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Risk-sensitivity Conditions for Stochastic Uncertain Model Validation ⋆

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Abstract: The paper presents sufficient and (under an additional technical assumption) necessary conditions that verify the relevance of given input and output processes to an assumed stochastic uncertain system model subject to an uncertainty constraint. The approach is to establish the existence of an admissible probability model under which dynamics of the proposed stochastic system model are consistent with the given input and measurement processes.

Keywords: Stochastic systems, Model validation, Input-output realizations, Uncertainty modeling, Uncertain systems, Relative entropy.

1. INTRODUCTION

In this paper, we establish conditions under which a given pair of processes can be realized as an input and a noisy output of a stochastic linear control system subject to uncertain perturbations. The class of stochastic disturbances under consideration is based on an interpretation of the system uncertainty as resulting from perturbations of reference Brownian motions; this allows one to account for a rich class of system uncertainties including some standard uncertainty models such as norm-bounded parametric disturbances and H_∞-norm bounded unmodeled dynamics [Ugrinovskii and Petersen, 2001, Petersen et al., 2000b]. Also, the uncertainty affecting probability laws of the system noises is directly included in this model, whereby uncertain systems driven by non-Gaussian noises can be accounted for. This approach to stochastic uncertainty modeling has been employed in a number of recent papers on robust stochastic control and filtering; e.g., see [Petersen et al., 2000a,b, Charalambous and Rezaei, 2007, Yoon et al., 2004]. The objective of this paper is to complement this recent stochastic robust control and robust filtering theory by addressing issues of model validation in relation to the stochastic uncertainty description used in the mentioned work.

The importance of model uncertainty in the derivation of a control system model consistent with available input and measurement data has been highlighted some time ago. While in the system identification theory the mismatch between the modeled and observed system outputs is often attributed to noise, or excitation (e.g., see Ljung and Soderstrom [1983]), it was argued in [Poolla et al., 1994, Smith and Doyle, 1992] and a number of other papers that in feedback control systems other types of uncertainty such as, e.g., unmodeled dynamics, coprime factor uncertainty or norm-bounded disturbances, must also be accounted for. This argument motivated a substantial research effort in the area of *deterministic* uncertain systems model validation; e.g., in addition to the above references see [Savkin and Petersen, 1996]. The deterministic model validation problem was formulated in these references as follows: Given an uncertain system model (consisting of a nominal model and a set of uncertain perturbations) and an experimental datum, does the uncertainty set include an uncertainty element such that the observed datum can be produced exactly (cf. Problem 3.1 in Smith and Doyle [1992])? The solution to this problem, as noted in Poolla et al. [1994], Smith and Doyle [1992], is instrumental for eliminating models which are inconsistent with observed measurements.

In stochastic systems, effects of process noise may further augment those attributed to the system uncertainty. While a common description of stochastic systems employs the classic Gaussian excitation model to describe dynamics which are difficult to predict accurately, in applications arising in the areas of hybrid systems control, mathematical finance and communications, it is often necessary to consider dynamics driven by non-Gaussian noises or noises whose probability distributions are state-dependent and/or affected by the system uncertainty. One approach to dealing with control problems for this type of uncertain systems, which has recently attracted attention (e.g., see Petersen et al. [2000a,b], Charalambous and Rezaei [2007] and references therein), is based on an interpretation of uncertain system noises as resulting from uncertain perturbations of reference Brownian motions. Thanks to its connection to the H_∞ control and risk-sensitive control theories (Fleming and McEneaney [1995], James et al. [1994], Dai Pra et al. [1996]), this approach has been shown to enable tractable solutions to a number of robust control and filtering problems involving uncertain stochastic systems. Aspects of stochastic system modeling however have not been addressed in this theory in a systematic manner. In this paper we complement the mentioned theory of stochastic robust control by proposing a formulation of the model validation problem which takes into account the presence of uncertainty in the system model and also accounts for a possibly non-Gaussian nature of excitations. Our aim is to pave the way for future research into issues of stochastic model reduction and stochastic systems realization for this class of stochastic uncertain systems, which would parallel the deter-

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ministic theory concerned with issues of system equivalence and system realizations [Beck et al., 1996, Petersen, 2007].

The model validation problem in this paper is concerned with establishing conditions under which given processes $u, \psi, \bar{y}$ (where $\bar{y}$ is a noisy version of ψ, as will be explained below) can be realized as an input and output of a stochastic linear control system subject to uncertain perturbations from a prescribed set. Specifically, consider a.s. bounded random signals $\psi(\cdot)$, $u(\cdot)$ adapted to the filtration generated by an exogenous process disturbance $w(\cdot)$ (we postpone rigorous definitions until Section 2). These processes will play the role of the input and 'perfect' output of a plant. In practice, it may be too expensive and difficult to measure the plant directly, and often cheaper but more noisy sensors are preferred, even though they provide measurements with a larger error. To reflect such a practical situation we augment our input-output pair by an Ito process $\bar{y}$

$$d\bar{y}(t) = \psi(t)dt + d\varrho(t), \quad \bar{y}(0) = 0; \qquad (1)$$

ϱ is a Wiener process independent of w, with a known increment covariance matrix $\Sigma(\cdot)$, which represents measurement noise. Since the probability law of ϱ is fixed, equation (1) imposes a coupling between $\psi(\cdot)$ and $\bar{y}(\cdot)$.

