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Global Set Stabilization of the Spacecraft Attitude Control Problem Based on Quaternion ⋆

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Abstract: In this paper, we develop a global set stabilization method for the attitude control problem of spacecraft system based on quaternion. The control law which uses both optimal control and finite-time control techniques can globally stabilize the attitude of spacecraft system to an equilibrium set. First for the kinematic subsystem, we design a virtual optimal angular velocity. Then for the dynamic subsystem, we design a finite-time control law which can force the angular velocity to track the virtual optimal angular velocity in finite time. It is rigorously proved that the closed loop system satisfies global set stability. The control method is more natural and energy-saving. The effectiveness of the proposed method is demonstrated by simulation results.

1. INTRODUCTION

Spacecraft requires attitude control system to provide attitude-maneuver, tracking, and pointing. However, due to inherent nonlinearity of attitude dynamic, control system design is a complex undertaking. Various nonlinear control methods have been proposed for solving the attitude stabilization problem. These methods include optimal control method(Kang (1995); Krstic et al. (1999); Park (2005); Debs et al. (1969); Tsiotras (1996b)) and other methods such as Lyapunov control method(Fragopoulos et al. (2004); Joshi et al. (1995)), sliding mode control method (Vadali (1986)), H-infinity control method (Kang (1995)), finite-time control method(Ding et al. (2007)), etc.

Optimal control of rigid spacecraft has a long history stemming. The main objective of optimal control is to determine a control law that will cause a process to satisfy physical constraints and minimize a performance criterion. Usually, the optimal control results in the literature can be mainly classified into the following two methods: direct optimal control method and inverse optimal method.

The former is to first select a performance index and then derive an optimal control law from it. Usually, the performance index includes a penalty term on the control effort. It is a very difficult thing to get the solution by solving the associated Hamilton-Jacobi equation. The latter is to first derive a control law, then construct a corresponding performance index for it.

Considering both of the above mentioned methods, the main obstruction stems from the difficulties in either solving complex differential equations for the former or constructing complex performance indices for the latter. Hence, some researchers have also developed optimal control laws only for dynamic subsystem(Debs et al. (1969)) or kinematic subsystem (Tsiotras (1996b)).

To enhance the flight envelope of a spacecraft system, global control result is always highly desirable for the attitude control problem. However, for those global stabilization results in the literature including Joshi et al. (1995); Park (2005); Vadali (1986), there exist two main problems.

One problem is that rigorously speaking, they are not global results. It is well known that to obtain a global attitude stabilization result, usually quaternion is used to ensure the global attitude description. Note that, the spacecraft attitude system belongs to multiple equilibria systems. For each system based on quaternion, there should have two equilibria. Observing these global attitude stabilization methods, one can find that the attraction domain of one of the equilibria is the global state space except one point. According to this fact, it should be pointed out that most of the global attitude stabilization results can not be really considered as global results. In fact, these results can be considered as almost global stabilization results like Chaturvedi et al. (2006). Moreover, for those methods using continuous feedbacks (Joshi et al. (1995); Park (2005)), it is rigorously proved in Bhat et al. (2000b) that the spacecraft attitude can not be globally stabilized through continuous feedback.

Another problem is that these control laws are not energy-saving. One reason for this is due to the fact that for any given initial value, the system states have to be driven to the desired equilibrium even if they are much closer to the other equilibrium of the system. This phenomenon is called “unwinding” phenomenon in Bhat et al. (2000b). Therefore, these nonlinear controllers show some stiffness and are not energy-saving for attitude control (See Section 2.2 for more detail discussion).

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To solve the above mentioned problems, one should first consider a question: Can both of the equilibria in the spacecraft attitude control system based on quaternion be designed to be stable under the same controller? The answer is positive. These problems may be overcome if we consider the stabilization of both equilibria instead of only one equilibrium. As pointed out in Rouche et al. (1977), the attractivity or asymptotic stability of a set are natural concepts fitting many practical applications. In this case, the stability involved is the stability with respect to a set (lin et al. (1994); Lin et al. (1995); Rouche et al. (1977)), which we call set stability here in agreement with Lin et al. (1995). In Fragopoulos et al. (2004), by constructing a Lyapunov function, a global control law has been derived for the set stabilization of the spacecraft attitude. However, the control law obtained is not continuous in the states and may cause chattering phenomenon.

