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Robustness of Distributed Multi-Agent Consensus ⋆

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Abstract: Distributed consensus schemes in the presence of measurement errors are analyzed for both first-order and second-order dynamic agents. The connection between consensus error of coordination variables and the measurement errors is derived. Analytic estimates of consensus error are given. A design problem based on minimizing the "error gain" is thus identified. One simple example illustrates our results.

1. INTRODUCTION

Reaching consensus amongst identical dynamic agents is a fertile research topic with many recent contributions, see, for instance, Jadbabaie et al. (2003), Lin et al. (2004), Moore et al. (2007), Moreau (2005), Moshtagh et al. (2007), Olfati-Saber et al. (2004), Olfati-Saber et al. (2007), Ren et al. (2005), Ren et al. (2005), Ren et al. (2007), Savkin (2004) and references herein. Both first order and second order dynamics are considered. In the first order case ($\dot{x}_i = u_i$), it has been demonstrated that the appropriate communication graph must contain a spanning tree in order to achieve consensus, as discussed in Olfati-Saber et al. (2004), Jadbabaie et al. (2003), Lin et al. (2004) and Ren et al. (2005). This result was extended by Ren et al. (2007) for the second-order case ($\dot{x}_i = v_i, \dot{v}_i = u_i$). Note that perfect measurements were assumed in most existing literature on deterministic consensus, though robustness issues have been addressed under stochastic setting, for example, see papers in Ren et al. (2005), Huang et al. (2007), Xiao et al. (2007) and Hatano et al. (2005). In this paper, we revisit the deterministic consensus problem in the context of unavoidable measurement errors that belong to some functional spaces, for example, $\mathcal{L}_\infty$, $\mathcal{L}_2$ or $\mathcal{L}_1$. The problem is that consensus by its very nature does not automatically posses "nice" robust properties. So, to what extent is consensus reached in the presence of measurement errors?

We establish that bounded measurement errors lead to bounded errors in the consensus. A finite gain property is established. The gain notably depends on the communication topology. This opens a new non-trivial design problem in an optimization sense: which communication structure provides the smallest gain from measurement errors to consensus errors.

⋆ This work was finished during Dr. Wang's visiting in Department of Electrical and Electronic Engineering, University of Melbourne, Australia.

The paper is organized as follows. Section 2 reviews the consensus results for both first and second order dynamics, and establishes the notation. Next we introduce the main results and explain their significance. Proofs are sketched in the Appendix. An example illustrates the results before the conclusion.

2. PRELIMINARIES AND PROBLEM FORMULATION

A digraph consists of a pair $(\mathcal{A}, \mathcal{N})$, where $\mathcal{A}$ is a finite non-empty set of nodes and $\mathcal{N}$ is a set of ordered pairs of nodes, called edges. A path is a sequence of ordered edges of the form $(v_{i1}, v_{i2}), (v_{i2}, v_{i3}), \cdots$, where $v_{ij} \in \mathcal{A}$, in a diagraph. We say a communication graph $\mathcal{G}$ has a spanning tree if there exists at least one node from which there is a path that can reach every other node. A node is called a spanning node if there exists a path from that node that can reach every other node in the graph.

The set of real numbers is denoted as $\mathbb{R}$. For a vector $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\|$ denotes the infinity norm. For a matrix $A \in \mathbb{C}^{n\times n}$, $\|A\|$ is the induced norm of the vector norm. Given a measurable function $\mathbf{w}(t) \in \mathbb{R}^n$, we define its infinity norm $\|\mathbf{w}\|_\infty := \operatorname{ess\,sup}_{t\geq 0} \|w(t)\|$. If $\|\mathbf{w}\|_\infty < \infty$, then we write $\mathbf{w} \in \mathcal{L}_\infty$.

