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Constrained Control Allocation for Linear Systems with Internal Dynamics ⋆

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Abstract: This paper presents a new control allocation design method for overactuated linear systems with internal dynamics and input constraints. The control inputs are designed to implement constrained control allocation and guaranteed stability of the closed-loop system. An LMI-based sufficient condition is provided to solve the control allocation problem. The proposed approach is demonstrated by a linear tailless aircraft model.

1. INTRODUCTION

In the recent decade, control design methods such as model following (Durham et al., 1994), dynamic inversion (Enns et al., 1994) and backstepping (Harkegard and Glad, 2000) have gained increasing attention. These control methods specify the generalized control inputs, such as forces/moments, instead of the individual actuator deflections. For systems with redundant actuators, control allocation, which maps the generalized control inputs into the actual actuator deflections, becomes inevitable.

Various formulations of the control allocation problem have been discussed in the literature, and several survey papers have been written to discuss the strengths and weaknesses of the existing approaches (Bodson, 2002; Page and Steinberg, 2000). In the design of control allocation for systems with internal dynamics, control allocation may potentially destabilize the internal dynamic (Buffington and Enns, 1996). Since the control design must account for closed-loop system stability and therefore cannot tolerate instability of internal dynamics, stability of the internal dynamics has to be addressed in the design of control allocation. Unfortunately, most research papers on control allocation (Bolender and Doman, 2004; Bodson, 2002; Harkegard and Glad, 2000; Petersen and Bodson, 2006) do not involve systems with internal dynamics. Buffington, etc (Buffington et al., 1998) presented a small gain condition to ensure asymptotic stability of internal dynamics. However, this condition is conservative due to its reliance on the small gain theorem and can only be used to analyze the effect of control allocation on the internal dynamics stability, and not as a design method.

The main contribution of this paper is a sufficient condition in the form of linear matrix inequalities (LMIs) (Boyd et al., 1994) for solving the control inputs which are required to not only implement the function of control allocation but also guarantee the closed-loop stability. This condition provides a synthesis tool for constrained control allocation of linear systems with internal dynamics.

The rest of this paper is organized as follows. Section 2 gives the formulation of constrained control allocation problems. Section 3 presents a approach to solve the constrained control allocation problem. Section 4 use a tailless aircraft model to demonstrate the proposed approach. Some conclusions are given in Section 5.

2. PRELIMINARIES

Consider the overactuated linear system with internal dynamics in the following form:

$$\begin{bmatrix}\dot{\mathbf{z}}\\ \dot{\mathbf{x}}\end{bmatrix} = \begin{bmatrix}\mathbf{A_{zz}} & \mathbf{A_{zx}}\\ \mathbf{A_{xz}} & \mathbf{A_{xx}}\end{bmatrix}\begin{bmatrix}\mathbf{z}\\ \mathbf{x}\end{bmatrix} + \begin{bmatrix}\mathbf{B_z}\\ \mathbf{B_x}\end{bmatrix}\mathbf{u} \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^{n_x}$ is the commanded state vector, $\mathbf{z} \in \mathbf{R}^{n_z}$ is the internal state vector, and $\mathbf{u} \in \mathbf{R}^m$ with $m > n_x$ are the control inputs and constrained by $\mathbf{u} \in \mathbf{\Omega}$ with

$$\mathbf{\Omega} \equiv \{\mathbf{u} \in \mathbf{R}^m \mid |u_i| \le \bar{u}_i, i = 1, 2, \ldots, m\} \tag{2}$$

In flight control, Equation (1) corresponds to the linearized equations of motion at some trim conditions, e.g. wing-level horizontal flight, where the internal state vector $\mathbf{z}$ may comprise the velocity components u, v, w, the commanded state vector $\mathbf{x}$ may comprise the angular velocity components p, q, r, and control vector $\mathbf{u}$ may comprise the actuator positions. In the case of small conventional aircraft, the number of actuators m typically equals three,

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i.e. the elevator, aileron and rudder commands. For modern and unconventionally configured aircraft, m can be as large as 11 or more (Buffington, 1999).

One situation in which the control allocation problem arises is the employment of dynamics inversion/model-reference control laws, where desired dynamics are provided via a generalized control that must be resolved into the actual control commands.

In this paper, it is desired to design controller $\mathbf{u}$, admissible within given control limits, for a system of the form (1) such that its $\mathbf{x}$-state tracks some desired dynamics, and that the closed-loop system remains stable.