In this paper, we concentrate on the design of global set stabilization law. We also consider the optimal control design only on the kinematic subsystem. First, by solving a Hamilton-Jacobi equation we try to develop an optimal angular velocity for the kinematic subsystem regarding the angular velocity as an input. However, the optimal solution is not unique. To get a global solution, we have to employ a special control law which can be regarded as a combination of open loop control and closed loop control. This optimal angular control law is only discontinuous in initial values, which will not cause chattering phenomenon in the closed loop system. Secondly, we design a global finite-time control law for dynamic subsystem. The attitude can be stabilized to a set consisting of two equilibria. We show that the control method avoids the "unwinding" phenomenon and is energy-saving. The effectiveness of the proposed method is illustrated by simulations.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries

Before designing the control law, we first give some definitions and lemmas.

Consider the system

$$\dot{x} = f(x), \tag{1}$$

where $x \in R^n$, $f : R^n \to R^n$ is continuous.

Definition 1:(Bhat et al. (2005, 2000a)) The equilibrium $x = 0$ of system (1) is finite-time stable if it is Lyapunov stable and finite time convergent in a neighborhood of $U_0 \subseteq R^n$ of the origin. The finite-time convergence means the existence of a function $T : U_0\backslash\{0\} \to (0, +\infty)$, such that, $\forall x_0 \in U_0 \subset R^n$, the solution of (1) satisfies $x(t, x_0) \in U_0\backslash\{0\}$ for $t \in [0, T(x_0))$, and $\lim_{t\to T(x_0)} x(t, x_0) = 0$ with $x(t, x_0) = 0$ for $t > T(x_0)$.

By combination of the definitions in Liao (1988),lin et al. (1994)and Rouche et al. (1977), the definition of globally asymptotically stable with a set M is given as follows.

Definition 2: Suppose M is a non-empty set. The solution of (1) is

• stable with respect to M if for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $t \geq 0$

$$d(x_0, M) < \delta(\varepsilon) \Rightarrow d(x(t, x_0), M) < \varepsilon;$$

• globally attractive with respect to M, if for each $x_0 \in R^n$,

$$\lim_{t\to\infty} d(x(t, x_0), M) = 0;$$

• globally bounded, if for all $x_0 \in R^n$, there exists a positive constant $K(x_0)$ such that for all $t \geq 0$

$$d(x(t, x_0), M) \leq K(x_0);$$

• globally asymptotically stable with respect to M, if it is globally bounded, stable and globally attractive with respect to M.

Lemma 1:(Bhat et al. (2005, 2000a)) Considering system (1), suppose there exists a continuous function $V : U \to R$ such that the following conditions hold:
(i)V is positive definite,
(ii)There exist real numbers $c > 0$ and $\alpha \in (0, 1)$ and an open neighborhood $U_0 \subset U$ of the origin such that

$$\dot{V}(x) + c(V(x))^\alpha \leq 0, x \in U_0\backslash\{0\}.$$

Then the origin is a finite-time stable equilibrium of system (1). If $U = U_0 = R^n$, then the origin is a globally finite-time stable equilibrium of (1).

Lemma 2:(Qian et al. (2001)) Let $i = 1, \cdots, n$ and $0 < p \leq 1$, for all $x_i \in R$, the following inequality holds:

$$(|x_1| + \cdots + |x_n|)^p \leq |x_1|^p + \cdots + |x_n|^p.$$

Lemma 3:(Fragopoulos et al. (2004); Liao (1988))Consider a positive-definite function $V(x)$ with respect to M for system (1), where M is a compact set and $V(x)$ is continuous in x on R^n, satisfying $\dot{V}(x) \leq 0$ for all $x \in R^n - M$. Then system (1) is stable with respect to M.

2.2 Problem formulation

The spacecraft attitude can be described by two sets of equations, namely, the kinematic equation and the dynamic equation. Available parameterization methods include Euler angles, Cayley-Rodrigues parameters, modified Rodrigues parameters and quaternion. Only quaternion description really is a global description since other parameterization methods have associated with singular problems. Therefore, we use quaternion to describe spacecraft attitude in this paper.