2.1 First-order dynamics

Let us consider the following collection of continuous-time first-order systems:

$$\dot{x}_i(t) = u_i(t), \qquad x_i(0) = x_{i,0}, \qquad \forall i = 1, 2, \cdots, n, \qquad (1)$$

where both x_i and u_i are scalar. The following consensus protocol has been proposed, (see, in Olfati-Saber et al. (2004), Jadbabaie et al. (2003), Lin et al. (2004) and Ren et al. (2005)):

$$u_i = -\sum_{j=1}^{n} g_{ij}k_{ij}(x_i - x_j), \quad i = 1, \ldots, n, \tag{2}$$

$$g_{ij} = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{No information flow from agents } j \text{ to } i \\ 1 & \text{Information flow from agents } j \text{ to } i \end{cases}. \tag{3}$$

Here k_{ij} is a weight on the information link from agent j to agent i. An adjacency matrix A of the information exchange topology is defined as $\{A\}_{i,j} = a_{ij}$, where $a_{ij} = g_{ij}k_{ij}$. The matrix A can be also written as $A = \left[A_1^T, \ldots, A_n^T\right]^T$, where $A_i = [a_{i1}, \cdots, a_{in}]$, $i = 1, 2, \cdots, n$.

The collective dynamics of the group of agents that follows system (1) and protocol (2) can be re-written as

$$\dot{\mathbf{x}} = -L\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{4}$$

where
$\mathbf{x} := \left[\, x_1,\, x_2,\, \cdots,\, x_n \,\right]^T$, $\mathbf{x}_0 := \left[\, x_{1,0},\, x_{2,0},\, \cdots,\, x_{n,0} \,\right]^T$, and L is the graph Laplacian of the network and its elements are defined as follows:

$$\{L\}_{i,j} = \ell_{ij} = \begin{cases} \sum_{i\neq k} a_{ik} & \text{if } i = j \\ -a_{ij} & \text{else} \end{cases}. \tag{5}$$

Given the graph Laplacian, $\alpha = [\alpha_1, \cdots, \alpha_n]^T$ is denoted as a left eigenvector of $-L$. That is $\alpha^T L = 0, \alpha \in \mathbb{R}^n$ with positive elements that sum to one. The graph Laplacian has the following property

Property 1. Assume that L is the graph Laplacian of the communication graph $\mathcal{G}$ that has a spanning tree. Then there exists a non-singular matrix P such that

$$-L = PJP^{-1}, \tag{6}$$

where $J = diag\{J_1, J_2, \cdots, J_s, 0\}$ is the Jordan canonical form of $-L$ with

$$J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix}_{n_i \times n_i}, \tag{7}$$

where $n_1 + n_2 + \cdots + n_s = n - 1$, and all $Re(\lambda_i) < 0$.

Moreover, P can be written as

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} T_{1n_1} & T_{1n_2} & \cdots & T_{1n_s} & t_0 \\ T_{2n_1} & T_{2n_2} & \cdots & T_{2n_s} & t_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{nn_1} & T_{nn_2} & \cdots & T_{nn_s} & t_0 \end{bmatrix}, \tag{8}$$

where T_{in_j} is a matrix with size $1 \times n_j$, $i = 1, 2, \cdots n$ and $j = 1, 2, \cdots, s$, and $t_0 \in \mathbb{R}$.

Sketch of proof: It is clear that $-L$ has a simple zero eigenvalue and other eigenvalues have negative real parts for any directed communication topology that has a spanning tree (Ren et al. (2005)). The Jordan decomposition of $-L$ takes the form as indicated. ◦

Definition 1. For the system (1) with the consensus protocol (2) (or (4) in the closed-loop form), the network is said to achieve consensus asymptotically if $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for each pair of agents (i, j), for all $i, j, = 1, \ldots, n$.

The following result provides a necessary and sufficient condition to ensure that the network determined by (1) and (2) (or (4) in the closed-loop form) reaches consensus asymptotically.

Result 1. (Ren et al. (2005)) Assume that the communication topology is time-invariant, i.e., all a_{ij} are constant. The system (1) with the consensus strategy (2) achieves global asymptotic consensus if and only if the communication graph $\mathcal{G}$ has a spanning tree. Moreover, $\lim_{t\to\infty} x_i(t) = \sum_{i=1}^{n} \alpha_i x_{i,0}$, where $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$ is the left eigenvector of $-L$.

2.2 Second-order dynamics

The following second-order system has been employed to study consensus when agents are moving vehicles, (see Ren et al. (2007) for more details).

$$\begin{cases} \dot{x}_i = v_i, & x_i(0) = x_{i,0} \\ \dot{v}_i = u_i, & v_i(0) = v_{i,0} \end{cases}, \qquad \forall i = 1, \cdots, n, \tag{9}$$

where scalars x_i and v_i are the position and the velocity of the agent i. $u_i \in \mathbb{R}$ is the control input. The following consensus protocol was proposed in Ren et al. (2007),

$$u_i = -\sum_{j=1}^{n} g_{ij}k_{ij}[(x_i - x_j) + \gamma(v_i - v_j)], \; i = 1, 2, \cdots, n, \tag{10}$$

where $g_{i,j}$ and $k_{ij} > 0$ are defined as (3). $\gamma > 0$ is a scaling factor.