Generally the desired dynamics is represented by a stable reference model as follows:

$$\dot{\mathbf{x}} = \mathbf{A}_m\mathbf{x} + \mathbf{B}_m\mathbf{r}, \tag{3}$$

where $\mathbf{r} \in \mathbf{R}^{n_x}$ is a reference input vector and $\mathbf{A}_m$ is Hurwitz. Since the derivative of $\mathbf{x}$ is given by (1), model matching follows if

$$\mathbf{B_x u} = \mathbf{a}_d(\mathbf{z}, \mathbf{x}, \mathbf{r}) \tag{4}$$

where $\mathbf{a}_d$ is the reduced-dimension, generalized control vector represented by

$$\mathbf{a}_d(\mathbf{z}, \mathbf{x}, \mathbf{r}) = (\mathbf{A}_m - \mathbf{A_{xx}})\mathbf{x} - \mathbf{A_{xz}}\mathbf{z} + \mathbf{B}_m\mathbf{r} \tag{5}$$

It is noted that $\mathbf{a}_d \in \mathbf{R}^{n_x}$ has the same dimension as the commanded state vector $\mathbf{x}$.

Suppose that the control $\mathbf{u}$ is in the form of

$$\mathbf{u} = \mathbf{K}_s \begin{bmatrix} \mathbf{z} \\ \mathbf{x} \end{bmatrix} + \mathbf{K}_r\mathbf{r} \tag{6}$$

Then the equation (4) is satisfied if

$$\mathbf{B_x K}_s = \mathbf{N} \tag{7}$$

$$\mathbf{B_x K}_r = \mathbf{B}_m \tag{8}$$

where

$$\mathbf{N} = [-\mathbf{A_{xz}} \quad (\mathbf{A}_m - \mathbf{A_{xx}})] \tag{9}$$

Obviously, the $\mathbf{x}$-state is driven to the desired dynamics by the control $\mathbf{u}$ as in (6) with $\mathbf{K}_s$ and $\mathbf{K}_r$ satisfying (7) and (8). The objective of control allocation is to minimize the error between $\mathbf{B_x u}$ and $\mathbf{a}_d(\mathbf{z}, \mathbf{x}, \mathbf{r})$ as in (4), as well as the control power. In other words, it is to minimize the cost function

$$J_c = J_1 + J_2 + J_3 + J_4 \tag{10}$$

where

$$J_1 = trace\left((\mathbf{B_x K}_s - \mathbf{N})^T \mathbf{H}_1 (\mathbf{B_x K}_s - \mathbf{N})\right) \tag{11}$$

$$J_2 = trace\left((\mathbf{B_x K}_r - \mathbf{B}_m)^T \mathbf{H}_2 (\mathbf{B_x K}_r - \mathbf{B}_m)\right) \tag{12}$$

$$J_3 = trace\left(\mathbf{K}_s^T \mathbf{H}_3 \mathbf{K}_s\right) \tag{13}$$

$$J_4 = trace\left(\mathbf{K}_r^T \mathbf{H}_4 \mathbf{K}_r\right) \tag{14}$$

subject to the control constraints $\mathbf{u} \in \mathbf{\Omega}$ and closed-loop stability. Here $\mathbf{H}_1 > 0$, $\mathbf{H}_2 > 0$, $\mathbf{H}_3 \geq 0$ and $\mathbf{H}_4 \geq 0$ are weighting matrices. J_1 and J_2 are together to minimize the error between $\mathbf{B_x u}$ and $\mathbf{a}_d(\mathbf{z}, \mathbf{x}, \mathbf{r})$. J_3 and J_4 are together to minimize the control power. Since J_1 and J_2 are main optimization objectives, while J_3 and J_4 are secondary optimization objectives, usually we choose the values of $\mathbf{H}_3$ and $\mathbf{H}_4$ far smaller than those of $\mathbf{H}_1$ and $\mathbf{H}_2$.

Now we consider the closed-loop stability of the system (1). Denote

$$\xi = \begin{bmatrix} \mathbf{z} \\ \mathbf{x} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A_{zz}} & \mathbf{A_{zx}} \\ \mathbf{A_{xz}} & \mathbf{A_{xx}} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B_z} \\ \mathbf{B_y} \end{bmatrix} \tag{15}$$

Substituting (6) into (1), we have the closed-loop system of the system (1) as

$$\dot{\xi} = (\mathbf{A} + \mathbf{BK}_s)\,\xi + \mathbf{BK}_r\mathbf{r} \tag{16}$$

It is well known that the system (16) is stable if $(\mathbf{A} + \mathbf{BK}_s)$ is Hurwitz.

In the following, we consider the control constraint $\mathbf{u} \in \mathbf{\Omega}$ in the design of the controller $\mathbf{u}$. Since unstable systems regulated by a constrained controller cannot be stabilized for all initial conditions, we introduce an invariant set as follows.