The dynamic equation can be described by(Kane et al. (1983))

$$J\dot{\omega} = s(\omega)J\omega + u \tag{2}$$

where $J = diag(J_1, J_2, J_3)$ is the inertia matrix, $\omega = [\omega_1, \omega_2, \omega_3]^T$ is the angular velocity, $u = [u_1, u_2, u_3]^T$ is the control signal, and $s(\omega)$ is the following matrix,

$$s(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

The kinematic equation can be described as follows(Kane et al. (1983))

$$\dot{q} = \frac{1}{2}E(q)\omega \tag{3}$$

where $q = [q_0, q_1, q_2, q_3]^T = [q_0, q_v^T]^T$ is the quaternion, and

$$E(q) = \begin{pmatrix} -q_v^T \\ -s(q_v) + q_0 I_3 \end{pmatrix},$$

where I_3 denotes the 3×3 identity matrix.

Actually, let Φ denote the principal angle and $e = [e_1, e_2, e_3]^T$ denote the principal axis associated with Euler's Theorem with $e_1^2 + e_2^2 + e_3^2 = 1$. Then the quaternion can be defined as

$$q_0 = cos(\Phi/2), q_i = e_i sin(\Phi/2), i = 1, 2, 3 \tag{4}$$

From (4), we obtain

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{5}$$

Furthermore, we have

$$E^T(q)E(q) = I_{3\times3}.$$

For the attitude stabilization problem, the desired quaternion is $(1, 0, 0, 0)^T$ or $(-1, 0, 0, 0)^T$ which means the desired principle angle should be $\Phi = 4m\pi, m \in Z$ or $\Phi = 4m\pi + 2\pi, m \in Z$ respectively according to (4).

In Joshi et al. (1995); Park (2005); Vadali (1986), the desired quaternion is designed to be $(1, 0, 0, 0)^T$ while $(-1, 0, 0, 0)^T$ is designed to be a unstable equilibrium, or a repeller. Then it is said that since in physical space both of the equilibria are the same, global asymptotical stability can be obtained.

However, it is not right. According to the stability theory, if $(-1, 0, 0, 0)^T$ is definitely unstable it can not be considered the same as the stable equilibrium $(1, 0, 0, 0)^T$. One can observe clearly from Fig. 1(a) that all the states near it just escape from it. Moreover, under those controllers, all the states have to be driven to $(1, 0, 0, 0)^T$ even if they are located very close to $(-1, 0, 0, 0)^T$. This phenomenon is called "unwinding" in Bhat et al. (2000b).

Our purpose in this paper is to design a set stabilization law to make both of the equilibria stable so that states can converge to the equilibrium which is closer to them (see Fig.1(b)).

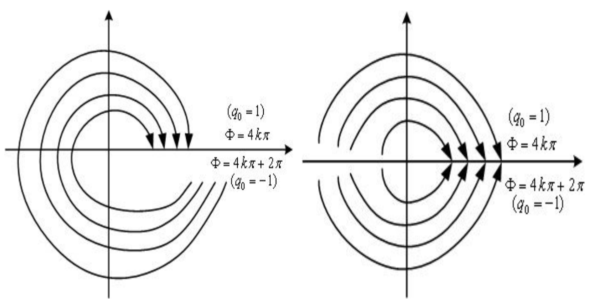

Fig. 1: The sketch maps of principle angle response (a) conventional stabilization method, (b) set stabilization method

3. CONTROL LAW DESIGN

From dynamic subsystem (2) and kinematic subsystem (3), we can regard the state ω as a virtual input of subsystem (3). Hence, the control method in this paper can be written as follows:

(i) Giving a performance index for kinematic subsystem (3) and regarding ω as the input, by solving a Hamilton-Jacobi equation, we can derive a virtual optimal angular velocity ω^* which stabilizes the kinematic subsystem and minimizes the performance index.

(ii) For dynamic subsystem (2), we use finite-time control techniques to design the control law such that ω will track the virtual optimal angular velocity ω^*.