The collective dynamics of the group of agents that follows system (9) and protocol (10) can be re-written as

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \Gamma \begin{bmatrix} x \\ v \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}, \tag{11}$$

where

$$x = [x_1, \cdots, x_n]^T, \quad v = [v_1, \cdots, v_n]^T$$
$$x_0 = [x_{1,0}, \cdots, x_{n,0}]^T, \quad v_0 = [v_{1,0}, \cdots, v_{n,0}]^T$$

and

$$\Gamma = \begin{bmatrix} 0_{n\times n} & I_n \\ -L & -\gamma L \end{bmatrix}, \tag{12}$$

where L is the graph Laplacian.

Similar to Property 1, we have the following property.

Property 2. Assume that Γ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. Then there exists a non-singular matrix Q such that

$$\Gamma = QJQ^{-1}, \tag{13}$$

where

$$J = \begin{bmatrix} 0 & 1 & 0_{1\times n_1} & \cdots & 0_{1\times n_k} \\ 0 & 0 & 0_{1\times n_1} & \cdots & 0_{1\times n_k} \\ 0_{n_1\times 1} & 0_{n_1\times 1} & J_1 & \cdots & 0_{n_1\times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_k\times 1} & 0_{n_k\times 1} & 0_{n_k\times n_1} & \cdots & J_k \end{bmatrix} \tag{14}$$

is the Jordan canonical form of Γ. J_i is defined by (7) which is a upper diagonal block matrix with size $n_i \times n_i$,

$n_1 + n_2 + \cdots + n_k + 2 = 2n$ and $i = 1, 2, \cdots, k$. λ_i is eigenvalue of Γ with $\mathrm{Re}\lambda_i < 0$, $\forall i = 1, 2, \cdots, k$.

Moreover, Q can be re-written as

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{2n} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & Q_{1n_1} & \cdots & Q_{1n_k} \\ q_{21} & q_{22} & Q_{2n_1} & \cdots & Q_{2n_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{2n1} & q_{2n2} & Q_{2nn_1} & \cdots & Q_{2nn_k} \end{bmatrix}, \quad (15)$$

where

$$\begin{array}{l} q_{11} = q_{21} = \cdots = q_{n1} = 1 \\ q_{(n+1)1} = q_{(n+2)1} = \cdots = q_{2n1} = 0 \\ q_{12} = q_{22} = \cdots = q_{n2} = m \, (m \in \mathbb{R}) \\ q_{(n+1)2} = q_{(n+2)2} = \cdots = q_{2n2} = 1 \end{array} \quad (16)$$

and Q_{in_j} is a matrix with size $1 \times n_j$ for $i = 1, \cdots, 2n$ and $j = 1, 2, \cdots, k$.

Sketch of proof: According to results in Ren et al. (2007), the eigenvalue zero of Γ has geometric multiplicity equal to one and algebraic multiplicity two. Then using a Jordan decomposition of Γ with a simple computation leads to (13). Let

$$Q_{ia} = \begin{bmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{ni} \end{bmatrix}, \quad Q_{ib} = \begin{bmatrix} q_{(n+1)i} \\ q_{(n+2)i} \\ \vdots \\ q_{2ni} \end{bmatrix}, \; i = 1, 2, \quad (17)$$

using(13) yields

$$\begin{bmatrix} 0_{n\times n} & I_n \\ -L & -\gamma L \end{bmatrix} \begin{bmatrix} Q_{1a} \\ Q_{1b} \end{bmatrix} = 0, \begin{bmatrix} 0_{n\times n} & I_n \\ -L & -\gamma L \end{bmatrix} \begin{bmatrix} Q_{2a} \\ Q_{2b} \end{bmatrix} = \begin{bmatrix} Q_{1a} \\ Q_{1b} \end{bmatrix}, \quad (18)$$

which implies that

$$Q_{1b} = 0, \; LQ_{1a} = 0, \; Q_{2b} = Q_{1a}, \; LQ_{2a} + \gamma LQ_{2b} = 0. \quad (19)$$