Lemma 1. (Blanchini, 1999) For a positive scalar ρ, the set

$$\mathbf{\Pi} \equiv \left\{\xi : \xi^T \mathbf{P} \xi < \rho\right\} \tag{17}$$

is positively invariant with respect to the system (16) where $\mathbf{r} \in \mathbf{\Phi}$ with

$$\mathbf{\Phi} \equiv \{\mathbf{r} : \mathbf{r}^T \mathbf{S} \mathbf{r} < \rho\} \tag{18}$$

if there exist positive-definite matrices $\mathbf{P} = \mathbf{P}^T > 0$ and $\mathbf{S} = \mathbf{S}^T > 0$ and a positive scalar α such that

$$\begin{bmatrix} \mathbf{P}(\mathbf{A}+\mathbf{BK}_s)+(\mathbf{A}+\mathbf{BK}_s)^T\mathbf{P}+\alpha\mathbf{P} & \mathbf{PBK}_r \\ \mathbf{K}_r^T\mathbf{B}^T\mathbf{P} & -\alpha\mathbf{S} \end{bmatrix} < 0 \tag{19}$$

Proof: Denote the Lyapunov function with respect to the system (16) as

$$V = \xi^T \mathbf{P} \xi \tag{20}$$

Then its derivative is given by

$$\begin{aligned} \dot{V} &= \xi^T \left[\mathbf{P}\left(\mathbf{A} + \mathbf{BK}_s\right) + \left(\mathbf{A} + \mathbf{BK}_s\right)^T \mathbf{P}\right] \xi \\ &\quad + \xi^T \mathbf{PBK}_r \mathbf{r} + \mathbf{r}^T \mathbf{K}_r^T \mathbf{B}^T \mathbf{P} \xi \\ &= \begin{bmatrix} \xi \\ \mathbf{r} \end{bmatrix}^T \begin{bmatrix} \mathbf{P}(\mathbf{A}+\mathbf{BK}_s) + (\mathbf{A}+\mathbf{BK}_s)^T\mathbf{P} & \mathbf{PBK}_r \\ \mathbf{K}_r^T\mathbf{B}^T\mathbf{P} & -\alpha\mathbf{S} \end{bmatrix} \begin{bmatrix} \xi \\ \mathbf{r} \end{bmatrix} + \alpha \mathbf{r}^T \mathbf{S} \mathbf{r} \end{aligned} \tag{21}$$

Substituting (19) into (21), we have

$$\dot{V} < -\alpha\xi^T \mathbf{P} \xi + \alpha \mathbf{r}^T \mathbf{S} \mathbf{r} \tag{22}$$

Since $\mathbf{r} \in \mathbf{\Phi}$, we obtain that

$$\dot{V} < -\alpha\xi^T \mathbf{P} \xi + \alpha\rho \tag{23}$$

If $V = \xi^T \mathbf{P} \xi \geq \rho$, from (23), we have $\dot{V} < 0$. Hence the state ξ converges to the positively invariant set $\mathbf{\Pi}$. □

Throughout this paper, we assume that

A1: $[\mathbf{A}, \mathbf{B}]$ is stabilizable.
A2: $\mathbf{A}_m$ is Hurwitz.

3. MAIN RESULTS

Theorem 1. Consider the system (16). For a given positive scalar ρ, if there exist symmetric positive-definite matrices $\mathbf{Y}$ and $\mathbf{S}$, matrices $\mathbf{Z}$ and $\mathbf{K}_r$, and a positive scalar α such that the following conditions are satisfied.

$$\begin{bmatrix} \mathbf{AY} + \mathbf{YA}^T + \mathbf{BZ} + \mathbf{Z}^T\mathbf{B}^T + \alpha\mathbf{Y} & \mathbf{BK}_r \\ \mathbf{K}_r^T\mathbf{B}^T & -\alpha\mathbf{S} \end{bmatrix} < 0 \quad (24)$$

$$\begin{bmatrix} \frac{\bar{u}_i^2}{2\rho} & \mathbf{E}_i\mathbf{Z} & \mathbf{E}_i\mathbf{K}_r \\ \mathbf{Z}^T\mathbf{E}_i^T & \mathbf{Y} & 0 \\ \mathbf{K}_r^T\mathbf{E}_i^T & 0 & \mathbf{S} \end{bmatrix} > 0 \quad (25)$$

$$i = 1, 2, \cdots, m$$

where $\mathbf{E}_i \in \mathbf{R}^m$ is a row vector with the ith element being 1 and the others being zero, then the control input

$$\mathbf{u} = \mathbf{K}_s\xi + \mathbf{K}_r\mathbf{r} \quad (26)$$

with

$$\mathbf{K}_s = \mathbf{ZP}, \quad \mathbf{P} = \mathbf{Y}^{-1} \quad (27)$$

can stabilize the closed-loop system (16) and satisfy the control constraint $\mathbf{u} \in \mathbf{\Omega}$ for any initial state $\xi(0) \in \mathbf{\Pi}$ as in (17) and any reference $\mathbf{r} \in \mathbf{\Phi}$ as in (18).

