since $f(x) = \sum_{i=1}^{s} f_j x^j$, the Hammerstein system (1) can be expressed as follows:

$$\begin{aligned} &y_{k+1} + a_1 y_k + \cdots + a_p y_{k+1-p} \\ &= b_1\sum_{j=1}^{s} f_j u_k^j + \cdots + b_q\sum_{j=1}^{s} f_j u_{k+1-q}^j + C(z)\omega_{k+1}. \end{aligned}$$

By setting

$$\begin{aligned} \theta^T =&[-a_1 \cdots -a_p\ (b_1 f_1)\cdots(b_1 f_s)\ \cdots \\ &(b_q f_1)\cdots(b_q f_s)\ c_1\cdots c_r], \end{aligned} \tag{5}$$

$$\begin{aligned} \varphi_k^{0T} =&[y_k \cdots y_{k+1-p}\ u_k \cdots u_k^s \cdots \\ &u_{k+1-q}\cdots u_{k+1-q}^s\ \omega_k \cdots \omega_{k+1-r}], \end{aligned} \tag{6}$$

the system is written in the compact form:

$$y_{k+1} = \theta^T\varphi_k^0 + \omega_{k+1}. \tag{7}$$

Thus, we have transformed the Hammerstein system into an ARMAX system. This makes it possible to apply the identification methods well-developed for linear systems to estimating unknown parameters.

We first introduce conditions to be used.

A1. $\{\omega_k, \mathcal{F}_k\}$ is a martingale difference sequence with

$$\sup_k E(\omega_{k+1}^2|\mathcal{F}_k) < \infty \ \text{ a.s.},\ \ R_\omega \triangleq \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\omega_k^2 > 0, \tag{8}$$

where $\{\mathcal{F}_k\}$ is a family of non-decreasing σ-algebras;

A2. There exists a nondecreasing sequence of positive numbers $\{d_k\}$ such that $d_{k+1} = O(d_k)$, $d_k = o(k)$ and $\omega_{k+1}^2 = O(d_k)$ a.s.;

A3. $C^{-1}(z) - \frac{1}{2}$ is strictly positive real (SPR), i.e., $C^{-1}(e^{i\lambda}) + C^{-1}(e^{-i\lambda}) > 1$, $\forall\ \lambda \in [0, 2\pi]$;

A4. $B(z)$ is of minimum phase, i.e., $B(z) \ne 0$, $\forall\ |z| \le 1$;

A5. $A(z)$, $zB(z)$ and $C(z)$ have no common factor and the row vector $[a_p\ b_q\ c_r] \ne 0$;

A6. $f_s = 1$ and s is an odd number.

Remark 1. Conditions **A1**, **A3**, and **A5** are commonly used for the LS-like algorithms Chen&Guo [1991], while **A4** is necessary for stability of the adaptive tracker Chen&Guo [1991], Goodwin&Sin [1984]. If s is even, for $f(x) = \sum_{j=1}^{s} f_j x^j$, then all control u_k solved from (4) may be complex. Condition **A6** is to exclude this possibility. Setting $f_s = 1$ is for identifiability.

With an arbitrary θ_0 and $P = \alpha_0 I$ for some $\alpha_0 > 0$, the WLS method defines $\{\theta_k\}_{k\ge1}$ and $\{P_k\}_{k\ge1}$ by the following algorithm:

$$\theta_{k+1} = \theta_k + a_k P_k\varphi_k\left(y_{k+1} - \theta_k^T\varphi_k\right), \tag{9}$$

$$P_{k+1} = P_k - a_k P_k\varphi_k\varphi_k^T P_k,\quad a_k = \frac{1}{\lambda_k^{-1} + \varphi_k^T P_k\varphi_k}, \tag{10}$$

$$\widehat{\omega}_{k+1} = y_{k+1} - \theta_{k+1}^T\varphi_k,\ \lambda_k \triangleq \frac{1}{\log^{1+\sigma} r_k},\ \text{some } \sigma > 0, \tag{11}$$

$$\begin{aligned} \varphi_k^T =&\big[y_k \cdots y_{k+1-p}\ u_k \cdots u_k^s \cdots \\ &u_{k+1-q}\cdots u_{k+1-q}^s\ \widehat{\omega}_k \cdots \widehat{\omega}_{k+1-r}\big], \end{aligned} \tag{12}$$