Since Γ has exactly two zero eigenvalues, it indicates that L has exactly one zero eigenvalue, i.e., $-L$ has only one linearly independent eigenvector Q_{1a}. Without loss of generality, we choose $Q_{1a} = \alpha_0 = [1, 1, \cdots, 1]^T$ and thus $Q_{2b} = \alpha_0, Q_{2a} = m\alpha_0$, where $m \in \mathbb{R}$. i.e., (16) holds. ◦

Definition 2. For the system (9) with the consensus protocol (10), the network is said to reach consensus asymptotically if $\begin{cases} \lim_{t\to\infty} |x_i(t) - x_j(t)| = 0 \\ \lim_{t\to\infty} |v_i(t) - v_j(t)| = 0 \end{cases}$, for each pair of agents (i, j), for all $i, j, = 1, \ldots, n$.

The necessary and sufficient condition that can achieve asymptotically consensus for the system (9) with (10) (or (11) in the closed-loop form) is presented in Result 2.

Result 2. (Ren et al. (2007)) System (9) with the consensus protocol (10) achieves consensus asymptotically if and only if the matrix Γ in (12) has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. Specifically, $x_i \to \sum_{i=1}^{n} \alpha_i x_{i,0} + t \sum_{i=1}^{n} \alpha_i v_{i,0}$ and $v_i \to \sum_{i=1}^{n} \alpha_i v_{i,0}$ for large t, where $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$ is a left eigenvector of $-L$.

2.3 Problem Formulation

In this paper, we assume that there exist measurement errors in (2)/or (10) for system (1)/ or (9). That is, for the first-order system (1), the input (2) becomes

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} (x_i - x_j - e_{ij}), \quad (20)$$

where e_{ij} represent measurement errors for x_i and/or x_j or disturbances on the communication channel from agent j to agent i. In the sequel, the closed-loop of the system (1) with (20) is written as

$$\dot{\mathbf{x}} = -L\mathbf{x} + B\mathbf{e}, \quad (21)$$

where L is as before, see (4) and

$$\mathbf{e} := \left[e_{11}, \cdots, e_{1n}, \cdots, e_{n1}, \cdots, e_{nn} \right]^T, \quad (22)$$

$$B = \begin{bmatrix} A_1 & 0_{1\times n} & \cdots & 0_{1\times n} \\ 0_{1\times n} & A_2 & \cdots & 0_{1\times n} \\ \cdots & \cdots & \cdots & \cdots \\ 0_{1\times n} & 0_{1\times n} & \cdots & A_n \end{bmatrix}. \quad (23)$$

Similarly, when the measurement errors are considered in (10), we write

$$u_i := -\sum_{j=1}^{n} a_{ij} \left[(x_i - x_j - e_{ij}) + \gamma (v_i - v_j - d_{ij}) \right], \quad (24)$$

where e_{ij} are measurement errors for position x_i and/or x_j and $d_{i,j}$ is the measurement error for velocity v_i and/or v_j on the communication channel from agent j to agent i. It follows that,

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \Gamma \begin{bmatrix} x \\ v \end{bmatrix} + \Lambda \eta, \quad (25)$$

where Γ is as before, see (12) and

$$\eta := \left[\mathbf{e}^T \; \mathbf{d}^T \right]^T, \quad (26)$$

$$\mathbf{d} := \left[d_{11}, \cdots, d_{1n}, \cdots, d_{n1}, \cdots, d_{nn} \right]^T, \quad (27)$$

$$\Lambda = \begin{bmatrix} 0_{n\times n^2} & 0_{n\times n^2} \\ B & \gamma B \end{bmatrix}, \quad (28)$$

where $\mathbf{e}$ is from (22) and B is defined in (23).

Let us assume that the consensus conditions of Result 1/or Result 2, as appropriate, are satisfied. When measurement errors are considered, what kind of "consensus" results can be obtained? In this paper, we will address these questions.

