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Quantum LQG Control with Quantum Mechanical Controllers ⋆

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Abstract: Based on a recently developed notion of physical realizability for quantum linear stochastic systems, we formulate a quantum LQG optimal control problem for quantum linear stochastic systems where the controller itself may also be a quantum system and the plant output signal can be fully quantum. This is distinct from previous works on the quantum LQG problem where measurement is performed on the plant and the measurement signals are used as input to a fully classical controller with no quantum degrees of freedom. The difference in our formulation is the presence of additional non-linear and linear constraints on the coefficients of the sought after controller, rendering the problem as a type of constrained controller problem. Due to the presence of these constraints our problem is inherently computationally hard and this distinguishes it in an important way from the standard LQG problem. We propose a numerical procedure for solving this problem based on an alternating projections algorithm and, as initial demonstration of the feasibility of this approach, we provide a fully quantum controller design example in which a numerical solution to the problem was successfully obtained.

1. INTRODUCTION

Recent successes in quantum and nano-technology have provided a great impetus for research in the area of quantum feedback control systems; e.g., see Belavkin [1983], Doherty and Jacobs [1999], Armen et al. [2002], Geremia et al. [2004]. One particular area in which significant theoretical and experimental advances have been achieved is *quantum optics*. In particular, *linear quantum optics* is one of the possible platforms being investigated for building future quantum computers Knill et al. [2001], [Nielsen and Chuang, 2000, Section 7.5], besides being an area of independent interest in physics. Interestingly, under appropriate assumptions, the dynamics of some quantum optical devices can be approximately modelled by linear quantum stochastic differential equations (QSDEs) driven by quantum Wiener processes Gardiner and Zoller [2000]. For details on QDSEs and quantum Wiener processes, see, e.g., Hudson and Parthasarathy [1984], Parthasarathy [1992], Bouten et al. [2007].

In this work we build on the ideas in James et al. [2007] and Shaiju et al. [2007] and formulate a quantum LQG optimal control problem for quantum linear stochastic systems represented by linear QSDEs. The distinguishing feature of our work compared to previous treatments of the quantum LQG problem in the literature is that we allow the controller to be another quantum system whereas previous works only consider the case where the controller is a classical system. We stress that this is an important distinction and leads to a more difficult problem which cannot be solved using the usual approach of quantum conditioning and dynamic programming. By viewing the problem as a polynomial matrix programming problem, we show that by utilizing a non-linear change of variables due to Scherer et al. [1997], the problem can be systematically converted to a rank constrained LMI problem. To demonstrate the feasibility of numerically solving this problem, we provide a design example of stabilization of a quantum plant for which a solution to the rank constrained LMI problem was successfully obtained using an alternating projections algorithm due to Orsi et al. [2006].

2. GENERAL QUANTUM LINEAR STOCHASTIC MODELS IN QUANTUM OPTICS

We follow the quantum probabilistic setup of [James et al., 2007, Section II] to describe the quantum stochastic models of interest. To this end, consider linear non-commutative stochastic systems of the form

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t); \quad x(0) = x_0 \\ dy(t) &= Cx(t)dt + Ddw(t) \end{aligned} \qquad (1)$$

where A, B, C and D are, respectively, real $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_w}$, $\mathbb{R}^{n_y\times n}$ and $\mathbb{R}^{n_y\times n_w}$ matrices (n, n_w, n_y are positive integers), and $x(t) = [\, x_1(t)\ \dots\ x_n(t)\,]^T$ is a vector of self-adjoint possibly non-commutative system variables. The

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initial system variables $x(0) = x_0$ are Gaussian with state ρ, and satisfy the *commutation relations*[1]

$$[x_j(0), x_k(0)] = 2i\Theta_{jk}, \quad j, k = 1, \ldots, n, \qquad (2)$$

where Θ is a real antisymmetric matrix with components Θ_{jk}, and $i = \sqrt{-1}$. Here, the commutator is defined by $[A, B] = AB - BA$. To simplify matters without loss of generality, we take the matrix Θ to be of one of the following forms: (i) *Canonical* if $\Theta = \mathrm{diag}(J, J, \ldots, J)$, or (ii) *Degenerate canonical* if $\Theta = \mathrm{diag}(0_{n' \times n'}, J, \ldots, J)$, where $0 < n' \leq n$. Here, J denotes the real skew-symmetric 2×2 matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the "diag" notation indicates a block diagonal matrix assembled from the given entries. To illustrate, the case of a system with one classical variable and two conjugate quantum variables is characterized by $\Theta = \mathrm{diag}(0, J)$, which is degenerate canonical. The vector quantity w describes the input signals and is assumed to admit the decomposition

