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Finite-Time Tracking Control of a Nonholonomic Mobile Robot ⋆

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Abstract: In this paper, the finite-time tracking problem is investigated for a nonholonomic wheeled mobile robot in a fifth-order dynamic model. We consider the whole tracking error system as a cascaded system. Two continuous global finite-time stabilizing controllers are designed for a second-order subsystem and a third-order subsystem respectively. Then finite-time stability results for cascaded systems are employed to prove that the closed-loop system satisfies the finite-time stability. Thus the closed-loop system can track the reference trajectory in finite-time when the desired velocities satisfy some conditions. In particular, we discuss the control gains selection for the third-order finite-time controller and give sufficient conditions by using Lyapunov and backstepping techniques. Simulation results demonstrate the effectiveness of our method.

1. INTRODUCTION

In the past decades, control problems for nonholonomic systems have been extensively studied (Kolmanovsky et al. 1995). Nonholonomic wheeled mobile robot(WMR) is one of the benchmark nonholonomic systems. Previous results on the control problem of nonholonomic mobile robots can be roughly categorized as either stabilization or tracking. It is known that stabilization of nonholonomic wheeled mobile robots is in general quite difficult. A well-known work of Brockett (1983) identifies that nonholonomic systems can not be stabilized via smooth or continuous time-invariant state feedback. To overcome this difficulty, researchers have proposed controllers that utilize smooth time-varying control laws(Samson, 1990; Pomet, 1992; Murray and Sastry, 1993; Tian and Li, 2002), discontinuous control laws (Wit and Sødalen, 1992; Astolfi, 1996) or hybrid control laws (Sørdalen and Egeland, 1995; M'Closkey and Murry, 1997; Hespanha and Morse, 1999).

From a practical perspective, the tracking control problem is a more interesting problem. Thus, the tracking control problem of nonholonomic WMR systems has attracted growing attention in recent years. A linearization-based local tracking control scheme is introduced for a mobile robot with a kinematic model (Kanayama et al. 1990). A linear time-varying feedback law is presented in Walsh et al.(1994) to solve the local uniform tracking control problem. Both local and global tracking problems are solved via time-varying state feedback based on the backstepping technique (Jiang and Nijmeijer 1997). Considering the input saturation case, Jiang et al. (2001) solves both global stabilization and global tracking problem for the kinematic model of a wheeled mobile robot. The paper of Tian and Cao (2007) considers the nonholonomic systems in chained form with target signals that may exponentially decay to zero. Smooth time-varying controllers render the tracking-error dynamics globally κ-exponentially stable and are applied to the tracking problem of a mobile robot. The work of Tian and Cao (2007) is generalized to a more general class of nonholonomic dynamic systems in Cao and Tian (2007).

While most of current research are focused on kinematic models of nonholonomic mobile robots, little attention has been attracted by dynamic models. The latter is of more practical significance since it considers not only system kinematics but also system dynamics. One way to solve the control design for nonholonomic mobile robots in dynamic models is an "indirect" method. Kinematic controllers are first designed or just borrowed from previous literature on the kinematic model, then torque controllers are designed by using integrator backstepping techniques. In this case, to ensure the global stability, global Lyapunov functions and stability analysis for the whole dynamic systems have to be given (Fierro and Lewis 1995; Jiang and Nijmeijer 1997; Fukao et al.2000). Another way is to construct the control law directly. A variable structure control law is proposed for a mobile robot and guarantees bounded tracking errors with initial bounded errors under bounded disturbances (Shim et al. 1995).

Finite-time control aims to make states reach their targets in finite time. Finite-time stability means finite-time convergence and Lyapunov stability. Finite-time stable systems might enjoy not only faster convergence but also better robustness and disturbance rejection properties (Bhat and Bernstein, 1998). In recent years, more and more attention has been paid to the finite-time control design and analysis based on continuous feedbacks. For the double

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integrator, finite-time stabilizers are derived (Bhat and Bernstein, 1998; Hong et al., 2001). For higher dimensional cases, formulae are given for the finite-time stabilization of a chain of power-integrators and a generalized class of nonlinear systems using a constructive method (Hong 2002). In Bhat and Bernstein (2005), a finite-time stabilizing feedback controller for a chain of integrators exists by giving an explicit construction involving a small valued parameter in $(0, 1)$. In Huang et al. (2005), the global finite-time stabilizability of uncertain nonlinear systems that are dominated by a lower-triangular system can be achieved by Hölder continuous state feedback. The problem of global finite-time stabilization by dynamic output feedback for a class of continuous but nonsmooth nonlinear systems is discussed in Li and Qian (2006). The paper of Hong and Jiang (2006) studies non-smooth finite-time stabilization of nonlinear systems with parametric and dynamic uncertainties.