We are concerned with establishing conditions under which the processes u, ψ, $\bar{y}$ can be represented by a stochastic linear uncertain system driven by the process noise w and the control input u and whose linear noisy output has the same probability law as that of $\bar{y}$. Such a representation would allow the designer to apply the recent stochastic robust control design methodology mentioned above (e.g., [Petersen et al., 2000b]), much of which is based on linear stochastic modeling of a system.

Motivated by this idea, we focus on admissible candidate linear models described by the linear stochastic differential equation

$$\begin{aligned} dx(t) &= (A(t)x(t) + B_1(t)u(t))dt + B_2(t)dw(t), \qquad (2)\\ dy(t) &= (C_2(t)x(t) + dv(t), \quad y(0) = 0, \quad x(0) = 0. \end{aligned}$$

A tacit feature of this problem formulation relates to the assumption that the probability space in which the above system is defined is not fixed. As will be explained below, this assumption allows one to consider a rich class of uncertain dynamics by describing them as the stochastic system (2) driven by perturbed process and measurement noises. This will enable us to establish the desired representation of the processes $u, \psi, \bar{y}$ by perturbing a nominal system of the form (2); such a nominal system corresponds to the case where w and v are Brownian motions. In this sense, our problem formulation is analogous to the deterministic formulation of the uncertain model validation problem considered by Poolla et al. [1994], Savkin and Petersen [1996], Smith and Doyle [1992], the main difference being that our approach directly accounts for stochastic perturbations and noises in the system. The set of admissible perturbations of the nominal system will be denoted Ξ_d; this set will be defined rigorously in Section 2.2.

The main result of the paper presented in Section 4 is a condition which guarantees the existence of an admissible probability measure from the given class Ξ_d under which $\bar{y}(\cdot) = y(\cdot)$ in the sense of probability laws, i.e., weakly. The condition, which is given in Theorem 2, is formulated in terms of a risk-sensitive performance cost defined on dynamics of the nominal system (2). The condition is sufficient, and, under some additional technical assumption, necessary. In somewhat lose terms, this risk-sensitive performance cost characterizes performance exhibited by this special system (2) when it 'attempts to track' ψ using the control input u. Our result in Theorem 2 compares this cost with a given bound on the energy in admissible disturbances. Such a bound is one of the parameters that define the class Ξ_d of admissible disturbances. The significance of this result is seen in connecting the realizability of the given processes $(u, \psi, \bar{y})$ with risk-sensitive performance of the reference system. In a number of related robust control problems involving a similar class of uncertain stochastic systems it was possible to evaluate corresponding risk-sensitive performance costs. Therefore the result of Theorem 2 sets a direction for future research into more tractable conditions for realizability of stochastic systems.

To conclude this section, we present a feedback control system setting which elucidates the assumptions concerning $(u, \psi, \bar{y})$ which will be used in this paper. Consider a stochastic open-loop control system

$$\bar{x}(t) = \mathcal{A}(u(\cdot)|_0^t, w(\cdot)|_0^t), \quad t \in [0, T], \qquad (3)$$

consisting of a causal mapping $\mathcal{A}\colon w(\cdot) \times u(\cdot) \to \bar{x}$ and a measurement equation (1) in which $\psi(t) \triangleq \Psi(\bar{x}(t))$. In (3), $w(\cdot)$ represents sample trajectories of an exogenous disturbance. The required properties of u and ψ can then be ascertained if we assume that $u(t)$ is an exogenous deterministic test signal and $\mathcal{A}$ and Ψ have appropriate measurability properties. E.g., when the mapping $\mathcal{A}$ is described by an Ito differential equation and $w(\cdot)$ is a semimartingale, assumptions which establish the existence and uniqueness of a solution $\bar{x}$ suffice. For a general discussion, we refer to [Ugrinovskii and Petersen, 2000].