$$\begin{aligned} \theta_k^T \triangleq&\big[-a_{1,k}\cdots -a_{p,k}\ (b_1 f_1)_k \cdots\ (b_1 f_{s-1})_k\ b_{1,k}\cdots \\ &(b_q f_1)_k \cdots\ (b_q f_{s-1})_k\ b_{q,k}\ c_{1,k}\cdots c_{r,k}\big], \end{aligned}$$

where $r_k \triangleq 1 + \sum_{i=0}^{k}\|\varphi_i\|^2$, and $r_k^0 \triangleq 1 + \sum_{i=0}^{k}\|\varphi_i^0\|^2$. According to (10), $P_{k+1}^{-1} = \sum_{i=0}^{k}\lambda_i\varphi_i\varphi_i^T + \frac{1}{\alpha_0}I$.

The real solution u_k with minimum magnitude to the following algebraic equation

$$(b_1f_1)_k u_k + \cdots + (b_1f_{s-1})_k u_k^{s-1} + b_{1,k}u_k^s \\ = y_{k+1}^* + \Big[(b_1f_1)_k u_k + \cdots + (b_1f_{s-1})_k u_k^{s-1} + b_{1,k}u_k^s - \theta_k^T\varphi_k\Big], \quad (13)$$

is defined as the adaptive control at time k. It is worth noting that the control terms in the square brackets at the right-hand side of (13) are with time indices$\le k-1$.

In the case $b_{1,k}=0$, u_k may not be defined from (13). To avoid this difficulty, we replace $b_{1,k}$ with any $\mathcal{F}_k$-measurable $\widehat{b}_{1,k}$ with properties as follows:

$$\widehat{b}_{1,k} \neq 0, \quad k \ge 1 \;\text{ and }\; \triangle\widehat{b}_{1,k} \triangleq \widehat{b}_{1,k} - b_{1,k} \underset{k\to\infty}{\longrightarrow} 0.$$

For example, $\widehat{b}_{1,k} \triangleq b_{1,k} + \text{sign}\{b_{1,k}\}\frac{1}{k}$, where $\text{sign}\{x\} \triangleq 1$, if $x \ge 0$ and $\text{sign}\{x\} \triangleq -1$, if $x<0$.

Further, u_k defined by (13) may not be sufficiently excited for identification. Motivated by Chapter 6 in Chen&Guo [1991], a diminishingly excited signal is added to the desired control solved from (13) aiming at obtaining strongly consistent estimates for θ without bringing down the control performance. For this, let us take a sequence $\{\varepsilon_k\}$ of bounded i.i.d. random variables with continuous distribution and $E\epsilon_k=0$, $E\epsilon_k^2=1$. Let $\{\varepsilon_k\}$ be independent of $\{\omega_k\}$.

We now define the WLS based diminishingly excited control. First, define $u_k^{(c)}$ from the following equality:

$$(b_1f_1)_k u_k^{(c)} + \cdots + (b_1f_{s-1})_k (u_k^{(c)})^{s-1} + \widehat{b}_{1,k}(u_k^{(c)})^s \\ = y_{k+1}^* + \Big[(b_1f_1)_k u_k + \cdots + (b_1f_{s-1})_k u_k^{s-1} + b_{1,k}u_k^s - \theta_k^T\varphi_k\Big], \quad (14)$$

where θ_k is generated by (9)-(12). Then, define the diminishingly excited control

$$u_k = u_k^{(c)} + v_k^{(d)}, \; v_k^{(d)} \triangleq \frac{\varepsilon_k}{k^{\frac{\epsilon}{2}}} \text{ with } \epsilon>0 \text{ suffciently small.} \quad (15)$$

u_k serves as the system input.

Remark 2. If $\lambda_k \equiv 1$, then the WLS algorithm (9)-(12) coincide with the extended least squares (ELS) algorithm. The optimal ELS based tracker for linear systems (i.e., $f(x)=x$) can be found in Chen&Guo [1991], Theorem 5.4, for which the key point in the proof is to show

$$u_k^2 = O\left(\log^2 r_{k-1}\Big(\sum_{i=0}^{p-1} y_{k-i}^2 + \sum_{i=1}^{q-1} u_{k-i}^2 + \sum_{i=0}^{r-1}\widehat{\omega}_{k-i}^2\Big)\right) \\ + O(\log r_{k-1}), \quad (16)$$

where u_k is defined from

$$\widehat{b}_{1,k}u_k = y_{k+1}^* + b_{1,k}u_k - \theta_k^T\varphi_k \quad (17)$$

with θ_k generated by the ELS algorithm and $\widehat{b}_{1,k} \triangleq b_{1,k} + \text{sign}\{b_{1,k}\}\frac{1}{\log^{\frac{1}{2}} r_{k-1}}$ (see Chen&Guo [1991], pp 175-181 for details). If θ_k in (14) is replaced by the ELS estimate, then it is not clear whether or not a relationship similar to (16) exists, because the left-hand side of (14) is nonlinear with respect to $u_k^{(c)}$. This is why we resort to the WLS algorithm.