Finite-time tracking controllers for the nonholonomic mobile robots have been studied (Wu et al., 2005; Li and Tian, 2007) respectively. In Wu et al. (2005), the relay switching technique and the terminal sliding mode control technique are used to design finite-time tracking control laws for the nonholonomic systems with extended chained form. The result is applied to a nonholonomic mobile robot in a dynamic model. However, only global finite-time convergence instead of stability is guaranteed. The work of Li and Tian (2007) consists of two aspects. First it is proved that for a cascaded time-varying system consisting of two uniformly finite-time stable subsystems, a forward completeness condition is sufficient to ensure the uniform global finite-time stability for the whole system. Second, two continuous finite-time tracking control laws are developed for two different cases of a nonholonomic mobile robot in a kinematic model. The global finite-time stability is guaranteed by using the cascaded results instead of constructing a global Lyapunov function.

In this paper, the finite-time tracking control result for the kinematic model in Li and Tian (2007) is extended to the dynamic model case. The main idea is to transform the tracking error model into a cascaded system and finite-time controllers of two subsystems are designed respectively. The forward completeness condition is verified and ensures the global finite-time stability of the closed loop system. Moreover, considering from the application aspect of finite-time control techniques, we give some useful results on the control gains selection for the third-order finite-time controller by a rigorous proof. As far as the authors know, though different finite-time control laws for the third-order case or higher order cases can be found in literatures (Hong, 2002; Bhat and Bernstein, 2005; Huang et al., 2005), the selection problem of control gains or fraction power is not well described yet. In Hong (2002), the control gains are estimated according to the proof of finite-time stability. In Bhat and Bernstein (2005), the control gains are just forced to obey a Hurwitz polynomial condition. However, the fraction power is not explicitly given though its existence is rigorously proved. In Huang et al. (2005), a construction procedure of finite-time control laws is given with the control gains all fixed to be constants.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries

Consider a time-varying system

$$\dot{x} = f(t, x), \ f(t, 0) = 0, \ \forall t \geq t_0 \geq 0, \tag{1}$$

where $f : R_{\geq 0} \times R^n \to R^n$ is continuous. The solution which starts from x_0 at time t_0 is denoted as $x(t, x_0)$.

Definition 1 (Bhat and Bernstein, 1997; Moulat and Perruquetti, 2003): The origin is said to be a uniformly finite-time stable (UFTS) equilibrium of (1) if

(i) the origin is uniformly Lyapunov stable in a neighborhood $U \subseteq U_0$ of the origin;
(ii) (uniform finite-time attractively) the origin is finite-time convergent in U. Namely, if there is a settling time function $T : U \to [0, \infty)$, such that, $\forall x_0 \in U$, the solution $x(t, x_0)$ of system (1) with x_0 as the initial conditions is defined and $x(t, x_0) \in U/\{0\}$ for $t \in [0, T(x_0))$ with the properties: $\lim_{t \to T(x_0)} x(t, x_0) = 0$ and $\lim_{x \to 0} T(x) = 0$.

When $U = U_0 = R^n$, then the origin is a uniformly globally finite-time stable (UGFTS) equilibrium.

Consider the following time-varying cascade system

$$\dot{x}_1 = f(t, x_1, x_2) = f_1(t, x_1) + g(t, x_1, x_2), \tag{2}$$

$$\dot{x}_2 = f_2(t, x_2), \tag{3}$$

where $x_1 \in R^n$, $x_2 \in R^m$, $f_1(t, x_1) = f(t, x_1, 0)$, $g(t, x_1, x_2) = f(t, x_1, x_2) - f(t, x_1, 0)$, which implies that $g(t, x_1, 0) = 0$, $\forall (t, x_1) \in (R_{\geq 0} \times R^n)$. Moreover, $f(t, x_1, x_2)$, $f_2(t, x_2)$ are locally Hölder continuous in x_1, x_2 and uniformly continuous in t.