2. DEFINITIONS

2.1 Uncertain stochastic systems

Let $T > 0$ be a constant which will denote the finite time horizon considered throughout the paper. Consider a complete probability space $(\Omega, \mathcal{F}, P^\dagger)$. In this probability space, consider mutually independent Wiener processes $w(\cdot) \in \mathbf{R}^r$, $y(\cdot) \in \mathbf{R}^q$ with covariance matrices $W(\cdot)$, $\Sigma(\cdot)$, respectively. We will assume $W(t) \geq \rho_W I$, $\Sigma(t) \geq \rho_\Sigma I$ for all $t \in [0, T]$, where ρ_W, ρ_Σ are positive constants. As in Dai Pra et al. [1996], the space Ω is thought of as the noise space $C([0,T], \mathbf{R}^r) \times C([0,T], \mathbf{R}^q)$, and the probability measure $P^\dagger$ is defined as the product-measure $P^{\dagger,w} \times P^{\dagger,y}$ where $P^{\dagger,w}$, $P^{\dagger,y}$ are standard Wiener measures on $C([0,T], \mathbf{R}^r)$, $C([0,T], \mathbf{R}^q)$, respectively. Also as in Dai Pra et al. [1996], we endow the space Ω with the filtrations $\{\mathcal{F}_t^w, t \geq 0\}$, $\{\mathcal{F}_t^y, t \geq 0\}$, and $\{\mathcal{F}_t, t \geq 0\}$, where $\mathcal{F}_t = \sigma\{\mathcal{F}_t^w \times \mathcal{F}_t^y\}$. These filtrations are generated by the following mappings $\Pi_t^w(w(\cdot))$, $\Pi_t^y(y(\cdot))$: $\Pi_0^w(w(\cdot)) \triangleq 0$, $\Pi_0^y(y(\cdot)) \triangleq 0$ and $\Pi_t(w(\cdot)) \triangleq w(t)$, $\Pi_t^y(y(\cdot)) = y(t)$ for $t > 0$. The filtrations are completed by including all corresponding sets of $P^\dagger$-probability zero. Without any loss of generality, we set $\mathcal{F}^w = \sigma\{\cup_{t\in[0,T]}\mathcal{F}_t^w\}$, $\mathcal{F}^y = \sigma\{\cup_{t\in[0,T]}\mathcal{F}_t^y\}$, and $\mathcal{F} = \sigma\{\cup_{t\in[0,T]}\mathcal{F}_t\} = \sigma\{\mathcal{F}^w \times \mathcal{F}^y\}$. The expectation with respect to $P^\dagger$ will be denoted $\mathbf{E}^\dagger$.

On the probability space defined above, we consider system dynamics driven by the noise input $w(\cdot)$ and governed by a control input $u(\cdot)$, as described by the stochastic differential equation (2). In equation (2), $x(t) \in \mathbf{R}^n$ is the state, and inputs $u(t)$ under consideration are assumed to be adapted to the filtration $\{\mathcal{F}_t^w, t \in [0, T]\}$. We will restrict the class of controls to those under which $P^\dagger\left(\int_0^T \|C_2(t)x(t)\|^2_{\Sigma^{-1}} dt < \infty\right) = 1$, where $C_2(\cdot)$ is a given deterministic time-varying matrix. Also, in the sequel we will make use of an auxiliary output of the system (2) defined by

$$z(t) = C_1(t)x(t) + D_1(t)u(t); \tag{4}$$

this output will be used in the definition of admissible uncertainties. Finally, all coefficients in equations (2) and (4) will be assumed to be deterministic bounded sufficiently smooth matrix valued functions mapping $[0, T]$ into the spaces of matrices of corresponding dimensions.

In this paper we do not fix the underlying probability measure and probability laws of the process and measurement noises w and v. Rather, following Petersen et al. [2000a,b], Ugrinovskii and Petersen [1999, 2001], dynamics of the system (2), (4) are considered under probability measures which are associated with feasible perturbations of the processes w, v. To describe the system (2) as an uncertain stochastic system in the context of this uncertain stochastic system framework, consider a probability measure Q which is absolutely continuous with respect to $P^\dagger$, $Q \ll P^\dagger$. Following Dai Pra et al. [1996], a pair of progressively measurable processes $\eta(t)$, $\xi(t)$ adapted to $\{\mathcal{F}_t, t \geq 0\}$ and Wiener processes $\{\tilde{w}(t), \tilde{v}(t), \mathcal{F}_t, t \geq 0\}$ can be associated with Q, so that under Q,

$$\begin{bmatrix} \tilde{w}(t) \\ \tilde{v}(t) \end{bmatrix} = \begin{bmatrix} w(t) \\ y(t) \end{bmatrix} - \int_0^t \begin{bmatrix} \eta(s) \\ C_2 x(s) + \xi(s) \end{bmatrix} ds.$$

In the probability space $(\Omega, \mathcal{F}, Q)$, dynamics and outputs of the system (2), (4), are governed by the Ito differential equations

$$\begin{aligned} dx(t) &= (A(t)x(t) + B_1(t)u(t) + B_2\eta(t))dt \\ &\qquad + B_2(t)d\tilde{w}(t), \quad x(0) = 0, \\ z(t) &= C_1(t)x(t) + D_1(t)u(t), \\ dy(t) &= (C_2(t)x(t) + \xi(t))dt + d\tilde{v}(t), \quad y(0) = 0. \end{aligned} \tag{5}$$