Remark 3. By Theorem 2.8 in Chen&Guo [1991], it is directly verified that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}(y_k - y_k^*)^2 = R_\omega \quad (18)$$

is equivalent to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}(y_k - y_k^* - \omega_k)^2 = 0 \;\text{ a.s.} \quad (19)$$

3. MAIN RESULTS

Since $\frac{x^2+\cdots+x^{2s}}{(f_1x+\cdots+f_{s-1}x^{s-1}+x^s)^2} \to 1$, as $|x|\to\infty$, for any $\epsilon>0$ there exists N large enough such that $x^2+\cdots+x^{2s} \le (1+\epsilon)f^2(x)$, $|x|\ge N$. Moreover, there exists $M>0$ such that $x^2+\cdots+x^{2s}\le M$, $\forall\ |x|\le N$. Hence, for u_k generated by (14) and (15) we have

$$u_k^2+\cdots+u_k^{2s} = O(1) + O\left(f^2(u_k)\right) \quad (20)$$

as $k\to\infty$.

Defining

$$U(k) \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \\ C_2^1 u_k^{(c)} & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ C_s^{s-1}(u_k^{(c)})^{s-1} & C_s^{s-2}(u_k^{(c)})^{s-2} & \cdots & 1 \end{bmatrix}$$

with $C_j^i = \frac{j!}{i!(j-i)!}$ $(j\ge i)$,

$$\overline{u}_k \triangleq \begin{bmatrix} u_k^{(c)} \\ (u_k^{(c)})^2 \\ \vdots \\ (u_k^{(c)})^s \end{bmatrix}, \quad \overline{v}_k \triangleq \begin{bmatrix} v_k^{(d)} \\ (v_k^{(d)})^2 \\ \vdots \\ (v_k^{(d)})^s \end{bmatrix},$$

we then have

$$\begin{bmatrix} u_k \\ u_k^2 \\ \vdots \\ u_k^s \end{bmatrix} = \begin{bmatrix} u_k^{(c)} + v_k^{(d)} \\ (u_k^{(c)} + v_k^{(d)})^2 \\ \vdots \\ (u_k^{(c)} + v_k^{(d)})^s \end{bmatrix} = \overline{u}_k + U(k)\overline{v}_k. \quad (21)$$

Denote by $\mathcal{F}'_{k-1}$ the σ-algebra generated by $\{\omega_i, 0\le i\le k; \varepsilon_j, 0\le j\le k-1\}$.

It is clear that $\overline{u}_k$ and $U(k)$ are $\mathcal{F}'_{k-1}$-measurable, $\overline{v}_k$ is $\mathcal{F}'_k$-measurable and $\left\{\overline{v}_k - E\overline{v}_k, \mathcal{F}'_k\right\}$ is a martingale difference sequence.

Noticing $v_k^{(d)}\to 0$, from the definitions of $\overline{u}_k$ and $U(k)$ we have

$$\|\overline{u}_k\|^2 = \left(u_k^{(c)}\right)^2 + \cdots + \left(u_k^{(c)}\right)^{2s} = O(1) + O\left((u_k)^{2s}\right), \quad (22)$$

$$\|U(k)\|^2 \le \text{tr}\Big\{U^T(k)U(k)\Big\} \\ = O(1) + O\left((u_k^{(c)})^2\right) + \cdots + O\left((u_k^{(c)})^{2(s-1)}\right) \\ = O(1) + O\left((u_k)^{2(s-1)}\right), \quad (23)$$

and

$$\mathrm{tr}\Big\{U(k)U^T(k)\Big\} = O(1) + O\left((u_k^{(c)})^{2(s-1)}\right)$$
$$= O(1) + O\left((u_k)^{2(s-1)}\right). \quad (24)$$