Lemma 1. (Li and Tian, 2007) The cascaded system (2)-(3) is UGFTS if assumptions A1, A2 and A3 hold:

A1: The subsystem $\dot{x}_1 = f_1(t, x_1)$ is uniformly globally finite-time stable(UGFTS).

A2: The subsystem (3) is UGFTS.

A3: (Uniform forward completeness) For any fixed and bounded x_2, there exists a finite-time bounded(FTB) function $B(t, x_1) : R_{\geq 0} \times R^n \to R_{\geq 0}$ positive definite, proper, radially unbounded and decrescent, which satisfies

$$\dot{B}(t, x_1) \mid_{(2)} \leq \beta_6(B(t, x_1)), \ \forall t \geq t_0 \geq 0, \tag{4}$$

where $\beta_6 : R_{>0} \to R_{>0}$ is a nondecreasing function satisfying the following condition for some constant $a > 0$

$$\beta_6(a) \geq 0; \ \int_a^\infty ds/(\beta_6(s)) = \infty. \tag{5}$$

For convenience, let $sig^\alpha x = sgn(x)|x|^\alpha$ for $\alpha > 0$, where $sgn(\cdot)$ is the sign function.

Lemma 2. (Bhat and Bernstein, 1997) The following control law

$$u = -k_1 sig^{\alpha_1} x_1 - k_2 sig^{\alpha_2} x_2, \tag{6}$$

where $k_1, k_2 > 0, 0 < \alpha_2 < 1, \alpha_1 = \alpha_2/(2 - \alpha_2)$, finite-time stabilizes the double integrator system $\dot{x}_1 = x_2, \quad \dot{x}_2 = u.$

2.2 Problem formulation

The problem we study here deals with a nonholonomic wheeled mobile robot with two-degrees-of-freedom. A simplified dynamic model of it is given by Jiang and Nijmeije (1997)

$$\begin{array}{l} \dot{x} = v\cos\theta,\ \dot{y} = v\sin\theta,\ \dot{\theta} = \omega, \\ \dot{v} = u_1,\ \dot{\omega} = u_2, \end{array} \tag{7}$$

where (x, y) are the Cartesian coordinates of the center of mass of the vehicle, θ is the angle between the heading direction and the x axis, v is the linear velocity and ω is the angular velocity of the mobile robot. u_1 and u_2 may be regarded as torques or generalized force variables of the two-degrees-of-freedom mobile robot.

The objective is the robot follows a reference robot, with position $(x_r, y_r, \theta_r)^T$ and inputs v_r and ω_r. The reference trajectory is given by the following equations.

$$\dot{x}_r = v_r\cos\theta_r,\ \dot{y}_r = v_r\sin\theta_r,\ \dot{\theta}_r = \omega_r. \tag{8}$$

Denoting the error coordinates by Kanayama et al.(1990)

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix} \tag{9}$$

which implies that

$$\begin{aligned} \dot{x}_e &= \omega y_e - v + v_r\cos\theta_e, \\ \dot{y}_e &= -\omega x_e + v_r\sin\theta_e, \\ \dot{\theta}_e &= \omega_r - \omega, \\ \dot{v} &= u_1, \\ \dot{\omega} &= u_2. \end{aligned} \tag{10}$$

3. FINITE-TIME TRACKING CONTROLLER DESIGN

We can treat the tracking error model (10) as a cascaded system consisting of two subsystems. One is the third-order subsystem with states x_e, y_e and v. The other is the second-order subsystem with θ_e and ω. First, a finite-time controller is designed for the second-order subsystem. Second, a finite-time controller is designed for the third-order subsystem. And then the sufficient conditions of finite-time stability for cascaded systems are verified so that the closed loop system is proved to be uniformly globally finite-time stable.

Considering a system with a chain of third-order integrator, the selection of control gains is also a very important problem, especially from the practical application aspect. To this end, we give sufficient conditions for selecting available control gains.