3.1 Optimal angular velocity design

Design procedure

The performance index for the kinematic subsystem can be selected as:

$$\Xi(0, q(0), \omega(0)) = \frac{1}{2}\int_0^{+\infty} \{\, q_v^T G^{-1} q_v + \omega^T G \omega \,\} dt \tag{6}$$

where $G = G^T$ is a 3×3 matrix and satisfies $G > 0$.

Hamilton function can be selected as

$$\begin{aligned} H(t, q, \omega) &= \frac{1}{2}\left(q_v^T G^{-1} q_v + \omega^T G \omega \right) + \\ &\frac{1}{2}\left(\frac{\partial \Xi^*(t, q, \omega^*)}{\partial q_v} \right)^T E_1(q)\omega \end{aligned} \tag{7}$$

where $\Xi^*(t, q, \omega^*)$ is the optimal value of performance index $\Xi(t, q, \omega)$, ω^* is the optimal velocity, and $E_1(q) = (-s(q_v) + q_0 I_{3\times3})$. Because the optimal control problem here is an unconstrained optimization problem, it is obtained that

$$\frac{\partial H}{\partial \omega} = G\omega + \frac{1}{2}E_1^T(q)\frac{\partial \Xi^*}{\partial q_v} = 0 \tag{8}$$

From (8), it follows that

$$\omega^*(q) = -\frac{1}{2}G^{-1}E_1^T(q)\frac{\partial \Xi^*}{\partial q_v} \tag{9}$$

We have Hamilton-Jacobi equation as follows:

$$-\frac{\partial \Xi^*}{\partial t} = H(t, q, \omega^*) \tag{10}$$

Since G is time invariant and since the optimization is for a process of infinite duration, it follows that Ξ^* will depend only on the state q. This implies that

$$-\frac{\partial \Xi^*}{\partial t} = 0 \tag{11}$$

Hence, according to (10),(11) and (7), we have

$$\begin{aligned} 0 = &\frac{1}{2}\left(\frac{1}{2}G^{-1}E_1^T(q)\frac{\partial \Xi^*}{\partial q_v} \right)^T G \left(\frac{1}{2}G^{-1}E_1^T(q)\frac{\partial \Xi^*}{\partial q_v} \right) \\ &+\frac{1}{2}q_v^T G^{-1} q_v - \frac{1}{4}\left(\frac{\partial \Xi^*}{\partial q_v} \right)^T E_1(q)G^{-1}E_1^T(q)\frac{\partial \Xi^*}{\partial q_v} \end{aligned} \tag{12}$$

From (12), we have

$$4q_v^T G^{-1} q_v - \left(\frac{\partial \Xi^*}{\partial q_v} \right)^T E_1(q)G^{-1}E_1^T(q)\frac{\partial \Xi^*}{\partial q_v} = 0 \tag{13}$$

Let

$$\frac{\partial \Xi^*}{\partial q_v} = \frac{2q_v(t)sgn(q_0(0))}{q_0} \tag{14}$$

where $q_0(0)$ denotes the initial state of q_0, and $sgn(x)$ is defined as follows:

$$sgn(x) = \begin{cases} 1, & for \;\; x \geq 0 \\ -1, & for \;\; x < 0 \end{cases}.$$

Since $s^T(q_v)q_v = (0,0,0)^T$, one obtains that $E_1^T(q)q_v = (-s^T(q_v) + q_0 I_{3\times3})q_v = q_0 q_v$. Hence, using (9) and (14) we obtain the optimal angular velocity as follows

$$\omega^* = -sgn(q_0(0))G^{-1}q_v \tag{15}$$

Verification and analysis

Noting that $q_0 = \pm\sqrt{1 - q_1^2 - q_2^2 - q_3^2}$ and solving partial equation (14), we obtain that

$$\Xi^*(t, q(t), \omega^*(t)) = c - 2sgn(q_0(0))q_0(t)$$

where c is a constant. Moreover, since $\Xi^*(0, q(0), \omega^*(0))$ should equal to 0 when the system initial states are at the equilibria, i.e., $q_0(0) = \pm 1$, we have $c = 2$ which means that

$$\Xi^*(t, q(t), \omega^*(t)) = 2 - 2sgn(q_0(0))q_0(t). \tag{16}$$

Thus if ω is optimal, from (16) we will have

$$\Xi(0, q(0), \omega(0)) = \Xi^*(0, q(0), \omega^*(0)) = 2 - 2|q_0(0)|.$$

Next let us verify it. Substituting control law (15) into (3) yields

$$\dot{q} = \frac{1}{2}E(q)\omega^* = -\frac{1}{2}\begin{pmatrix} -q_v^T \\ -s(q_v) + q_0 I_{3\times3}\end{pmatrix} sgn(q_0(0))G^{-1}q_v \tag{17}$$