The system (5) has the form of a perturbed stochastic system governed by the disturbance η, and the process y under Q describes the noisy output of the system (5) perturbed by ξ. Under certain mild conditions on (η, ξ), which will be presented later in this section, the measures Q and $P^\dagger$ are related by

$$Q(dw \times dy) = \zeta^{\eta,\xi}(T)P^\dagger(dw \times dy),$$

where the process $\{\zeta^{\eta,\xi}(t), \mathcal{F}_t, t \in [0, T]\}$ defined below can be shown to be a martingale [Bensoussan, 1992],

$$\begin{aligned} \zeta^{\eta,\xi}(t) &:= e^{\int_0^t \eta'(s)W^{-1}(s)dw(s) - \frac{1}{2}\|\eta(s)\|^2_{W^{-1}(s)}ds} \\ &\times e^{\int_0^t (C_2(s)x(s)+\xi(s))'\Sigma^{-1}dy(s) - \frac{1}{2}\int_0^t \|C_2(s)x(s)+\xi(s)\|^2_{\Sigma^{-1}}ds}. \end{aligned}$$

A particular probability measure P, associated with the pair of uncertainty inputs $(\eta, \xi) = (0, 0)$ is regarded as the *nominal* probability measure. Under this probability measure, w and v become Wiener processes, and the system (2) becomes a standard stochastic linear system driven by Wiener processes, with linear noisy output. This system therefore is referred to as a nominal system. In addition, since the process $\zeta^\dagger(t) \triangleq \zeta^{0,0}(t)$, is a martingale and due to the assumption that solutions of the system (2) are $P^\dagger$-a.s. L_2-integrable, we observe that the probability measures P and $P^\dagger$ are equivalent, $P \sim P^\dagger$ [Liptser and Shiryayev, 1977, Theorem 7.1], i.e.,

$$P^\dagger(dw \times dy) = (\zeta^\dagger(T))^{-1}P(dw \times dy).$$

As in Petersen et al. [2000a], Ugrinovskii and Petersen [1999], the foregoing discussion leads us to define the set of feasible uncertain systems by associating with it a set $\mathcal{P}$ of all probability measures Q with an additional that property

$$h(Q\|P) < \infty. \tag{6}$$

In equation (6), $h(Q\|P)$ denotes the *relative entropy* between a probability measure Q and the nominal probability measure P (Dupuis and Ellis [1997]):

$$h(Q\|P) := \begin{cases} \mathbf{E}^Q \log\left(\frac{dQ}{dP}\right) & \text{if } Q \ll P \text{ and} \\ & \log\left(\frac{dQ}{dP}\right) \in L_1(dQ), \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\mathbf{E}^Q$ is the expectation with respect to Q. Also, using localizations one can express the relative entropy between Q and P as

$$h(Q\|P) = \frac{1}{2}\mathbf{E}^Q \int_0^T (\|\eta(t)\|^2_{W^{-1}} + \|\xi(t)\|^2_{\Sigma^{-1}})dt; \tag{7}$$

see Dai Pra et al. [1996]. Hence, the satisfaction of condition (6) included in the definition of the set $\mathcal{P}$ implies that $\mathbf{E}^Q \int_0^T (\|\eta(t)\|^2_{W^{-1}} + \|\xi(t)\|^2_{\Sigma^{-1}})dt < \infty$. That is, in fact condition (7) shows that we deal with a class of L_2-integrable perturbations of the nominal system.

2.2 Admissible uncertainty

In the previous section, we observed that L_2-integrable disturbances can be associated with probability measures of the set $\mathcal{P}$. While the set $\mathcal{P}$ represents a rich class of feasible disturbances, in a practical problem not all uncertainties of this class may be realized. For instance, the uncertainty in a practical problem under consideration may correspond to an unmodeled dynamic and may be known to be bounded in magnitude, e.g., its H_∞ norm may be known to satisfy certain constraint. Therefore, only uncertainties satisfying this constraint must be considered as admissible in the problem, and the rest must be ruled out. Following Petersen et al. [2000a,b], Ugrinovskii and Petersen [1999], our characterization of admissible uncertainties bounds the size of admissible perturbations by constraining the relative entropy between associated perturbation probability measures Q and the nominal probability measure P. As follows from (7), such a constraint essentially describes a bound on the energy in admissible disturbances.

Definition 1. Given a constant $d > 0$, a probability measure $Q \in \mathcal{P}$ is said to define an admissible uncertainty if the following *stochastic uncertainty constraint* is satisfied:

$$h(Q\|P) \leq \frac{1}{2}\mathbf{E}^Q \int_0^T \|z(t)\|^2 dt + d. \tag{8}$$

In (8), $x(\cdot)$, $z(\cdot)$ are defined by equation (2) considered under the probability measure Q.

We denote the set of probability measures defining the admissible uncertainties by Ξ_d. Elements of the set Ξ_d are also called admissible probability measures.