3. MAIN RESULTS

3.1 First-order dynamics

Our first result is stated as follows

Theorem 1. Assume that system (1) with the communication topology $\mathcal{G}$ defined by (20) contains a spanning tree, then there exist positive constants M and δ such that the system (1) with the communication topology (20) (or in closed-loop form (21)) satisfies

$$\|x_i(t) - x_j(t)\| \le Ke^{-\delta t}\|\mathbf{x_0}\| + \frac{2}{\delta}M\Pi\|e\|_\infty, \tag{29}$$

where $\mathbf{x}_0$ is from (4) and

$$K := 2M(\max_i \|P_i\|)\|P^{-1}\|, \Pi := (\max_i \|P_i\|)\|P^{-1}B\|.$$

The non-singular matrix P is from the Jordan decomposition of $-L$ (see (6) and (8) in Property 1).

3.2 Second-order dynamics

Our second result is for the second-order dynamics.

Theorem 2. Assume that the matrix Γ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. Then there exist positive constants M and δ such that the system (9) with the communication topology (24) (or (25) in the closed-loop form) satisfies

$$||x_i(t) - x_j(t)|| \le Ke^{-\delta t}\left\|\begin{bmatrix}\mathbf{x}_0\\ \mathbf{v}_0\end{bmatrix}\right\| + \frac{2}{\delta}M\Xi\,\|\eta\|_\infty \tag{30}$$

and

$$||v_i(t) - v_j(t)|| \le Ke^{-\delta t}\left\|\begin{bmatrix}\mathbf{x}_0\\ \mathbf{v}_0\end{bmatrix}\right\| + \frac{2}{\delta}M\Xi\,\|\eta\|_\infty\,, \tag{31}$$

where $\mathbf{x}_0$ and $\mathbf{v}_0$ is from (11) and

$$K := 2M(\max_i ||Q_i||)\|Q^{-1}\|, \Xi := (\max_i ||Q_i||)\|Q^{-1}\Lambda\|.$$

The non-singular matrix Q is the Jordan decomposition of Γ (see (13) and (15) in Property 2).

Remark 1. Theorem 1 and Theorem 2 provide noteworthy properties of consensus in the presence of measurement errors for first-order/second-order dynamics. Because at first glance, due to one zero eigenvalue of L (first-order dynamics) /two zero eigenvalues of Γ (second-order dynamics), one would not expect a robustness property in general. Due to the specific structure of the errors, which preserve the communication topology, this robustness is realized.

Remark 2. Theorem 1 and Theorem 2 also estimate the constant gain between the measurement error and errors in consensus. This gain is notably determined by communication topology (see (29) for the first-order dynamics/or (30) and (31) for the second-order dynamics). Therefore the estimate gain in (29)/or (30) and (31) open a new venue to design the optimal communication topology, which can minimize the gain from measurement errors to consensus errors. We will address this interesting design issue in the future.

Remark 3. Although in Theorem 1/Theorem 2, the measurement errors can be either constant or time-varying functions that belong to $\mathcal{L}_\infty$ (see (29)/(30) and (31)), our results are applicable to measurement errors ($\mathbf{e}$ or η) that are in $\mathcal{L}_2$ or $\mathcal{L}_1$ with a slight modification of the gain from the measurement errors to consensus errors.

4. NUMERICAL EXAMPLE

In this paper, we just consider a simple example due to space limitation.

The information exchange topology is a leader-follower used in Ren et al. (2007), whose structure is shown in Figure 1. First-order dynamics and second-order dynamics are both considered.

In the example, $k_{ij} = 1, \forall i, j = 1, \cdots, 4$, for both the first-order dynamics and the second-order dynamics.

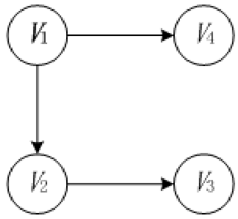

Fig. 1. A digraph with a leader-follower topology

4.1 First-order dynamics

It is easy to test that the conditions of Theorem 1 are satisfied. The initial condition is chosen as $(1.5, 3, 5, 0.5)^T$.

For the first-order systems (21), measurement errors of communication channel in the example are set to be constants, i.e.,

$$\mathbf{e} = \mathbf{e_0} = [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]^T, \tag{32}$$

which belongs to $\mathcal{L}_\infty$. Figure 2 shows the evolution of each state x_i, $i = 1, \cdots, 4$ by using protocol (20) under given the information exchange topology. Obviously, $|x_i - x_j|$ is bounded, for any $i = 1, \cdots, 4$ and $j = 1, \cdots, 4$. This result verifies the first main result (Theorem 1) of this paper.