Proof: By substituting (27) into (24) and making congruence transformation by multiplying $diag(\mathbf{P}, \mathbf{I})$ on its left and right sides, we prove that (24) is equivalent to (19). From the inequality (19), we have

$$\mathbf{P}(\mathbf{A} + \mathbf{BK}_s) + (\mathbf{A} + \mathbf{BK}_s)^T\mathbf{P} + \alpha\mathbf{P} < 0 \quad (28)$$

Since $\alpha > 0$ and $\mathbf{P} > 0$, we have

$$\mathbf{P}(\mathbf{A} + \mathbf{BK}_s) + (\mathbf{A} + \mathbf{BK}_s)^T\mathbf{P} < 0 \quad (29)$$

Obviously, the matrix $\mathbf{A} + \mathbf{BK}_s$ is Hurwitz. Hence, the control as in (26) can stabilize the closed-loop system (16).

As (24) is equivalent to (19), according to Lemma 1, the set $\mathbf{\Pi}$ is a positively invariant set with respect to the closed-loop system (16) for $\mathbf{r} \in \mathbf{\Phi}$. Hence, for any initial state $\xi(0) \in \mathbf{\Pi}$, we have

$$\xi(t)^T\mathbf{P}\xi(t) < \rho, \quad t \geq 0 \quad (30)$$

Since $\mathbf{r} \in \mathbf{\Phi}$, i.e.,

$$\mathbf{r}^T\mathbf{Sr} < \rho \quad (31)$$

we obtain that

$$\xi^T\mathbf{P}\xi + \mathbf{r}^T\mathbf{Sr} < 2\rho \quad (32)$$

This can be rewritten as

$$\begin{bmatrix} \xi \\ \mathbf{r} \end{bmatrix}^T \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{S} \end{bmatrix} \begin{bmatrix} \xi \\ \mathbf{r} \end{bmatrix} < 2\rho \quad (33)$$

which is equivalent to

$$\begin{bmatrix} \xi \\ \mathbf{r} \end{bmatrix} \begin{bmatrix} \xi \\ \mathbf{r} \end{bmatrix}^T < 2\rho \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{S}^{-1} \end{bmatrix} \quad (34)$$

From (26), we have

$$\begin{aligned} u_i^2 &= \mathbf{E}_i(\mathbf{K}_s\xi + \mathbf{K}_r\mathbf{r})(\mathbf{K}_s\xi + \mathbf{K}_r\mathbf{r})^T\mathbf{E}_i^T \\ &= \mathbf{E}_i[\mathbf{K}_s \;\; \mathbf{K}_r] \begin{bmatrix} \xi \\ \mathbf{r} \end{bmatrix} \begin{bmatrix} \xi \\ \mathbf{r} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_s^T \\ \mathbf{K}_r^T \end{bmatrix} \mathbf{E}_i^T \\ &< 2\rho\mathbf{E}_i[\mathbf{K}_s \;\; \mathbf{K}_r] \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{K}_s^T \\ \mathbf{K}_r^T \end{bmatrix} \mathbf{E}_i^T \\ &= 2\rho\mathbf{E}_i[\mathbf{Z} \;\; \mathbf{K}_r] \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{Z}^T \\ \mathbf{K}_r^T \end{bmatrix} \mathbf{E}_i^T \end{aligned}$$

By Schur complement formula, the inequalities (25) is equivalent to

$$\bar{u}_i^2 > 2\rho\mathbf{E}_i[\mathbf{Z} \;\; \mathbf{K}_r] \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{Z}^T \\ \mathbf{K}_r^T \end{bmatrix} \mathbf{E}_i^T$$

$$i = 1, 2, \cdots, m \quad (35)$$

Thus we have

$$u_i^2 < \bar{u}_i^2, \quad i = 1, 2, \cdots, m \quad (36)$$

namely, $\mathbf{u} \in \mathbf{\Omega}$ for $\xi(0) \in \mathbf{\Pi}$ and $\mathbf{r} \in \mathbf{\Phi}$. This completes the proof. □

Remark 1. Theorem 1 gives a sufficient condition for solving the constrained controller $\mathbf{u}$. The controller $\mathbf{u}$ consists of two parts: the part of state feedback is used to stabilize the closed-loop system while the part with forward reference signal is used to implement tracking of reference signal.

As (24) involves the items $\alpha\mathbf{Y}$ and $\alpha\mathbf{S}$ where α, $\mathbf{Y}$ and $\mathbf{S}$ are variables, it is not a regular LMI. However, if α is given, (24) is an LMI. The optimization of α will be presented subsequently.