Lemma 2. If **A1** and **A3** hold, then the WLS estimate θ_k defined by (9)-(12) is bounded and has the following properties,

$$\widetilde{\theta}_{n+1}^T P_{n+1}^{-1} \widetilde{\theta}_{n+1} = O(1) \quad \text{a.s.} \quad (25)$$

$$\sum_{k=1}^{\infty} \lambda_k \xi_k^2 < \infty \quad \text{a.s.}, \quad \sum_{k=1}^{\infty} \frac{\left(\widetilde{\theta}_k^T \varphi_k\right)^2}{\lambda_k^{-1} + \varphi_k^T P_k \varphi_k} < \infty \quad \text{a.s.} \quad (26)$$

$$\frac{1}{\log^{1+\sigma} r_n} \sum_{k=1}^{n} \xi_k^2 = o(1) \quad \text{a.s.}, \quad \sum_{k=1}^{\infty} \delta_k < \infty \quad \text{a.s.} \quad (27)$$

$$\left(\widetilde{\theta}_k^T \varphi_k\right)^2 = O\left(\log^{1+\sigma} r_k\right) + O\left(\delta_k \|\varphi_k\|^2\right) \quad \text{a.s.} \quad (28)$$

$$\sum_{k=1}^{n} \left(\widetilde{\theta}_k^T \varphi_k\right)^2 = O\left(\log^{1+\sigma} r_n\right) + O\left(\sup_{0 \le k \le n} \|\varphi_k\|^2\right) \quad (29)$$

where $\xi_{k+1} \triangleq y_{k+1} - \theta_{k+1}^T \varphi_k - \omega_{k+1}$, $\widetilde{\theta}_{k+1} \triangleq \theta - \theta_{k+1}$, and $\delta_k \triangleq \mathrm{tr}(P_k - P_{k+1})$.

Proof: The boundedness of θ_k, and (25) and (26) are proved in Lemma 1 in Guo [1996]. The proof of (27), (28), and (29) is based on (25) and (26). ■

Theorem 3. If **A1-A4**, and **A6** hold, then the WLS-based adaptive tracker defined by (14) and (15) is optimal. Precisely,

$$\|\varphi_k\|^2 = O\left(d_k + \log^{1+\sigma} k\right), \quad \text{a.s.} \quad (30)$$

$$\sum_{k=1}^{n} (y_{k+1} - y_{k+1}^* - \omega_{k+1})^2 = o(n) \quad \text{a.s.} \quad (31)$$

The proof of Theorem 3 is motivated by the analysis for the ELS based adaptive tracker in Chen&Guo [1991].

Remark 4. The WLS-based adaptive tracker is still optimal if $v_k^{(d)} \equiv 0$.

Denote by $\lambda_{\max}(n)$, $\lambda_{\min}(n)$, $\lambda_{\max}^0(n)$, and $\lambda_{\min}^0(n)$ the maximal and minimal eigenvalues of $\sum_{i=0}^n \varphi_i \varphi_i^T + \frac{1}{\alpha_0} I$ and $\sum_{i=0}^n \varphi_i^0 \varphi_i^{0T} + \frac{1}{\alpha_0} I$, respectively. For strong consistency of the WLS estimate, we have the following lemma, which is inspired by Theorem 4.2 in Chen&Guo [1991] for the ELS algorithm.

Lemma 4. If **A1**, **A3** hold and

$$\left(\log \lambda_{\max}^0(n)\right)^{1+\sigma} = o\left(\lambda_{\min}^0(n)\right) \quad \text{a.s.}, \quad (32)$$

then the WLS estimate θ_n is strongly consistent with the following rate of convergence:

$$\left\|\theta_{n+1} - \theta\right\|^2 = O\left(\frac{\left(\log \lambda_{\max}^0(n)\right)^{1+\sigma}}{\lambda_{\min}^0(n)}\right) \quad \text{a.s.} \quad (33)$$

where $\sigma > 0$ is defined in (11).

For further results, we strengthen condition **A2** to

A7. There is a nondecreasing sequence of positive numbers $\{d_k\}$ such that $\omega_{k+1}^2 = O(d_k)$, $d_k = O(k^\delta)$, $v_k^{(d)} = \varepsilon_k / k^{\frac{\epsilon}{2}}$ for some $\delta > 0$ and small enough $\epsilon > 0$ such that the interval

$$\Bigg(\frac{1}{2}\Big(1 + \frac{2s-1}{s}\delta\Big),$$
$$1 - (\mu+1)s\epsilon - (\mu+1)\delta - (\mu+1)\frac{(s-1)^2}{s}\delta\Bigg], \quad (34)$$

is nonempty, where $\mu \triangleq p + \max\{q, r\} - 1$.