Hence, we have $E(q)G^{-1}q_v = -2sgn(q_0(0))\dot{q}$. Note that $\dot{q}_0 = \frac{1}{2}sgn(q_0(0))q_v^T G^{-1}q_v$. If $q_0(0) \geq 0$, we have $\dot{q}_0(t) \geq 0$, which yields $q_0(t) \geq 0, \forall t \geq 0$. Also in this case, we have $\lim_{t\to\infty} q_0(t) = 1$, which we will prove it later. Similarly, if $q_0(0) < 0$, we also have $q_0(t) < 0, \forall t \geq 0$ and $\lim_{t\to\infty} q_0(t) = -1$.

Now suppose that $q_0(0) > 0$. Substituting control law (15) into (6) yields

$$\Xi(0, q(0), \omega(0)) = \int_0^{+\infty} q_v^T G^{-1} q_v \, dt$$

Noting that $E^T(q)E(q) = I_{3\times3}, E^T(q) = (-q_v, s(q_v) + q_0 I_{3\times3})$, then we have

$$\begin{aligned}
&\Xi(0, q(0), \omega(0)) \\
&= \int_0^{+\infty} q_v^T G^{-1} q_v \, dt = \int_0^{+\infty} q_v^T E^T(-2sgn(q_0(0))\dot{q}) \, dt \\
&= -2\int_0^{+\infty} (-q_v^T q_v \dot{q}_0 + q_0 q_1 \dot{q}_1 + q_0 q_2 \dot{q}_2 + q_0 q_3 \dot{q}_3) \, dt \\
&= 2\int_{q_0(0)}^{1} (1 - q_0^2) \, dq_0 - \int_0^{+\infty} \sqrt{1 - q_1^2 - q_2^2 - q_3^2} \frac{d(q_1^2 + q_2^2 + q_3^2)}{dt} \, dt \\
&= 2\int_{q_0(0)}^{1} (1 - q_0^2) \, dq_0 - \int_{1-q_0^2(0)}^{0} \sqrt{1-s} ds \\
&= 2 - 2q_0(0)
\end{aligned}$$

If $q_0(0) < 0$, by a similar calculation procedure, we will have $\Xi(0, q(0), \omega(0)) = 2 + 2q_0(0)$. Therefore we have $\Xi(0, q(0), \omega(0)) = \Xi^*(0, q(0), \omega^*(0)) = 2 - 2|q_0(0)|$.

Now let us verify the Lyapunov stability of the kinematic subsystem (3) under the optimal angular velocity (15). We can select a candidate Lyapunov function as follows:

$$V_q = (q_0 - sgn(q_0(0)))^2 + q_v^T q_v.$$

Using the property that $q_v^T s(q_v) = (0,0,0)$, according to (17) we have the derivative of V_q along system (3) and (15) as follows

$$\dot{V}_q = 2(q_0 - sgn(q_0(0)))\dot{q}_0 + 2q_v^T \dot{q}_v = -q_v^T G^{-1} q_v \tag{18}$$

Note that $G > 0$, we have $G^{-1} > 0$. Hence, it is obtained that $\dot{V}_q < 0$ for $q_v \neq 0$ such that q_v can converge to zero. If $q_0(0) \geq 0$, we have $\lim_{t\to\infty} q_0(t) = 1$. If $q_0(0) < 0$, we have $\lim_{t\to\infty} q_0(t) = -1$.