3. STOCHASTIC MODEL VALIDATION PROBLEM

We are now in a position to formalize our stochastic model validation problem. Along with the uncertain system consisting of the system (2) and the set of probability measures Ξ_d satisfying the uncertainty constraint (8), consider signals $\bar{y}(\cdot)$, $u(\cdot)$ and $\psi(\cdot)$. As described in Section 1, we assume that u and

ψ are adapted with respect to the filtration $\{\mathcal{F}_t^w, t \in [0,T]\}$, and $u \in \mathbf{R}^m$, and $\bar{y}, \psi \in \mathbf{R}^q$. Also, the processes $\bar{y}(\cdot)$ and $\psi(\cdot)$ are connected as described in equation (1), where the measurement noise $\varrho(t)$ takes values in $\mathbf{R}^q$.

The problem under consideration is to verify whether there exists an admissible probability measure $Q \in \Xi_d$ such that under this measure, probability distributions of the output $y(\cdot)$ of the system (2) considered in the probability space $(\Omega, \mathcal{F}, Q)$ (or equivalently system (5) considered in the same probability space) match those of the given process $\bar{y}(\cdot)$, i.e., $y(t) = \bar{y}(t)$ in the weak sense, provided the measurement noise process $\varrho(t)$ governing the measurement model (1) has the same probability law as the process $\tilde{v}(t)$ governing the system (5) considered under Q. That is, $\{\varrho, \mathcal{F}_t^y, t \in [0,T]\}$ must be Wiener under Q. A probability measure satisfying this requirements will be denoted $\bar{Q}$.

The requirement that the processes $y(\cdot)$, $\bar{y}(\cdot)$ must have the same probability distributions under $\bar{Q}$ can be regarded as a constraint on an uncertain system which is additional to that described in Section 2.2. To investigate the structure of uncertain probability measures $\bar{Q}$ satisfying this constraint, let us consider uncertain dynamics of the system (2) governed by u, then for the identity $y(\cdot) = \bar{y}(\cdot)$ to hold weakly, the process $\tilde{\varrho}(t) \triangleq y(t) - \int_0^t \psi(t)dt$ must be a Wiener process under $\bar{Q}$, with increment covariance Σ. To satisfy this constraint, we now consider a subset of feasible measures constructed via probability measure transformations, by restricting perturbations $\eta(\cdot)$ in the definition of the martingale $\zeta^{\eta,\xi}$ to belong to the class of $\mathcal{F}_t^w$-adapted disturbances. This will ensure that in the presence of such disturbances the state process $x(\cdot)$ of the uncertain system (2) (equivalently, the system (5)) is $\mathcal{F}_t^w$-adapted, which is necessary to be able to realize the $\mathcal{F}_t^w$-measurable process $\psi(t)$ using an $\mathcal{F}_t^w$-adapted process $C_2(t)x(t)$.

In summary, the requirement that under $\bar{Q}$, $y(\cdot)$ must be an Ito process with the differential $dy = \psi dt + d\tilde{\varrho}$, and the state process must be $\mathcal{F}_t^w$-adapted and must be driven by $\mathcal{F}_t^w$-adapted disturbances, leads us to constrain the consideration to probability measures $\bar{Q}$ (and corresponding uncertainties) which have the following structure:

$$\bar{Q}(dw \times dv) = Q^w(dw) \times \bar{Q}(dy|w(\cdot)), \qquad (9)$$

where according to the Girsanov Theorem,

$$\begin{aligned} Q^w(dw) &\triangleq P^{\dagger,w}(dw) \\ &\quad \times e^{\int_0^T \eta'(s)W^{-1}(s)dw(s) - \frac{1}{2}\int_0^T \|\eta(s)\|^2_{W^{-1}(s)}ds}, \qquad (10) \\ \bar{Q}(dy|w(\cdot)) &\triangleq P^{\dagger,y}(dy) \\ &\quad \times e^{\int_0^T \psi(s)'\Sigma^{-1}dy(s) - \frac{1}{2}\int_0^T \|\psi(s)\|^2_{\Sigma^{-1}}ds}. \qquad (11) \end{aligned}$$

In the above definition, Q^w is a probability measure defined on sets of the σ-field $\mathcal{F}^w$, generated by $w(s)$, $0 \le s \le t$. Also, $\bar{Q}(\cdot|w(\cdot))$ denotes the conditional probability measure defined on sets of $\mathcal{F}^y$ given a realization of $w(\cdot)$. In (9) we have used the fact that $P^\dagger(dy|w(\cdot)) = P^{\dagger,y}(dy)$ since w and y are independent under $P^\dagger$.

The foregoing discussion leads us to formulate the stochastic model validation problem as follows: *Derive conditions under which there exists a probability measure $\bar{Q} \in \mathcal{P}$ of the form (9) such that $\bar{Q} \in \Xi_d$.*

Definition 2. Given an uncertain system (2) subject to the constraint (8). A triple $(u, \psi, \bar{y})$, in which $\bar{y}(\cdot)$, $\psi(\cdot)$ are connected via (1), is said to be realizable via a stochastic uncertain system (2), (8) (or simply realizable) if there exists $\bar{Q} \in \mathcal{P}$ of the form (9) that satisfies the relative entropy uncertainty constraint (8):

$$h((\bar{Q}\|P) < \frac{1}{2}\mathbf{E}^{\bar{Q}} \int_0^T \|z(s)\|^2 ds + d. \qquad (12)$$

4. THE MAIN RESULT

It is easy to see from the chain rule for the relative entropy [Dupuis and Ellis, 1997] that the condition (7) implies $h(Q^w\|P^{\dagger,w}) < \infty$. Therefore, since $\bar{Q} \in \mathcal{P}$, the search for an admissible $\bar{Q}$ satisfying (12) must be be carried out among probability measures of the form (9) in which $h(Q^w\|P^{\dagger,w}) < \infty$. The set of feasible probability measures Q^w satisfying the latter condition will be denoted $\mathcal{P}^w$.