Once the closed-loop stability and control constraints are guaranteed, the control allocation becomes a constrained optimization problem: for a given positive scalar α,

$$\min_{\mathbf{Y},\mathbf{S},\mathbf{Z},\mathbf{K}_r,\mathbf{W}_1,\mathbf{W}_3,\mathbf{Q}_i,i=1,2,3,4,5} \gamma$$

subject to the LMIs (24), (25) and

$$\sum_{i=1}^{4} trace(\mathbf{Q}_i) + \lambda_1 trace(\mathbf{Q}_5) + \lambda_2 trace(\mathbf{S}) < \gamma \quad (37)$$

$$\begin{bmatrix} \mathbf{W}_1 & (\mathbf{B_x Z} - \mathbf{NY})^T\mathbf{H}_1^{\frac{1}{2}} \\ \mathbf{H}_1^{\frac{1}{2}}(\mathbf{B_x Z} - \mathbf{NY}) & \mathbf{I} \end{bmatrix} > 0 \quad (38)$$

$$\begin{bmatrix} 2\mathbf{Y} - \mathbf{W}_1 & \mathbf{I} \\ \mathbf{I} & \mathbf{Q}_1 \end{bmatrix} > 0 \quad (39)$$

$$\begin{bmatrix} \mathbf{Q}_2 & (\mathbf{B_x K}_r - \mathbf{B}_m)^T\mathbf{H}_2^{\frac{1}{2}} \\ \mathbf{H}_2^{\frac{1}{2}}(\mathbf{B_x K}_r - \mathbf{B}_m) & \mathbf{I} \end{bmatrix} > 0 \quad (40)$$

$$\begin{bmatrix} \mathbf{W}_3 & \mathbf{Z}^T\mathbf{H}_3^{\frac{1}{2}} \\ \mathbf{H}_3^{\frac{1}{2}}\mathbf{Z} & \mathbf{I} \end{bmatrix} > 0 \quad (41)$$

$$\begin{bmatrix} 2\mathbf{Y} - \mathbf{W}_3 & \mathbf{I} \\ \mathbf{I} & \mathbf{Q}_3 \end{bmatrix} > 0 \quad (42)$$

$$\begin{bmatrix} \mathbf{Q}_4 & \mathbf{K}_r^T \mathbf{H}_4^{\frac{1}{2}} \\ \mathbf{H}_4^{\frac{1}{2}} \mathbf{K}_r & \mathbf{I} \end{bmatrix} > 0 \quad (43)$$

$$\begin{bmatrix} \mathbf{Q}_5 & \mathbf{I} \\ \mathbf{I} & \mathbf{Y} \end{bmatrix} > 0 \quad (44)$$

where $\mathbf{N}$ is defined as in (9), $\mathbf{H}_1 > 0$, $\mathbf{H}_2 > 0$, $\mathbf{H}_3 \geq 0$ and $\mathbf{H}_4 \geq 0$ are known weighting matrices, $\lambda_1 > 0$ and $\lambda_2 > 0$ are known weighting scalars, and $\rho > 0$ is a given scalar. Then we have $\mathbf{K}_s = \mathbf{ZP}$ and $\mathbf{P} = \mathbf{Y}^{-1}$ as in (27).

In the above LMI constraints, (38) and (39) are used to find the upper bound of J_1 as in (11). By Schur complement formula, (38) is equivalent to

$$\mathbf{W}_1 > \mathbf{Y}(\mathbf{B_x K}_s - \mathbf{N})^T \mathbf{H}_1 (\mathbf{B_x K}_s - \mathbf{N})\mathbf{Y} \quad (45)$$

and (39) is equivalent to

$$2\mathbf{Y} - \mathbf{Q}_1^{-1} > \mathbf{W}_1 \quad (46)$$

Since $(\mathbf{Q}_1^{-1} - \mathbf{Y})\mathbf{Q}_1(\mathbf{Q}_1^{-1} - \mathbf{Y}) > 0$, it is obvious that

$$\mathbf{YQ}_1\mathbf{Y} > 2\mathbf{Y} - \mathbf{Q}_1^{-1} \quad (47)$$

From (45)-(47), we have $\mathbf{Q}_1 > (\mathbf{B_x K}_s - \mathbf{N})^T \mathbf{H}_1 (\mathbf{B_x K}_s - \mathbf{N})$. Hence $J_1 < trace(\mathbf{Q}_1)$. Similarly, (41) and (42) are used to find the upper bound of J_3 as in (13).

Remark 2. The purpose of minimizing $trace(\mathbf{Q}_1)$, $trace(\mathbf{Q}_2)$, $trace(\mathbf{Q}_3)$ and $trace(\mathbf{Q}_4)$ is to minimize J_1, J_2, J_3 and J_4, respectively. While the purpose of minimizing $trace(\mathbf{Q}_5)$ is to minimize trace($\mathbf{P}$) such that the set $\mathbf{\Pi}$ as in (17) is as large as possible for a positive constant ρ. Similarly, $trace(\mathbf{S})$ is minimized to make the set $\mathbf{\Phi}$ as in (18) as large as possible for a positive constant ρ.

Remark 3. The controller $\mathbf{u} = \mathbf{K}_s\xi + \mathbf{K}_r\mathbf{r}$ solved from the proposed off-line optimization approach can implement the control allocation and guarantee the closed-loop stability subject to given control constraints.