It should point out that the optimal solution for the performance index (6) is not unique. Two trivial solutions can also be obtained. To make Eq.(13) satisfied, we can also choose

$$\frac{\partial \Xi^*}{\partial q_v} = \frac{2q_v(t)}{q_0} \quad or \quad \frac{\partial \Xi^*}{\partial q_v} = -\frac{2q_v(t)}{q_0}$$

which means the following control laws

$$\omega^* = -G^{-1}q_v \tag{19}$$

and

$$\omega^* = G^{-1}q_v \tag{20}$$

are also optimal solutions under the performance index (6). The optimal performance indices in such cases are $\Xi^*(0, q(0), \omega^*(0)) = 2 - 2q_0(0)$ for the former and $\Xi^*(0, q(0), \omega^*(0)) = 2 + 2q_0(0)$ for the latter. These results can be obtained under a similar design and verification procedure.

The question arises: among these three optimal control laws, which is the best one? In both cases, almost global control results can be obtained and the control laws employ continuous feedback. However, both cases can merely obtain local minima, not a global minimum for the performance index (6).

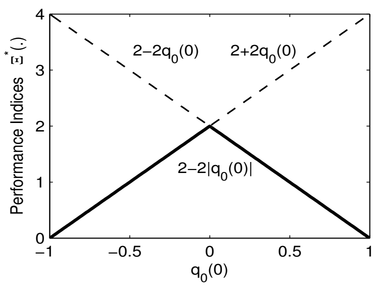

Fig. 2: Performance indices

The reason for such circumstance is that here we are handling the control problem of two equilibria. The optimal control laws (19) and (20) only consider one of the two equilibria which can only locally optimize the closed loop system. From Fig. 2, one can observe that the performance index $2 - 2q_0(0)$ is only minimal in the right half area, i.e.,

$0 \le q_0(0) \le 1$ and $2+2q_0(0)$ is only minimal in the left half area, i.e., $-1 \le q_0(0) < 0$. A natural way to get a globally optimal solution is to combine these two local properties together, i.e.,

$$\Xi^* = \min_{q_0(0)}\{2-2q_0(0), 2+2q_0(0)\} = 2-2|q_0(0)|.$$

As shown in Fig. 2, the solid curve which consists of two lines is a globally optimal solution. That is, to design a set control law considering both equilibria. To this end, we employ a nontrivial optimal control law (15) to get a global minimum $2-2|q_0(0)|$ for the performance index (6).

3.2 Finite-time control law design

In this section, We design a finite-time control law u such that the angular velocity ω will track the optimal angular velocity ω^*. Then we give a rigorous set stability analysis about the attitude control system. Let

$$\varepsilon(t) = (\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t))^T = \omega - \omega^*, \tag{21}$$

Combining (21)with (2) and (3), we have the following error model:

$$\dot{q} = \frac{1}{2}E(q)\omega = \frac{1}{2}E(q)\omega^* + \frac{1}{2}E(q)\varepsilon \tag{22}$$

$$J\dot{\varepsilon} = s(\omega)J\omega - J\dot{\omega}^* + u \tag{23}$$

where $\dot{\omega}^* = \frac{d\omega^*}{dt}$.

System (22) and (23) can be regarded as a cascaded system. The interconnection term is $\frac{1}{2}E(q)\varepsilon$. Now, we have the following main theorem:

Theorem 1: For spacecraft attitude control system (2) and (3), if the control law u is chosen as

$$u = -s(\omega)J\omega + J\dot{\omega}^* - k sig^\alpha(\varepsilon) \tag{24}$$

where $k > 0, sig^\alpha(\varepsilon_i) = |\varepsilon_i|^\alpha sign(\varepsilon_i), 0 < \alpha < 1, i = 1,2,3, sig^\alpha(\varepsilon) = (sig^\alpha(\varepsilon_1), sig^\alpha(\varepsilon_2), sig^\alpha(\varepsilon_3))^T$, and $sign$ the sign function. Then system (2)-(3) is globally asymptotically stable with respect to the equilibrium set $M_1 = \{(q^T, \varepsilon^T) : (-1,0,0,0,0,0,0), (1,0,0,0,0,0,0)\}$.