The following theorem shows that the realizability property introduced in Definition 2 can be recast as an optimization problem.

Theorem 1. Given an uncertain system (2) subject to the constraint (8). A given control-output-measurement triple $u(\cdot)$, $\psi(\cdot)$, $\bar{y}(\cdot)$ is realizable using the uncertain system (2), (8) if and only if

$$\inf_{Q^w \in \mathcal{P}^w} \left[h(Q^w\|P^{\dagger,w}) - \frac{1}{2}\mathbf{E}^{Q^w} \int_0^T (\|z(s)\|^2 - \|\psi(s) - C_2x(s)\|^2_{\Sigma^{-1}} ds \right] < d. \qquad (13)$$

Proof First we observe using the chain rule for the relative entropy [Dupuis and Ellis, 1997] that

$$\begin{aligned} h(\bar{Q}\|P) &= h\left(Q^w\|P^w\right) + \mathbf{E}^{Q^w}\left[h\left(\bar{Q}\left(\cdot|w(\cdot)\right)\|P\left(\cdot|w(\cdot)\right)\right)\right] \\ &= h\left(Q^w\|P^{\dagger,w}\right) + \frac{1}{2}\mathbf{E}^{Q^w}\int_0^T \|\psi(s) - C_2x(s)\|^2_{\Sigma^{-1}}ds; \end{aligned}$$

the last identity holds since $x(\cdot)$ is independent of y from the definition of $\bar{Q}$. For the same reason, the expectation $\mathbf{E}^{\bar{Q}}$ in (12) can be replaced with $\mathbf{E}^{Q^w}$. Also, in the proof of the above identity we have used the fact that according to the definition of P, its restriction on $\mathcal{F}^w$ equals $P^{\dagger,w}$.

The above calculation implies that for any $\bar{Q} \in \mathcal{P}$ of the form (9), condition (12) of Definition 2 reduces to the condition

$$h(Q^w\|P^{\dagger,w}) \le \frac{1}{2}\mathbf{E}^{Q^w}\int_0^T \left[\|z(s)\|^2 - \|\psi(s) - C_2(s)x(s)\|^2_{\Sigma^{-1}}\right] ds + d. \qquad (14)$$

The claim of the theorem can now be established. Clearly, if there exists $\bar{Q}$ of the form (9) which satisfies (12), then it also satisfies (14) and hence, (13) holds. Conversely, suppose (13) holds. Then since (13) is a strict inequality, there must exist an infinitesimally small $\epsilon > 0$ such that

$$\inf_{Q^w \in \mathcal{P}^w} \left[h(Q^w\|P^{\dagger,w}) - \frac{1}{2}\mathbf{E}^{Q^w}\int_0^T (\|z(s)\|^2 - \|\psi(s) - C_2x(s)\|^2_{\Sigma^{-1}} ds \right] < d - \epsilon. \qquad (15)$$

Hence, for any $0 < \epsilon_1 < \epsilon/2$ there exists $Q^w \in \mathcal{P}^w$ such that

$$h(Q^w\|P^w) - \frac{1}{2}\mathbf{E}^{Q^w}\int_0^T \left[\|z(s)\|^2 - \|\psi(s) - C_2(s)x(s)\|^2_{\Sigma^{-1}}\right] ds < d - \epsilon/2. \quad (16)$$

To verify (12), one must construct $\bar{Q}$ using this Q^w in accordance with (9). □

The representation of our model validation problem in the form of the optimization problem (13) allows us to relate the property of realizability to a 'risk-sensitive tracking' performance of the system (2) considered under the original probability measure $P^\dagger$ (and in fact under $P^{\dagger,w}$ since $u(t)$ is adapted to $\{\mathcal{F}_t^w\}$). The performance index is defined as follows

$$V = \log \mathbf{E}^{\dagger,w}\exp(F), \quad (17)$$

$$F \triangleq \frac{1}{2}\int_0^T \left[\|z(s)\|^2 - \|\psi(s) - C_2x(s)\|^2_{\Sigma^{-1}}\right] ds; \quad (18)$$

here as suggested above, $x(\cdot)$ is a solution to the system (2) considered under $P^{\dagger,w}$; recall that in this case the disturbance $w(\cdot)$ in (2) is a Wiener process.