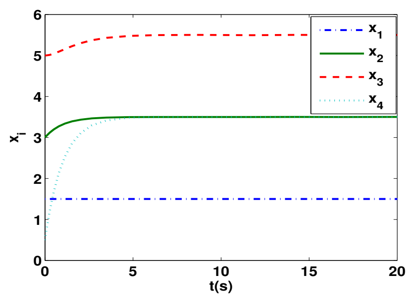

Fig. 2. The evolution of each state x_i, $i = 1, \cdots, 4$ with communication errors $\mathbf{e}_0$.

4.2 Second-order dynamics

γ is chosen to be one in the second-order dynamics (10). The initial condition of system (25) is chosen to be $(1, 3, 5, 0.5, 0.5, 2, 3.5, 2)^T$.

Conditions of Theorem 2 are also satisfied for the given communication topology. Time-varying communication errors are considered under such a situation:

$$\mathbf{e}(t) = \sin(t)\mathbf{e}_0 \tag{33}$$

$$\mathbf{d}(t) = \cos(t)\mathbf{e}_0, \tag{34}$$

which implies that $\eta(t) \in \mathcal{L}_\infty$. Figure 3 shows the evolution of the state x_i and v_i, $i = 1, \cdots, 4$, by using protocol (24) under given communication topology.

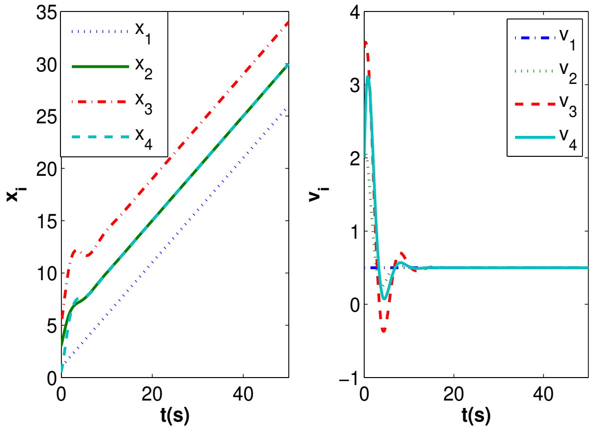

Fig. 3. The evolution of the states x_i and v_i, $i = 1, \cdots, 4$ with communication errors η.

From Figure 3, it is apparent that both $|x_i - x_j|$ and $|v_i - v_j|$ are bounded under given communication topology. This illustrates the robustness of consensus as discussed in Theorem 2.

5. CONCLUSION

In this paper, robustness properties of distributed multi-agent consensus for the first-order dynamics and the second-order dynamics are addressed. Our results show that consensus errors are proportional to measurement errors and the gain between them is determined by the communication topology of the system. Therefore, these results provide a useful tool to design communication topology that can minimize the gain from the measurement errors to consensus errors.

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Appendix A. SKETCH OF PROOF OF MAIN RESULTS

Sketch of proof: Theorem 1:

The solution of equation (21) can be expressed as

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \Phi(t)\int_0^t \Phi(s)^{-1}B\mathbf{e}(s)ds, \qquad \text{(A.1)}$$

where

$$\Phi(t) = e^{-Lt} = Pe^{Jt}P^{-1} = P\begin{bmatrix} e^{J_1 t} & & & & \\ & e^{J_2 t} & & & \\ & & \ddots & & \\ & & & e^{J_s t} & \\ & & & & 1 \end{bmatrix}P^{-1}.$$

Let

$$P^{-1}\mathbf{x}_0 = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_s \\ \xi_0 \end{bmatrix}, \quad P^{-1}B\mathbf{e}(s) = \begin{bmatrix} h_1(s) \\ h_2(s) \\ \vdots \\ h_s(s) \\ h_0(s) \end{bmatrix} \qquad \text{(A.2)}$$

where ξ_i and $h_i(s)$ are respectively vector and vector function with dimension n_i, $\forall i = 1, 2, \cdots, s$. ξ_0 and $h_0(s)$ are respectively scalar and scalar function. Then

$$\mathbf{x}(t) = P \begin{bmatrix} e^{J_1 t}\xi_1 \\ \vdots \\ e^{J_s t}\xi_s \\ \xi_0 \end{bmatrix} + P\int_0^t \begin{bmatrix} e^{J_1(t-s)}h_1(s) \\ e^{J_2(t-s)}h_2(s) \\ \vdots \\ e^{J_s(t-s)}h_s(s) \\ h_0(s) \end{bmatrix} ds. \quad \text{(A.3)}$$