3.1 Finite-time control law for a triple integrator system

In this section, we aim at deriving a finite-time control law for a system with a chain of a triple integrator as:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dot{x}_3 = u. \tag{11}$$

Theorem 3. The system (11) is globally finite-time stabilized by the following control law

$$u = -l_3(x_3^{1/(2q_1-1)} + l_2^{1/(2q_1-1)}(x_2^{1/q_1} + l_1^{1/q_1}x_1))^{3q_1-2}, \tag{12}$$

where $q_1 = p/q \in (2/3, 1)$, $p > 0$ and $q > 0$ are odd integers, $k_i(i = 1, 2) > 0$ and $l_j(j = 1, 2, 3) > 0$ satisfy the following conditions:

$$\begin{aligned} (i)\ & -l_1 + \frac{2^{1-q_1}}{1+q_1} + \frac{(2-q_1)2^{1-q_1}}{k_1(1+q_1)} l_1^{1+1/q_1} q_1 \\ & + \frac{(3-2q_1)(1-q_1)}{k_2 q_1(1+q_1)} l_1^{1/q_1-1} (2^{2-2q_1} l_2^{2q_1/(2q_1-1)} \\ & + 2^{4-4q_1} l_2^{1/(2q_1-1)}) + \frac{(3-2q_1)q_1}{k_2(1+q_1)} 2^{2-2q_1} l_1^{1+1/q_1} l_2^{1/(2q_1-1)} \\ & < 0, \end{aligned} \tag{13}$$

$$\begin{aligned} (ii)\ & \frac{2^{1-q_1}q_1}{1+q_1} - \frac{l_2}{k_1} + \frac{(2-q_1)l_1^{1/q_1}}{k_1}(\frac{2^{1-q_1}l_1}{1+q_1} + 2^{2-2q_1}) \\ & + \frac{(3-2q_1)q_1}{k_2(1+q_1)} 2^{3-3q_1} l_1^{1/q_1} l_2^{1/(2q_1-1)} + \frac{(2-q_1)2^{2-2q_1}}{k_1(1+q_1)} \\ & + \frac{(3-2q_1)l_2^{1/(2q_1-1)}}{k_2 q_1(1+q_1)} (2^{2-2q_1} l_1^{1/q_1-1} l_2(2q_1-1) \\ & + 2^{2-q_1} q_1 l_2 + 2^{4-3q_1}(1-q_1)) < 0, \end{aligned} \tag{14}$$

$$\begin{aligned} (iii)\ & \frac{(2q_1-1)2^{2-2q_1}}{k_1(1+q_1)} + \frac{(3-2q_1)l_2^{1/(2q_1-1)}}{k_2 q_1(1+q_1)} (2^{2-2q_1} l_1^{1/q_1-1} l_2 \\ & + 2^{2-q_1} l_2 + 2^{5-4q_1} q_1 l_1^{1/q_1-1} + 2^{5-3q_1} q_1) \\ & + \frac{(3-2q_1)}{k_2(1+q_1)} l_1^{1/q_1} l_2^{1/(2q_1-1)} (2^{2-2q_1} l_1 + 2^{3-3q_1}) - \frac{l_3}{k_2} \\ & < 0. \end{aligned} \tag{15}$$

Proof: For page limits, the proof is omitted here.

Remark 1: It should be pointed out that the controller structure (12) as well as the proof procedure of Theorem 1 are similar to that in Huang et al. (2005). The only difference is that here we do not restrict the control gains as well as the coefficients of Lyapunov functions to be constants. Hence, a sufficient condition for the selection of control gains can be extra obtained.

Remark 2: It can be observed from the proof that the fraction power q_1 directly influences the convergence rate of the closed loop system. From the proof, we know that the states converge faster while q_1 is smaller. Moreover, when q_1 approaches 1, the response of the closed loop system approaches exponential convergence and when q_1 approaches $2/3$, the controller approaches a variable structure controller.