The main result of this paper presented below in Theorem 2 shows that there exists a connection between the realizability of the processes $(u, \psi, \bar{y})$ and the value of V. Owing to Theorem 1, such a connection is prompted by the variational formula for the relative entropy (Dupuis and Ellis [1997], Dai Pra et al. [1996]). However, the variational formula cannot be applied directly to replace the quantity on the left hand side of (13) with V. This is because the variational formula requires the mapping $F\colon \Omega \to \mathbf{R}$ to be either bounded from above or bounded from below. The proof of Theorem 2 circumvents this technical difficulty by using suitable approximations.

Theorem 2. Consider the uncertain system (2) and the constraint (8). If

$$V > -d, \quad (19)$$

then $(u, \psi, \bar{y})$ is a realizable using an uncertain system (2), (8).

Conversely, if $(u, \psi, \bar{y})$ is realizable and additionally, under the reference probability measure $P^\dagger$, dynamics of the system (2) governed by $u(\cdot)$ satisfy the condition

$$\mathbf{E}^\dagger \exp\left(\frac{1}{2}\int_0^T \|z(t)\|^2\right) < \infty; \quad (20)$$

then (19) holds.

Before we proceed to the proof of Theorem 2, we note that the additional technical condition (20), known as the Novikov condition [Liptser and Shiryayev, 1977], is only needed to prove the 'only if' part of Theorem 2; the 'if' claim does not require this condition. It is worthwhile to note that this additional condition concerns properties of the reference system which is not subject to uncertainty, therefore properties of z are entirely dependent on the properties of the input process u. Particularly, if the properties of u establish that $z(t)$ is Markov and mean-square integrable on $[0, T]$, then condition (20) can be verified using the same approach which was used to prove the satisfaction of the Novikov condition in Lemma 1.1 of [Stummer, 1993]. Two practically important cases where the Markov property of z holds justify the introduction of condition (20). In one case the system (2) is driven by a deterministic input u (the open loop control). The second case arises where the input u is realized by a linear state feedback, and therefore, the system (2) is a closed loop linear state-feedback control system. In both cases, the solution $x(t)$ of the system (2) is a Markov process, and so is $z(t)$.

Proof of Theorem 2 First we prove the "if" claim. We will proceed by establishing a contradiction. Suppose $(u, \psi, \bar{y})$ is not realizable, and therefore for any $\bar{Q}$ of the form (9), condition (12) fails. Owing to the argument based on the chain rule and the assumptions and properties noted in the proof of Theorem 1, this fact implies that

$$\inf_{Q^w \in \mathcal{P}^w}\left[h(Q^w\|P^{\dagger,w}) - \mathbf{E}^{Q^w}[F]\right] \geq d. \quad (21)$$

Let us introduce an approximation of the quantity V,

$$V_\delta = \log \mathbf{E}^{\dagger,w}\exp(F_\delta), \quad (22)$$

$$F_\delta \triangleq -\frac{1}{2}\int_0^T \|\psi(t) - C_2x(t)\|_{\Sigma^{-1}(t)}dt + \frac{r}{1+\delta r},$$

$$r \triangleq \frac{1}{2}\int_0^T \|z(t)\|^2 dt, \quad (23)$$

and $\delta > 0$ is an infinitesimally small constant. Note that the mapping $F_\delta : \Omega \to \mathbf{R}$ is bounded from above: $F_\delta \leq \frac{r}{1+\delta r} \leq \frac{1}{\delta}$. This property allows us to apply the variational formula for the relative entropy mentioned above; e.g., see Dupuis and Ellis [1997], Dai Pra et al. [1996]. When applied to the system (2) and the risk sensitive cost (22), the variational formula yields

$$-V_\delta = \inf_{Q^w \in \mathcal{P}^w}\left\{h(Q^w\|P^{\dagger,w}) - \mathbf{E}^{Q^w}[F_\delta]\right\}. \quad (24)$$

Together with our assumptions leading us to conclude that (21) holds, this condition implies $V_\delta \leq -d$ for all $\delta > 0$, i.e., V_δ is bounded from above uniformly in δ. Observe that F_δ is monotone increasing as $\delta \downarrow 0$ and $F_\delta \uparrow F$ a.s.. Then Lebesgue's Monotone Convergence Theorem yields $V \leq -d$. The contradiction with the condition that $V > -d$ implies that if (19) holds then $(u, \psi, \bar{y})$ must be realizable.

Conversely, to prove the "only if" claim of the theorem, suppose $(u, \psi, \bar{y})$ is realizable. To prove that this fact implies condition (19), we employ an approximation similar to that used in the first part of the proof.