By (A.1), (A.3) and (8) the state of i^{th} agent can be expressed as

$$x_i(t) = T_{in_1}e^{J_1 t}\xi_1 + \cdots + T_{in_s}e^{J_s t}\xi_s + t_0\xi_0 + \int_0^t \left[T_{in_1}e^{J_1(t-s)}h_1(s) + \cdots + T_{in_s}e^{J_s(t-s)}h_s(s) + t_0h_0(s)\right] ds. \quad \text{(A.4)}$$

In the sequel

$$x_i(t) - x_j(t) = \Upsilon\Psi(t,0)RT^{-1}x_0 + \int_0^t \Upsilon\Psi(t,s)RT^{-1}Be(s)ds \quad \text{(A.5)}$$

where

$$R = [I_{n-1} \quad 0_{(n-1)\times 1}],$$
$$\Upsilon = \left[\, T_{in_1} - T_{jn_1} \ \cdots \ T_{in_s} - T_{jn_s} \,\right],$$
$$\Psi(t,s) = \begin{bmatrix} e^{J_1(t-s)} & & \\ & \ddots & \\ & & e^{J_s(t-s)} \end{bmatrix}.$$

Since all the eigenvalues of $J_1, J_2, \cdots, J_s$ have negative real parts there exist scalars $M > 0$ and $\delta > 0$ such that

$$\|\Psi(t,s)\| \le Me^{-\delta(t-s)} \quad \text{(A.6)}$$

Therefore (29) follows from (A.5) by using (A.6). This completes the proof. □

Sketch of proof: Theorem 2:

Note that

$$e^{\Gamma t} = Qe^{Jt}Q^{-1} = Q\begin{bmatrix} 1 & t & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & e^{J_1 t} & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & e^{J_k t} \end{bmatrix} Q^{-1}.$$

Let

$$Q^{-1}\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_{n_1} \\ \vdots \\ \zeta_{n_k} \end{bmatrix}, \quad Q^{-1}\Lambda\begin{bmatrix} e(s) \\ d(s) \end{bmatrix} = \begin{bmatrix} h_0(s) \\ h_1(s) \\ h_{n_1}(s) \\ \vdots \\ h_{n_k}(s) \end{bmatrix}$$

where ζ_0, ζ_1 and $h_0(s)$,$h_1(s)$ are respectively scalars and scalar functions. ζ_{n_i} and $h_{n_i}(s)$ are respectively vector and vector function with dimension n_i, $i = 1, 2, \cdots, k$. Then the solution of (25) can be expressed as

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{bmatrix} = e^{\Gamma t}\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{v}_0 \end{bmatrix} + \int_0^t e^{\Gamma(t-s)}\Lambda\begin{bmatrix} \mathbf{e}(s) \\ \mathbf{d}(s) \end{bmatrix} ds = Q\begin{bmatrix} \zeta_0 + t\zeta_1 \\ \zeta_1 \\ e^{J_1 t}\zeta_{n_1} \\ \vdots \\ e^{J_k t}\zeta_{n_k} \end{bmatrix} + Q\int_0^t \begin{bmatrix} 1 & t-s & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & e^{J_1(t-s)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{J_k(t-s)} \end{bmatrix}\begin{bmatrix} h_0(s) \\ h_1(s) \\ h_{n_1}(s) \\ \vdots \\ h_{n_k}(s) \end{bmatrix} ds. \quad \text{(A.7)}$$