3.2 Discussion for the selection of control gains

The three nonlinear inequalities of Theorem 1 are complicated. To this end, we try to find some available methods to ensure the inequalities (13), (14) and (15) all have solutions for $k_i(i = 1, 2)$ and $l_j(j = 1, 2, 3)$. One way is to use a trial method. That is, first selecting the values according to the characteristics of the three inequalities, then testing them by substituting them into the three inequalities. It is not easy to take values according to them. This method requires patience and experience due to the complexity of inequalities. Moreover, though one can finally find a feasible solution, a good comprehension of the solution space for the control gains can not be obtained.

Another way is to use an exhaustive algorithm. First, q_1 is chosen. Second, we restrict the searching intervals for k_1, k_2, l_1 and l_2 respectively. Third, by using a nested circulation technique, we can get all the feasible solutions in the given intervals satisfying (13) and (14), and then the infimum of l_3, denoted as l_{3inf} can thus be obtained by (15). Hence, a group value for l_1, l_2 and l_{3inf} can be chosen from the saved data sets S_{inf}. The numerical feasible solution set S in the given intervals can thus be expressed as follows

$$S = \{(l_1, l_2, l_3)|(l_1, l_2, l_{3inf}) \in S_{inf}, l_3 > l_{3inf}\}.$$

One can also find some further descriptions and results in the simulation section.

3.3 Finite-time tracking control for the dynamic model

In this section, we design a finite-time tracking controller for the dynamic model of a nonholonomic wheeled mobile robot. Let us consider the error dynamic model (10) and assume that the velocities v and ω are subject to the following constraints:

B1: ω_r, $\dot{\omega}_r$, $\ddot{\omega}_r$ exist and $0 < \omega_r^{min} \leq |\omega_r(t)| \leq \omega_r^{max}$, $|\dot{\omega}_r(t)| < \omega_1^{max}$ for each $t \geq t_0 \geq 0$.

B2: v_r and $\dot{v}_r$ exist and $|v_r(t)| \leq v_r^{max}$ for each $t \geq t_0 \geq 0$.

If we denote $\omega = \omega - \omega_r + \omega_r$, (10) can be rephrased as follows

$$\begin{aligned}
&\dot{x}_e = \omega_r y_e - v + v_r - \bar{\omega} y_e + v_r(\cos\theta_e - 1),\\
&\dot{y}_e = -\omega_r x_e + \bar{\omega} x_e + v_r \sin\theta_e,\\
&\dot{v} = u_1,\\
&\dot{\theta}_e = \bar{\omega},\\
&\dot{\bar{\omega}} = \bar{u}_2,
\end{aligned} \tag{16}$$

where $\bar{\omega} = \omega_r - \omega$ and $\bar{u}_2 = \dot{\omega}_r - u_2$. Consider a state transformation defined by

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_e \\ -\omega_r x_e \\ -\omega_r^2 y_e + \omega_r(v - v_r) + \dot{\omega}_r x_2/\omega_r \end{bmatrix},$$

$$X_2 = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \theta_e \\ \bar{\omega} \end{bmatrix}. \tag{17}$$

The derivatives of X_1 and X_2 are

$$\dot{X}_1 = f(X_1, \bar{u}_1) + g(X_1, X_2), \tag{18}$$

$$\dot{X}_2 = \begin{bmatrix} x_5 \\ \bar{u}_2 \end{bmatrix}, \tag{19}$$

where

$$\bar{u}_1 = -\omega_r\dot{\omega}_r x_1 + \left(\frac{\ddot{\omega}_r}{\omega_r} - \frac{\omega_r^2 - 2\dot{\omega}_r^2}{\omega_r^2}\right)x_2 + 2\frac{\dot{\omega}_r x_3}{\omega_r} + \omega_r u_1 - \omega_r \dot{v}_r,$$

$$f(X_1, \bar{u}_1) = \begin{bmatrix} x_2 \\ x_3 \\ \bar{u}_1 \end{bmatrix},$$

$$g(X_1, X_2) = \begin{bmatrix} -x_2 x_5/\omega_r + v_r \sin x_4 \\ x_1 x_5 \omega_r - \omega_r v_r(\cos x_4 - 1) \\ x_2 x_5 \omega_r - \omega_r^2 v_r \sin x_4 + \dot{\omega}_r x_1 x_5 - \dot{\omega}_r v_r(\cos x_4 - 1) \end{bmatrix}.$$

Thus, the tracking error model (10) is transformed to a cascaded system (18) and (19).