Proof: Substituting the control law (24) into system (23) yields

$$J\dot{\varepsilon} = -k sig^\alpha(\varepsilon) \tag{25}$$

(a)Local stability with respect to set M_1

The candidate Lyapunov function for system (22) and (25) can be selected as

$$V(q,\varepsilon) = (q_0 - sgn(q_0(0)))^2 + q_v^T q_v + \frac{1}{2}\varepsilon^T J \varepsilon. \tag{26}$$

The derivative of V along system (22) and (25) yields

$$\dot{V}(q,\varepsilon) = -k\varepsilon^T sig^\alpha(\varepsilon) - q_v^T G^{-1} q_v + sgn(q_0(0)) q_v^T \varepsilon. \tag{27}$$

Note that $q_i\varepsilon_i \le \gamma|q_i|^2 + \varepsilon_i^2/(4\gamma), i = 1,2,3$, where $\gamma > 0$ is the minimum eigenvalue of G^{-1}. We have

$$q_v^T\varepsilon \le \gamma q_v^T q_v + \varepsilon^T\varepsilon/(4\gamma).$$

Hence, we have the follow inequality:

$$\dot{V}(q,\varepsilon) \le -k\varepsilon^T sig^\alpha(\varepsilon) + \varepsilon^T\varepsilon/(4\gamma). \tag{28}$$

Let $\Omega = \{(q^T, \varepsilon^T)^T : |\varepsilon_i| \le (4k\gamma)^{\frac{1}{1-\alpha}}\}$. From (28), for all states in the set Ω, we have $\dot{V}(q,\varepsilon) \le 0$. From Lemma 3, system (22) and (25) is local stable with respect to set M_1 .

(b)Global attractivity with respect to set M_1.

Let's first consider the stability of error $\varepsilon(t)$. The candidate Lyapunov function for error $\varepsilon(t)$ can be selected as

$$V_\varepsilon = \frac{1}{2}\varepsilon^T J \varepsilon.$$

Substituting the control law (24) into the derivative of V_ε with (25) yields

$$\dot{V}_\varepsilon = \varepsilon^T[s(\omega)J\omega - J\dot{\omega}^* + u] = -k\varepsilon^T sig^\alpha(\varepsilon) \tag{29}$$

From (29), it follows that

$$\dot{V}_\varepsilon = -k(|\varepsilon_1|^{1+\alpha} + |\varepsilon_2|^{1+\alpha} + |\varepsilon_3|^{1+\alpha}).$$

By using Lemma 2, we have

$$(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)^{(1+\alpha)/2} \le |\varepsilon_1|^{1+\alpha} + |\varepsilon_2|^{1+\alpha} + |\varepsilon_3|^{1+\alpha}. \tag{30}$$

Let $J_{max} = max\{J_1, J_2, J_3\}$, from (30), we have

$$|\varepsilon_1|^{1+\alpha} + |\varepsilon_2|^{1+\alpha} + |\varepsilon_3|^{1+\alpha} \ge \frac{2^{(1+\alpha)/2} V_\varepsilon^{(1+\alpha)/2}}{J_{max}^{(1+\alpha)/2}}.$$

Hence, it is obtained that

$$\begin{aligned} \dot{V}_\varepsilon &= -k(|\varepsilon_1|^{1+\alpha} + |\varepsilon_2|^{1+\alpha} + |\varepsilon_3|^{1+\alpha}) \\ &\le -k\frac{2^{(1+\alpha)/2} V_\varepsilon^{(1+\alpha)/2}}{J_{max}^{(1+\alpha)/2}}. \end{aligned} \tag{31}$$

Let $c = \frac{2^{(1+\alpha)/2}}{J_{max}^{(1+\alpha)/2}}k$. Using (31), we have

$$\dot{V}_\varepsilon + cV_\varepsilon^{(1+\alpha)/2} \le 0. \tag{32}$$

Note that $0 < (1+\alpha)/2 < 1$. Using Lemma 1, we obtain that $\varepsilon(t)$ can be stabilized to zero in finite time.

Supposing that $\varepsilon(t)$ converge to zero at the moment $t = T_1$, we obtain $\omega = \omega^*$ for $t \ge T_1$. Hence, (3) reduces to system (17). According to the description in Section 3.1.2, we know that q_v converges to zero, q_0 converges to 1 or -1 and ω can also converge to zero since $\omega = \omega^* = -sgn(q_0(0)))G^{-1}q_v$ at time $t > T_1$. Hence we obtain that the states $(q_0, q_v^T, \varepsilon^T)$ can converge to the set $M_1 = \{(-1,0,0,0,0,0,0), (1,0,0,0,0,0,0)\}$ and ω can also converge to zero.