Substituting (15) into (A.7), with the help of property , it yeilds

$$\mathbf{x}(t) = \Delta_1 + \int_0^t \Delta_2 ds, \quad \text{(A.8)}$$

where

$$\Delta_1 = \begin{bmatrix} \zeta_0 + t\zeta_1 + m\zeta_1 + Q_{1n_1}e^{J_1 t}\zeta_{n_1} + \cdots + Q_{1n_k}e^{J_k t}\zeta_{n_k} \\ \zeta_0 + t\zeta_1 + m\zeta_1 + Q_{2n_1}e^{J_1 t}\zeta_{n_1} + \cdots + Q_{2n_k}e^{J_k t}\zeta_{n_k} \\ \vdots \\ \zeta_0 + t\zeta_1 + m\zeta_1 + Q_{nn_1}e^{J_1 t}\zeta_{n_1} + \cdots + Q_{nn_k}e^{J_k t}\zeta_{n_k} \end{bmatrix}, \quad \text{(A.9)}$$

$$\Delta_2 = \begin{bmatrix} h(t,s) + Q_{1n_1}e^{J_1(t-s)}h_{n_1}(s) + \cdots + Q_{1n_k}e^{J_k(t-s)}h_{n_k}(s) \\ h(t,s) + Q_{2n_1}e^{J_1(t-s)}h_{n_1}(s) + \cdots + Q_{2n_k}e^{J_k(t-s)}h_{n_k}(s) \\ \vdots \\ h(t,s) + Q_{nn_1}e^{J_1(t-s)}h_{n_1}(s) + \cdots + Q_{nn_k}e^{J_k(t-s)}h_{n_k}(s) \end{bmatrix}, \quad \text{(A.10)}$$

in which $h(t,s) = h_0(s) + (t-s)h_1(s) + mh_1(s)$, and

$$\mathbf{v}(t) = \Delta_3 + \int_0^t \Delta_4 ds, \quad \text{(A.11)}$$

where

$$\Delta_3 = \begin{bmatrix} \zeta_1 + Q_{(n+1)n_1}e^{J_1 t}\zeta_{n_1} + \cdots + Q_{(n+1)n_k}e^{J_k t}\zeta_{n_k} \\ \zeta_1 + Q_{(n+2)n_1}e^{J_1 t}\zeta_{n_1} + \cdots + Q_{(n+2)n_k}e^{J_k t}\zeta_{n_k} \\ \vdots \\ \zeta_1 + Q_{2nn_1}e^{J_1 t}\zeta_{n_1} + \cdots + Q_{2nn_k}e^{J_k t}\zeta_{n_k} \end{bmatrix} \quad \text{(A.12)}$$

$$\Delta_4 = \begin{bmatrix} h_1(s) + Q_{(n+1)n_1}e^{J_1(t-s)}h_{n_1}(s) + \cdots + Q_{(n+1)n_k}e^{J_k(t-s)}h_{n_k}(s) \\ h_1(s) + Q_{(n+2)n_1}e^{J_1(t-s)}h_{n_1}(s) + \cdots + Q_{(n+2)n_k}e^{J_k(t-s)}h_{n_k}(s) \\ \vdots \\ h_1(s) + Q_{2nn_1}e^{J_1(t-s)}h_{n_1}(s) + \cdots + Q_{2nn_k}e^{J_k(t-s)}h_{n_k}(s) \end{bmatrix} \quad \text{(A.13)}$$

Therefore from (A.8)- (A.13) we have

$$x_i(t) - x_j(t) = N_1\Theta(t,0)NQ^{-1}\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} + \int_0^t N_1\Theta(t,s)NQ^{-1}\Lambda\begin{bmatrix} e(s) \\ d(s) \end{bmatrix} ds, \quad \text{(A.14)}$$

and

$$v_i(t) - v_j(t) = N_2\Theta(t,0)NQ^{-1}\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} + \int_0^t N_2\Theta(t,s)NQ^{-1}\Lambda\begin{bmatrix} e(s) \\ d(s) \end{bmatrix} ds \quad \text{(A.15)}$$

for $i, j = 1, 2, \cdots, n$, where

$$N = [0_{(2n-2)\times 1} \quad 0_{(2n-2)\times 1} \quad I_{2n-2}]$$
$$N_1 = \left[\, Q_{in_1} - Q_{jn_1} \ \cdots \ Q_{in_k} - Q_{jn_k} \,\right],$$
$$N_2 = \left[\, Q_{(n+i)n_1} - Q_{(n+j)n_1} \ \cdots \ Q_{(n+i)n_k} - P_{(n+j)n_k} \,\right],$$

and

$$\Theta(t,s) = \begin{bmatrix} e^{J_1(t-s)} & & \\ & \ddots & \\ & & e^{J_k(t-s)} \end{bmatrix}.$$

Following the same step in the proof of Theorem 1, the proof completes. □