Theorem 4. If assumptions B1 and B2 hold, then the system (10) is uniformly globally finite-time stable, i.e., the system (7) can globally track the reference trajectory (8) in finite time, under the following control laws

$$\begin{aligned}
u_1 = &-\frac{1}{\omega_r} l_3 (x_3^{1/(2q_1-1)} + l_2^{1/(2q_1-1)}(x_2^{1/q_1} + l_1^{1/q_1} x_1))^{3q_1-2}\\
&+ \dot{\omega}_r x_1 + \left(-\frac{\ddot{\omega}_r}{\omega_r^2} + \omega_r + \frac{2}{\omega_r^3}\dot{\omega}_r^2\right)x_2 - \frac{2}{\omega_r^2}\dot{\omega}_r x_3 + \dot{v}_r,
\end{aligned} \tag{20}$$

$$u_2 = \dot{\omega}_r + l_4 sig^{\alpha_1}(x_4) + l_5 sig^{\alpha_2}(x_5). \tag{21}$$

where $l_i (i = 1, \ldots, 5)$, α_1 and α_2 satisfy the conditions in Theorem 3 and Lemma 2 respectively.

Proof: It can be verified that subsystems $\dot{X}_1 = f(X_1, \bar{u}_1)$ and (19) under control laws (20) and (21) will be finite-time stable by using Theorem 3 and Lemma 2. Next we will show that the closed loop system (18) and (19) will also satisfy the uniform forward completeness condition.

Consider the finite-time bounded function $B(X_1) = x_1^2/2 + x_2^2/2 + x_3^2/2$, which is obviously a decrescent function. Taking the derivative of $B(X_1)$ along solutions of (18) yields

$$\begin{aligned}
\dot{B}(X_1) = &\, x_1(x_2 - x_2 x_5/\omega_r + v_r \sin x_4) + x_2(x_3 + \omega_r x_1 x_5\\
&-\omega_r v_r(\cos x_4 - 1)) + x_3(\bar{u}_1 + \omega_r x_2 x_5\\
&-\omega_r^2 v_r \sin x_4 + \dot{\omega}_r x_1 x_5 + \dot{\omega}_r v_r(\cos x_4 - 1))\\
\leq &\, x_1 x_2 - x_1 x_2 x_5/\omega_r + |v_r x_1| + x_2 x_3 + \omega_r x_1 x_2 x_5\\
&+|\omega_r v_r x_2| + x_3\bar{u}_1 + \omega_r x_2 x_3 x_5 + |\omega_r^2 v_r x_3|\\
&+\dot{\omega}_r x_1 x_3 x_5 + |\dot{\omega}_r v_r x_3|.
\end{aligned} \tag{22}$$

Substituting (12) into (22) yields

$$\begin{aligned}
\dot{B}(X_1) \leq &\, x_1 x_2 + (\omega_r - 1/\omega_r) x_1 x_2 x_5 + |v_r x_1| + x_2 x_3\\
&+|\omega_r v_r x_2| + \omega_r x_2 x_3 x_5 - l_3 x_3 (x_3^{1/(2q_1-1)}\\
&+l_2^{1/(2q_1-1)}(x_2^{1/q_1} + l_1^{1/q_1} x_1))^{3q_1-2} + |\omega_r^2 v_r x_3|\\
&+\dot{\omega}_r x_1 x_3 x_5 + |\dot{\omega}_r v_r x_3|.
\end{aligned}$$

Since the subsystem (19) is uniformly globally finite-time stable, x_5 is bounded. We denote $|x_5(t)| \leq x_5^{max}$, where $x_5^{max} > 0$ is a constant. Let $\eta_1 = \|X_1(t)\| = \sqrt{x_1^2 + x_2^2 + x_3^2} \geq \eta > 1$, then we have $|x_i(t)| \leq \eta_1 \leq \eta_1^2$ and $|x_i(t) x_j(t)| \leq \eta^2/2$, $i = 1, 2, 3$. With this in mind, we obtain