In the design of control allocation, to determine the positive optimal scalar α, an iterative method is proposed:

Step 1: Give the initial value of ν and the maximum number of iteration n.
Step 2: At jth iteration, set $\alpha = Tj$ where T is a given small constant.
Step 3: Minimize $\gamma^{(j)}$ over $\mathbf{Y}^{(j)}$, $\mathbf{S}^{(j)}$, $\mathbf{Z}^{(j)}$, $\mathbf{K}_r^{(j)}$, $\mathbf{W}_1^{(j)}$, $\mathbf{W}_3^{(j)}$ and $\mathbf{Q}_i^{(j)}$ $(i = 1, 2, 3, 4, 5)$ subject to LMIs (24), (25) and (37)-(44).
Step 4: If $\gamma^{(j)} < \nu$, set $\nu = \gamma^{(j)}$ and $L = j$.
Step 5: If $j = n$, we have $\alpha_{opt} = TL$ and stop. Otherwise, $j = j + 1$ and go to Step 2.

Once α_{opt} is obtained, by minimizing γ subject to LMIs (24), (25), (37)-(44), we can obtain $\mathbf{K}_r$, $\mathbf{K}_s$ and $\mathbf{P}$.

4. EXAMPLE

In this section, we use a modified tailless aircraft model (Buffington and Enns, 1996) to demonstrate the proposed control allocation approach. The linear approximation of the lateral/directional dynamics of the tailless aircraft is given as follows.

$$\begin{bmatrix} \dot\beta \\ \hline \dot p \\ \dot r \end{bmatrix} = \left[\begin{array}{c|cc} 0.1 & 0.154 & -0.988 \\ \hline -8.210 & -0.785 & 0.117 \\ -0.889 & -0.030 & -0.016 \end{array}\right] \begin{bmatrix} \beta \\ \hline p \\ r \end{bmatrix}$$

$$+ \left[\begin{array}{ccccc} 0.006 & -0.004 & -0.009 & 0.013 & 0.011 \\ \hline 3.650 & -0.614 & 7.570 & -4.970 & 0.079 \\ -0.416 & 0.188 & 0.091 & -0.181 & -0.804 \end{array}\right] \begin{bmatrix} \delta_t \\ \delta_f \\ \delta_e \\ \delta_s \\ \delta_{ytv} \end{bmatrix}$$

Here the internal state is the side-slip angle β expressed in degree, the commanded states are the body-axis roll rate p and yaw rate r expressed in degree per second. The control input consists of the differential all moving tips δ_t, differential outboard leading-edge flaps δ_f, differential elevons δ_e, differential spoiler/slot-deflectors δ_s and yaw thrust vectoring δ_{ytv}. They are all expressed in degrees and satisfy the following constraints:

$$|\delta_e| \leq 20, |\delta_s| \leq 20, |\delta_t| \leq 25, |\delta_{ytv}| \leq 30, |\delta_f| \leq 30 \quad (48)$$

It is noted that there is an unstable eigenvalue 0.4709 in the open-loop tailless aircraft system. In this example, it is desired to command the roll rate p and yaw rate r. The desired closed-loop dynamics is given by

$$\begin{bmatrix} \dot p_c \\ \dot r_c \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_c \\ r_c \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_p \\ r_r \end{bmatrix} \quad (49)$$

Set $\rho = 1$, $\lambda_1 = 5.0 \times 10^{-5}$, $\lambda_2 = 0.25$, $\mathbf{H}_1 = \mathbf{I}_{2\times 2}$, $\mathbf{H}_2 = 20\mathbf{I}_{2\times 2}$ and $\mathbf{H}_3 = \mathbf{H}_4 = 0$. Using the optimization approach proposed in Section 3, we obtain $\alpha = 0.14$ and

$$\mathbf{K}_s = \begin{bmatrix} -2.1709 & -0.3204 & 2.0264 \\ 2.6370 & 0.3792 & -2.4077 \\ 1.1551 & -0.0072 & -0.1500 \\ -1.7916 & -0.2454 & 1.5670 \\ 0.9044 & 0.2353 & -0.5276 \end{bmatrix}$$

$$\mathbf{K}_r = \begin{bmatrix} -1.8959 & 12.3191 \\ 2.2595 & -14.6438 \\ 0.2360 & -0.7050 \\ -1.4835 & 9.5714 \\ 1.8677 & -13.2619 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0.0318 & 0.0045 & -0.0287 \\ 0.0045 & 0.0010 & -0.0051 \\ -0.0287 & -0.0051 & 0.0314 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 0.0233 & -0.1432 \\ -0.1432 & 0.9292 \end{bmatrix}$$