Lemma 5. If **A1-A7** hold, then

$$\sum_{k=1}^{\infty} \frac{1}{k^{1-s\epsilon-\frac{(s-1)^2}{s}\delta}} \Bigg(U(k)(\overline{v}_k - E\overline{v}_k)(\overline{v}_k - E\overline{v}_k)^T U^T(k)$$
$$- U(k)E\Big[(\overline{v}_k - E\overline{v}_k)(\overline{v}_k - E\overline{v}_k)^T\Big]U^T(k)\Bigg) < \infty \quad \text{a.s.}$$
$$(35)$$

$$\frac{1}{k^{1-s\epsilon-\frac{(s-1)^2}{s}\delta}} \sum_{i=1}^{k} U(i)(\overline{v}_i - E\overline{v}_i)(\overline{v}_i - E\overline{v}_i)^T U^T(i) \ge c_0 I$$
$$(36)$$

for all large enough k, where $c_0 > 0$ may depend on sample paths.

For proving the lemma, (30) and the convergent theorem for martingale difference sequences play the key role.

Theorem 6. If **A1-A7** hold, then the WLS estimate is strongly consistent with the following convergence rate,

$$\left\|\theta_{n+1} - \theta\right\|^2 = O\left(\frac{\log^{1+\sigma} n}{n^\alpha}\right) \quad \text{a.s.} \quad (37)$$

for any $\alpha \in \left(\frac{1}{2}\left(1 + \frac{2s-1}{s}\delta\right), 1 - (\mu+1)s\epsilon - (\mu+1)\delta - (\mu+1)\frac{(s-1)^2}{s}\delta\right]$.

Instead of detailed proof we only outline the basic steps of the proof. First, we prove that $\lambda_{\min}^0(n) \ge c_1 n^\alpha$, where α belongs to the interval (34). Second, we prove that $\lambda_{\max}^0(n) \le c_2 n$. The positive numbers c_1 and c_2 may depend on sample paths. Finally, based on the inequalities established in the previous two steps, the conclusion of Theorem 6 follows from Lemma 4 incorporating with (30).

Remark 5. Since $b_1 \ne 0$ by **A4**, $(b_1 f_j)_k / b_{1,k}$, $j = 1, \cdots, s-1$ may serve as the estimates for $f_1, \cdots, f_{s-1}$.

Remark 6. If $f(x) = x$, i.e., $s = 1$, then the Hammerstein system under consideration is reduced to an ARMAX system and the interval defined by (34) is reduced to $\left(\frac{1}{2}(1+\delta), 1 - (\mu+1)\epsilon - (\mu+1)\delta\right]$ which coincides with the one used in Theorem 6.2 of Chen&Guo [1991] for ARMAX systems.

4. NUMERICAL EXAMPLE

Consider the following SISO Hammerstein system

$$y_{k+1} + a_1 y_k = b_1(f_1 u_k + f_2 u_k^2 + u_k^3) + \omega_{k+1} + c_1 \omega_k,$$

where $a_1 = 0.5, b_1 = 2, f_1 = 1, f_2 = 1.5, c_1 = -0.3$, and $\{\omega_k\}_{k \ge 0}$ is a sequence of i.i.d. random variables $\omega_k \in \mathcal{N}(0, \sigma_\omega^2)$ with $\sigma_\omega^2 = 0.25$.

For $l = 0, \cdots, 9$ and $k = 1, \cdots, 10000$, define the reference signal

$$y_k^* = \begin{cases} +1, & k \in \big[1000l + 1, \cdots, 1000l + 500\big] \\ -1, & k \in \big[1000l + 501, \cdots, 1000l + 1000\big] \end{cases}$$

It is directly verified that **A1-A6** hold.