(c)Global boundedness

Note that $q_i, i = 0,1,2,3$ is always bounded. We only need to prove that ε is bounded. From (25), we know that ε is also bounded.

By Definition 2, we conclude that system (22)-(23) is globally asymptotically stable with respect to set M_1. Note that $\omega^* = -sgn(q_0(0)))G^{-1}q_v$ also converges to zero. The Lyapunov function (26) can also be considered as a set Lyapunov function for system (2)-(3), which implies the local set stability of system (2)-(3). It can be easily verified that system (2)-(3) also satisfies the global set attractivity and global boundedness. Hence, we can finally conclude that system (2)-(3) is globally asymptotically stable with respect to set M_1. This completes the proof.

4. SIMULATION RESULTS

In this section, we illustrate the previous theoretical results by means of numerical simulations. The parameters of model are given as: $J_1 = 72kg.m^2, J_2 = 60kg.m^2, J_3 = 50kg.m^2$. The parameters of control law (24) are selected as : $\alpha = 0.5, G = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0.6 \end{pmatrix}$. Hence, we have $G^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 5/3 \end{pmatrix}$, which implies $\gamma = 0.382, \| G^{-1} \| = 2.618$. We select k as $k = 8$. Substituting $\omega^* = -sgn(q_0(0))G^{-1}q_v$ into (24) yields:

$$\begin{aligned} u = & -s(\omega)J\omega - \frac{1}{2}sgn(q_0(0))JG^{-1}(q_0\omega - s(q_v)\omega) - \\ & ksig^{\alpha}(\omega + sgn(q_0(0))G^{-1}q_v) \end{aligned} \quad (33)$$

Let $e(0) = (0.4896, 0.2030, 0.8480)^T, \Phi(0) = 4m\pi + 2.4647rad, m \in Z$ (not unique). Then, the corresponding initial quaternion is $q(0) = [0.332, 0.4618, 0.1915, 0.7999]^T$. Let $\omega(0) = [-0.2, 0.3, 0.5]^T$. Note that $q_0(0) \geq 0$. Under the control law (33), we conclude that q_0 will be stabilized to 1. Fig.3 shows the response curves of attitude in the case of $q_0(0) \geq 0$. In Fig.3, we can see that q_0 converges to 1 since $q_0(0) \geq 0$.

5. CONCLUSION

In this paper, a control method based on optimal control and finite-time control techniques has been proposed for spacecraft attitude stabilization. To avoid the unwinding phenomenon and stabilize the states of the closed loop system to both of the two equilibria, we use set stabilization method to design the control law. In the design procedure of optimal angular velocity, we have demonstrated that only the stabilization law based on set control idea can obtain a globally optimal solution. The states can be stabilized to an equilibrium set. We have shown that the control method based on set control idea is more natural. The effectiveness of the proposed method has been demonstrated by simulation results.

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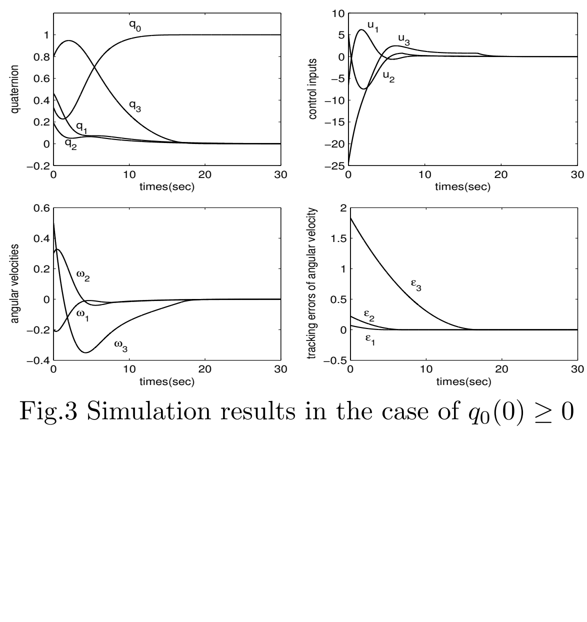

Fig.3 Simulation results in the case of $q_0(0) \geq 0$