Then the closed-loop tailless aircraft system are

$$\begin{bmatrix} \dot\beta \\ \hline \dot p \\ \dot r \end{bmatrix} = \left[\begin{array}{c|cc} 0.0527 & 0.1500 & -0.9503 \\ \hline -0.0328 & -1.0034 & 0.0271 \\ 0.2122 & 0.0291 & -1.1848 \end{array}\right] \begin{bmatrix} \beta \\ \hline p \\ r \end{bmatrix}$$

$$+ \left[\begin{array}{cc} -0.0213 & 0.1174 \\ \hline 0.9997 & 0.0018 \\ 0.0018 & 0.9883 \end{array}\right] \begin{bmatrix} r_p \\ r_r \end{bmatrix}$$

with eigenvalues of -0.1452, -0.9911 and -0.9992. Thus the control $\mathbf{u} = \mathbf{K}_s\xi + \mathbf{K}_r\mathbf{r}$ with $\mathbf{u} = [\delta_t, \delta_f, \delta_e, \delta_s, \delta_{ytv}]^T$, $\xi = [\beta, p, r]^T$ and $\mathbf{r} = [r_p, r_r]^T$ can stabilize the closed-loop tailless aircraft system and satisfy the control constraint (48) for any initial state $\xi(0) \in \mathbf{\Pi}$ and the reference $\mathbf{r} \in \mathbf{\Phi}$.

Set the initial states $\beta(0) = 1$, $p(0) = 0$ and $r(0) = 0$ and the reference

$$r_p = \begin{cases} 0, \ 0 \le t < 5 \text{ sec} \\ 15, \ 5 \le t < 15 \text{ sec} \\ -15, \ 15 \le t < 25 \text{ sec} \\ 0, \ 25 \text{ sec} \le t \end{cases}$$

$$r_r = \begin{cases} 0, \ 0 \le t < 5 \text{ sec} \\ 2.3, \ 5 \le t < 15 \text{ sec} \\ -2.3, \ 15 \le t < 25 \text{ sec} \\ 0, \ 25 \text{ sec} \le t \end{cases}$$

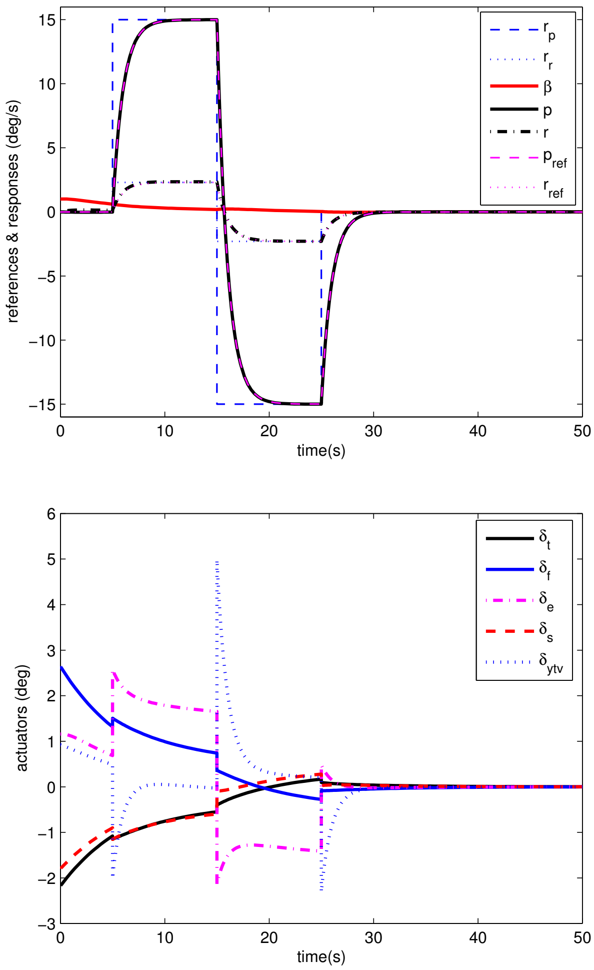

Fig. 1. Simulation results of tailless aircraft with consideration of internal dynamics stability

The simulation results are given in Figure 1 which shows that the proposed control allocation approach can effectively match the reference model (49) and stabilize the sideslip angle β.

For the purpose of comparison, Figure 2 gives the simulation result of the general on-line control allocation (Petersen and Bodson, 2006) where the stability of the internal dynamics is not considered. From Figure 2, it is observed that at the beginning the reference model (49) can still be effectively tracked although the sideslip angle β is divergent. However, when β is large enough, it makes the actuators saturated. As a result, the responses of p and r cannot track p_{ref} and r_{ref} which are responses of the reference model (49) and p and r begin to diverge.