$$\begin{aligned}
\dot{B}(X_1) \leq &\, \frac{\eta_1^2}{2} + \frac{\eta_1^2}{2}|\omega_r - \frac{1}{\omega_r}| x_5^{max} + |v_r|\eta_1 + \frac{\eta_1^2}{2} + |\omega_r v_r|\eta_1\\
&+\frac{\eta_1^2}{2}|\omega_r| x_5^{max} + \omega_r^2 |v_r|\eta_1 + \frac{\eta_1^2}{2}|\dot{\omega}_r| x_5^{max} + |\dot{\omega}_r v_r|\eta_1^2\\
&+l_3\eta_1|\eta_1^{1/(2q_1-1)} + l_2^{1/(2q_1-1)}\eta_1^{1/q_1}\\
&+l_2^{1/(2q_1-1)} l_1^{1/q_1}\eta_1|^{3q_1-2}
\end{aligned}$$

$$\leq \frac{\eta_1^2}{2}(2+|\omega_r-\frac{1}{\omega_r}|x_5^{max}+2|v_r|+2|\omega_r v_r| \\ +|\omega_r|x_5^{max}+2\omega_r^2|v_r|+|\dot{\omega}_r|x_5^{max}+2|\dot{\omega}_r v_r| \\ +2l_3(1+l_2^{1/(2q_1-1)}+l_2^{1/(2q_1-1)}l_1^{1/q_1})^{3q_1-2}).$$

Consider assumptions B1 and B2, then $\dot{B}(X_1)$ can be rewritten as

$$\dot{B}(X_1) \leq KB(X_1) \tag{23}$$

where

$$K = 2+(\omega_r^{max}+\frac{1}{\omega_r^{min}})x_5^{max}+2\omega_r^{max}v_r^{max}+\omega_r^{max}x_5^{max} \\ +2v_r^{max}+2(\omega_r^{max})^2v_r^{max}+\omega_1^{max}x_5^{max}+2\omega_1^{max}v_r^{max} \\ +2l_3(1+l_2^{1/(2q_1-1)}+l_1^{1/q_1}l_2^{1/(2q_1-1)})^{3q_1-2}>0.$$

On the other hand, if $\eta_1 \leq 1$, there exists a constant $L > 0$ such that $\dot{B}(X_1) \leq L$. Thus, we obtain

$$\dot{B}(X_1) \leq KB(X_1)+L.$$

Let $\beta_6(s) = Ks + L$. Obviously, the forward completeness condition is satisfied.

By virtue of Lemma 1, the closed loop system (18) and (19) is uniformly globally finite-time stable. Then under the control laws (20) and (21), the closed loop system (10) is uniformly globally finite-time stable, which means that the system (7) can globally track the reference trajectory (8) in finite time. Thus, the proof is completed.

4. SIMULATION RESULTS

In this section, we demonstrate the previous theoretical results by means of numerical simulation. The reference velocities are given as

$\omega_r = 1+2t/(t+10)\ m\cdot s^{-1}$, $v_r = 1.5-1.5t/(t+10)\ rad\cdot s^{-1}$.

For subsystem (19), we select $\alpha_2 = 3/5$, $\alpha_1 = \alpha_2/(2-\alpha_2) = 3/7$, $l_4 = 10$ and $l_5 = 8$. For subsystem $f(X_1, \bar{u}_1) = [x_2\ x_3\ \bar{u}_1]^T$, we choose parameter value as $q_1 = 11/13$.

According to previous descriptions, we can use an exhaustive program to get feasible control gains. First, we restrict $k_1 \in [0.1, 4]$, $k_2 \in [1, 300]$, $l_1 \in [0.61, 5]$ and $l_2 \in [1, 10]$. The step sizes are all set to 0.1. By using a nested circulation technique, all the feasible solutions of l_1 and l_2 satisfying (13) and (14) in the restricted area can be found. Thus we obtain the corresponding infimums of l_3 from (15). To have a good comprehension of the feasible solution, we give a plot of (l_1, l_2, l_{3inf}) in the case of fixing k_1 and k_2 as $k_1 = 2.1$ and $k_2 = 294.6$. It is shown in Fig. 1.