Let $N > 0$ be fixed, and define $\tau_N = \inf\{t \leq T : \int_0^t \|z\|^2 dt = N\}$ with $\tau_N = T$ if $\int_0^T \|z\|^d t < N$. Also let $\delta > 0$ be an infinitesimally small constant and define

$$\tilde{V}_{\delta,N} = \log \mathbf{E}^{\dagger,w} e^{\tilde{F}_{\delta,N}}, \quad (25)$$

$$\tilde{F}_{\delta,N} := -\frac{f}{1+\delta f} + \frac{1}{2}\left[\int_0^{\tau_N} \|z(t)\|^2 dt\right],$$

$$f := \frac{1}{2}\int_0^T \|\psi(t) - C_2(t)x(t)\|^2_{\Sigma^{-1}(t)}dt, \quad (26)$$

The quantity $\tilde{F}_{\delta,N}$ is bounded from below uniformly in N, $\tilde{F}_{\delta,N} \geq -\frac{f}{1+\delta f} \geq -\frac{1}{\delta}$, so we can apply the variational formula:

$$-\tilde{V}_{\delta,N} = \inf_{Q^w \in \mathcal{P}^w)}\left\{h(Q^w\|P^{\dagger,w}) - \mathbf{E}^{Q^w}[\tilde{F}_{\delta,N}]\right\}. \quad (27)$$

Hence, since $(u, \psi, \bar{y})$ is realizable, there must exist Q^w and a sufficiently small $\varepsilon > 0$ such that

$$
\begin{aligned}
d - \varepsilon &> h(Q^w \| P^{\dagger,w}) - \mathbf{E}^{Q^w}[F] \\
&= h(Q^w \| P^{\dagger,w}) - \mathbf{E}^{Q^w}[\tilde{F}_{\delta,N}] + \mathbf{E}^{Q^w}\left[\frac{\delta f^2}{1+\delta f}\right] \\
&\geq h(Q^w \| P^{\dagger,w}) - \mathbf{E}^{Q^w}[\tilde{F}_{\delta,N}] \\
&\geq \inf_{Q^w \in \mathcal{P}^w} \left\{ h(Q^w \| P^{\dagger,w}) - \mathbf{E}^{Q^w}[\tilde{F}_{\delta,N}] \right\} \\
&= -\tilde{V}_{\delta,N}.
\end{aligned}
$$

That is, $\tilde{V}_{\delta,N} > -d + \varepsilon$ for any sufficiently small $\varepsilon > 0$ and an arbitrary $N > 0$.

Next we note that $\tilde{F}_{\delta,N} \downarrow \tilde{F}_N$ a.s. and is monotone decreasing as $\delta \downarrow 0$; here $\tilde{F}_N = -f + \frac{1}{2}\int_0^{\tau_N} \|z(t)\|^2 dt$. Also, $0 \leq e^{\tilde{F}_{\delta,N}} \leq \exp\left\{\frac{1}{2}\int_0^{\tau_N} \|z(t)\|^2 dt\right\}$ and

$$
\mathbf{E}^{\dagger,w}\left[\exp\left\{\frac{1}{2}\int_0^{\tau_N} \|z(t)\|^2 dt\right\}\right] \leq e^{N/2} < +\infty.
$$

Therefore, by the Lebesgue's Dominated Convergence Theorem $\lim_{\delta\downarrow 0} \tilde{V}_{\delta,N} = \log \mathbf{E}^{\dagger,w}[e^{\tilde{F}_N}] \geq -d + \varepsilon$.

To conclude the proof, we note that $\tilde{F}_N \uparrow F$ as $N \to \infty$, and $\tilde{F}_N \leq F \leq \frac{1}{2}\int_0^T \|z(t)\|^2 dt$. Due to (20), the last inequality implies that the sequence of random variables $e^{\tilde{F}_N}$ is uniformly bounded from above by an integrable random variable $\exp\left[\frac{1}{2}\int_0^T \|z(t)\|^2 dt\right]$. Therefore, according Lebesgue's Monotone Convergence Theorem, e^F is integrable and also

$$
V = \log \mathbf{E}^{\dagger,w}[e^F] = \lim_{N\to\infty} \log \mathbf{E}^{\dagger,w}[e^{\tilde{F}_N}] \geq -d + \varepsilon.
$$

Since $\varepsilon > 0$ was chosen arbitrarily small, we have $V > -d$ as required.

5. CONCLUSIONS

The paper has presented a sufficient condition for realizability of given a process triple $u, \psi, \bar{y}$ (in which ψ and $\bar{y}$ are coupled by a Brownian motion) using perturbations of a linear stochastic system. The class of admissible perturbations includes uncertainties satisfying a relative entropy constraint. Under an additional technical assumption, the proposed condition is also necessary.

Our result establishes a connection between the realizability of $u, \psi, \bar{y}$, on one hand, and a risk-sensitive tracking performance of the nominal system and the size of admissible disturbances, on the other hand. Such a connection is seen as an important stepping stone towards obtaining more explicit and tractable conditions. In a number of related robust control problems involving relative entropy constraints of the form (8), it was possible to evaluate risk-sensitive performance costs associated with those problems in an explicit form. Therefore the result of Theorem 2 sets a direction for further research into more tractable conditions for realizability of stochastic systems. A numerical approach [Kushner and Dupuis, 2001] to computing the risk-sensitive realizability index V is also worth pursuing.

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