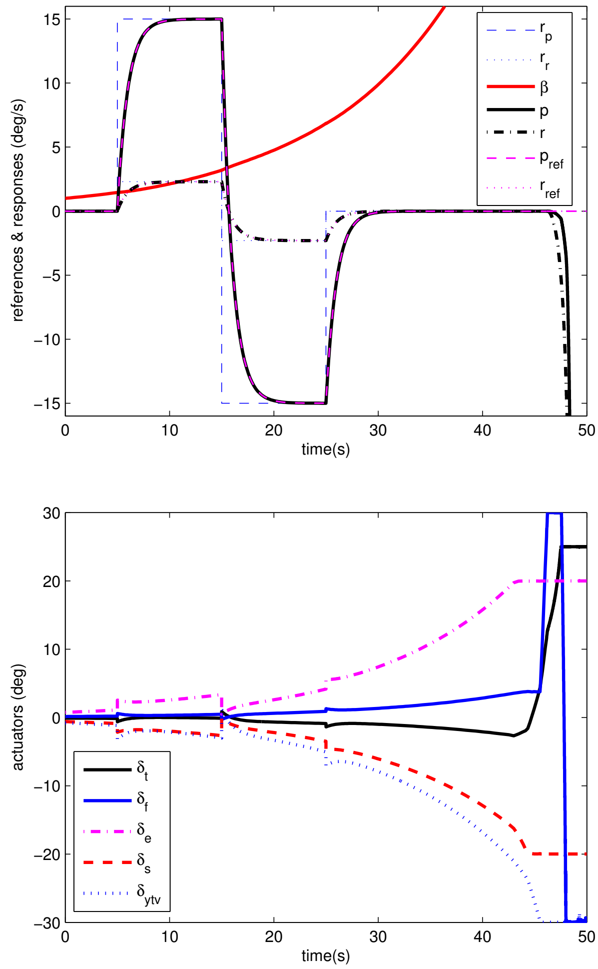

Fig. 2. Simulation results of tailless aircraft without consideration of internal dynamics stability

Although it is easy to use robust optimization control (Zhou and Doyle, 1998) approaches to stabilize an unstable system and optimize its tracking performance where control input can be constrained within given control limits by adjusting weighting matrices, it is not easy to achieve the desired dynamics of the closed-loop. This is verified by Figure 3 that shows the responses of the tailless aircraft using H_2 optimal control with the state-feedback and the integral of tracking errors $\mathbf{e} = \mathbf{r} - \mathbf{x}$. From Figure 3, it is observed that p and r can track p_{ref} and r_{ref} with zeros steady-state tracking error, the response of β is stable, and all the actuator defelections satisfy the constraints (48). However, it is also observed that the dynamics of p and r cannot match those of p_{ref} and r_{ref}.

5. CONCLUSIONS

This paper presents a new control allocation method for linear systems with internal dynamics. In this method, closed-loop systems are stabilized by state-feedback control, and positively ellipsoidal invariant sets are used to guarantee that the specified control constraints be satisfied within a range of references and initial states. An LMI-based off-line optimization approach is then proposed to solve for a fixed controller which achieves the desired dynamics. Simulation results show that the proposed method

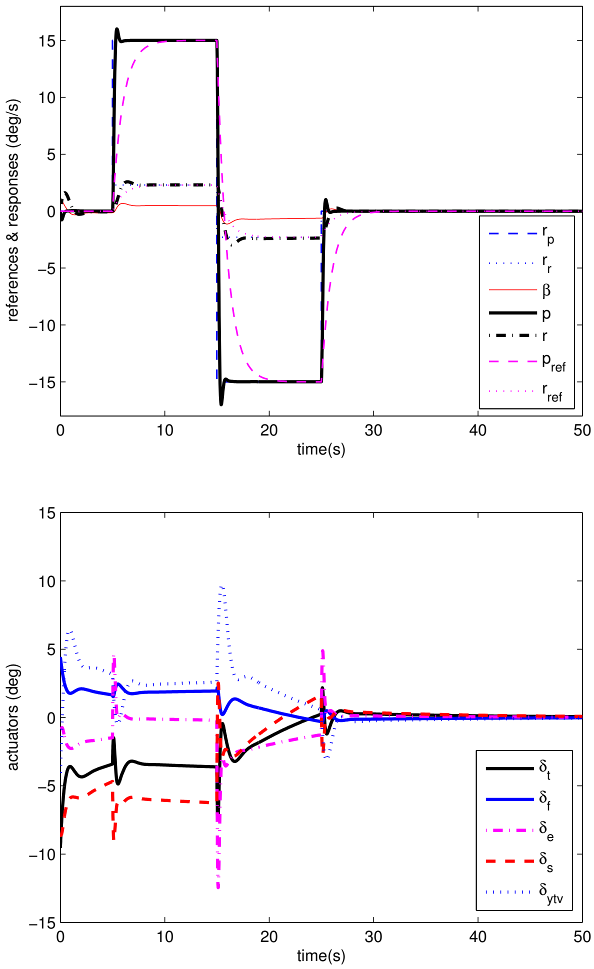

Fig. 3. Simulation results of tailless aircraft based on H_2 optimal control

is better than the existing control allocation approach where the closed-loop stability is not considered. Compared to the robust optimization approach, the proposed approach can achieve the desired dynamics of closed-loop system more easily.

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