Thus, a group of control parameters is selected as $l_1 = 0.81$, $l_2 = 3.9$ and $l_3 = 210 > l_{3inf} = 196.92$. Thus, two finite-time controllers are constructed as

$$u_1 = -\frac{210}{\omega_r}(x_3^{13/9}+3.9^{13/9}(x_2^{13/11}+0.81^{13/11}x_1))^{7/13} \\ +\dot{\omega}_r x_1+(-\frac{\ddot{\omega}_r}{\omega_r^2}+\omega_r+\frac{2}{\omega_r^3}\dot{\omega}_r^2)x_2-\frac{2}{\omega_r^2}\dot{\omega}_r x_3+\dot{v}_r \tag{24}$$

$$u_2 = \dot{\omega}_r+10sig^{3/7}x_4+8sig^{3/5}x_5. \tag{25}$$

From Theorem 2, the system (7) can globally track the reference trajectory (8) in finite time.

In order to get the states in the dynamic model, the following transformations are needed.

$$\theta = \theta_r - x_4,\ x = x_r + \frac{cos\theta}{\omega_r}x_2 + x_1 sin\theta, \\ y = y_r + \frac{sin\theta}{\omega_r}x_2 - x_1 cos\theta,\ v = \frac{x_3}{\omega_r}+\omega_r x_1 + v_r - \frac{\dot{\omega}_r x_2}{\omega_r^2}, \\ \omega = \omega_r - x_5.$$

In simulation we take the initial conditions as $[x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)] = [0.54, -0.25, -1.4, -0.5, 0.5]$ and $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, -1]$.

We have simulated the nonholonomic WMR with the tracking controllers (24) and (25). The simulation results are shown in Fig. 2-4. Fig. 2 shows the time response of x_e, y_e, θ_e. Fig. 3 shows the time history of control input u_1 and u_2. Fig. 4 shows the phase plot on $x-y$ plane.

5. CONCLUSION

A finite-time tracking control design method for the nonholonomic wheeled mobile robot in a dynamic model has been presented. The closed loop system can track the desired trajectory in finite time. A sufficient condition for the control gains of the third-order finite-time controller has been derived for the triple integrator subsystem. The results of simulations have validated the efficiency of the proposed control method.

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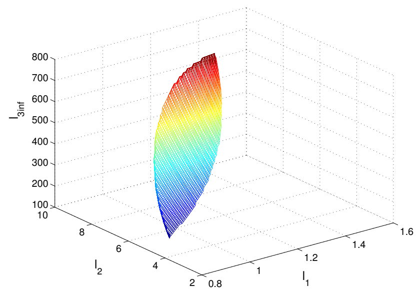

Fig. 1. The plot of l_1, l_2 and l_{3inf} in the case of $k_1 = 2.1$ and $k_2 = 294.6$

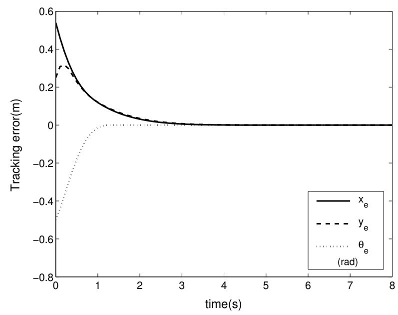

Fig. 2. Tracking errors of x_e, y_e, θ_e

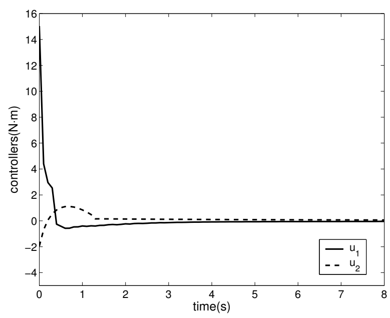

Fig. 3. Control signals u_1 and u_2

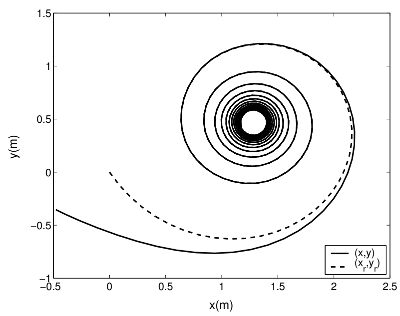

Fig. 4. Moving path of the WMR