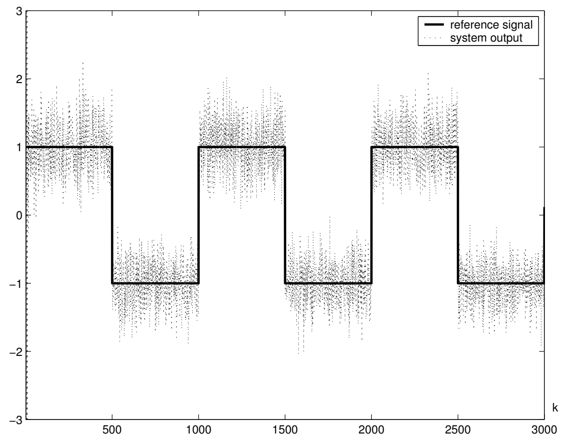

Fig. 2. System output vs reference signal

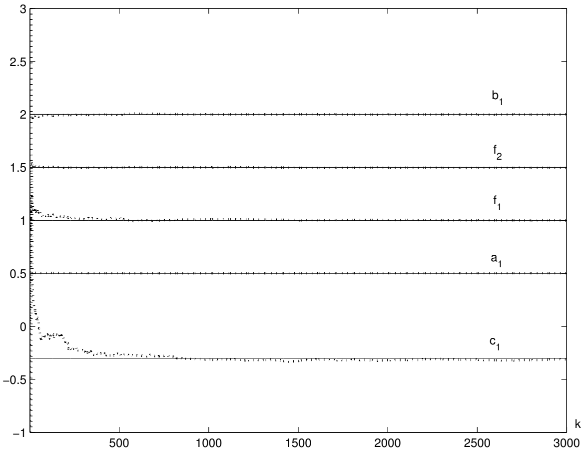

Fig. 3. Estimates for unknown parameters

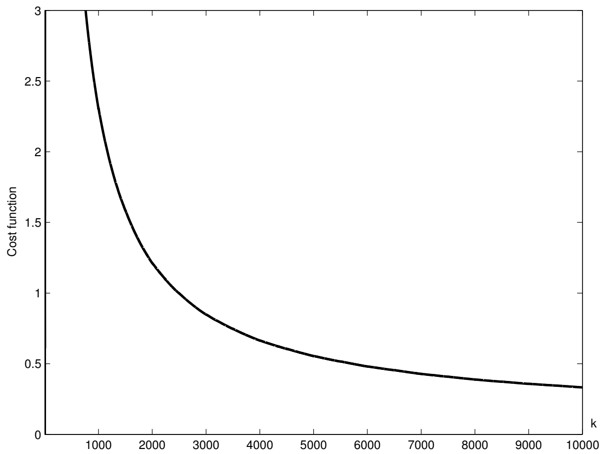

Fig. 4. Cost function of the adaptive tracker

Let $\{\epsilon_k\}$ be a sequence of i.i.d. random variables uniformly distributed over $[-0.2, 0.2]$ and let $\{\epsilon_k\}$ be independent of $\{\omega_k\}$. Let $\{u_k^{(c)}\}$ be defined by (14) and $\{u_k = u_k^{(c)} + v_k^{(d)} = u_k^{(c)} + \epsilon_k/k^{\frac{1}{30}}\}_{k\geq 1}$ serve as the system input. We can verify that **A7** also holds. Fig.2, Fig.3, and Fig.4 show the performance of the WLS-based tracker with $\{v_k^{(d)} = \epsilon_k/k^{\frac{1}{30}}\}$.

In Fig.2, the dotted line denotes the system output while the solid line the reference signal. In Fig.3, the dotted lines denote the estimates for $a_1,\ b_1,\ f_1,\ f_2$, and c_1, while the solid lines the values of the true parameters. Fig.4 shows the asymptotic properties of $\frac{1}{k}\sum_{i=1}^{k}(y_i - y_i^*)^2$ as time k increases.

The simulation results are consistent with the theoretical analysis.

5. CONCLUDING REMARKS

In this work, a WLS based adaptive tracker is designed for a class of Hammerstein systems, and its optimality is proved. By using the diminishing excitation technique, the strong consistency of the WLS estimate is achieved as well.

In the Hammerstein model under consideration, it is assumed that $f(x) = \sum_{j=1}^{s} f_j x^j$, where s is known and finite. But for a continuous function $f(x)$, by the Weierstrass approximation theorem $f(x) = \sum_{j=1}^{s} f_j x^j + h(x)$, where $h(x)$ is the approximation error and $h(x)$ is small if s is large enough. Hence, in general, the Hammerstein system is expressed as

$$A(z)y_{k+1} = B(z)\sum_{j=1}^{s} f_j u_k^j + C(z)\omega_{k+1} + B(z)h(u_k).$$

Adaptive control and identification for such kind of systems belong to further research.

It is also of interest to consider the robustness of the proposed adaptive tracker, more complex control performance indices, including the LQG problem and others.

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