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t) \qquad (3)$$

where $\tilde{w}(t)$ is the noise part of $w(t)$ and $\beta_w(t)$ is a self adjoint, adapted process (see Hudson and Parthasarathy [1984], Parthasarathy [1992], Bouten et al. [2007] for a discussion of adapted quantum processes). The process $\beta_w(t)$ serves to model variables of other systems which may be passed to the system (1) via an interconnection. It is also represented on the same quantum probability space, enlarged if necessary. Consequently, we assume that components of $\beta_w(t)$ commute with those of $dw(t)$. Furthermore, we will also assume that components of $\beta_w(t)$ commute with those of $x(t)$; this will simplify matters for the present work. The noise $\tilde{w}(t)$ is a vector of self-adjoint quantum noises with Ito table $d\tilde{w}(t)d\tilde{w}^T(t) = F_{\tilde{w}}dt$,, where $F_{\tilde{w}}$ is a non-negative Hermitian matrix; e.g., see Parthasarathy [1992], Belavkin [1991]. This determines the following commutation relations for the noise components $d\tilde{w}(t)d\tilde{w}^T(t) - (d\tilde{w}(t)d\tilde{w}^T(t))^T = 2T_{\tilde{w}}dt$, where we use the notation $S_{\tilde{w}} = \frac{1}{2}(F_{\tilde{w}} + F_{\tilde{w}}^T)$, $T_{\tilde{w}} = \frac{1}{2}(F_{\tilde{w}} - F_{\tilde{w}}^T)$ so that $F_{\tilde{w}} = S_{\tilde{w}} + T_{\tilde{w}}$. For instance, $F_{\tilde{w}} = \mathrm{diag}(1, I + iJ)$ describes a noise vector with one classical component and a pair of conjugate quantum noises (here I is the 2×2 identity matrix).

For simplicity, we also adopt the conventions of James et al. [2007] to put the system (1) into a standard form. Therefore, we assume that (1): (i) n_y is even, and (ii) $n_w \geq n_y$. Furthermore, we also assume that $F_{\tilde{w}}$ is of the *canonical* form $F_{\tilde{w}} = I + i\mathrm{diag}(J, \ldots, J)$. Hence n_w has to be even. Note that if $F_{\tilde{w}}$ is not canonical but of the form $F_{\tilde{w}} = I + i\mathrm{diag}(0_{n' \times n'}, \mathrm{diag}(J, \ldots, J))$ with $n' \geq 1$, we may enlarge $w(t)$ (and hence also $\tilde{w}(t)$) and B as before such that the enlarged noise vector, say $\tilde{w}'$, can be taken to have an Ito matrix $F_{\tilde{w}'}$ which is canonical.

3. FORMULATION OF QUANTUM LQG PROBLEM

We consider *plants* described by non-commutative stochastic models of the following form:

[1] In the case of a single degree of freedom quantum particle, $x = (x_1, x_2)^T$ where $x_1 = q$ is the position operator, and $x_2 = p$ is the momentum operator. The annihilation operator is $a = (q + ip)/2$. The commutation relations are $[a, a^*] = 1$, or $[q, p] = 2i$.

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdu(t) + B_w dw(t); \quad x(0) = x; \\ dy(t) &= Cx(t)dt + D_w dw(t); \\ z(t) &= C_z x(t) + D_z \beta_u(t). \end{aligned} \qquad (4)$$

Here $x(t)$ is a vector of plant variables, $w(t)$ is a quantum Wiener disturbance vector, $\beta_u(t)$ is an adapted, self-adjoint process commuting with $x(t)$ (i.e., $\beta_u(t)x(t)^T - (x(t)\beta_u(t)^T)^T = 0$), and $u(t)$ is a *control* input of the form

$$du(t) = \beta_u(t)dt + d\tilde{u}(t) \qquad (5)$$

where $\beta_u(t)$ is the "signal part" and $\tilde{u}(t)$ is the noise part of $u(t)$. The vectors $w(t)$ and $\tilde{u}(t)$ are independent quantum noises (meaning that they live on distinct Fock spaces) with Ito matrices F_w and $F_{\tilde{u}}$ which are all non-negative Hermitian. We also assume that $x(0)x(0)^T - (x(0)x(0)^T)^T = \Theta$.

Controllers are assumed to be non-commutative stochastic systems of the form

$$\begin{aligned} d\xi(t) &= A_K\xi(t)dt + B_{K1}dw_{K1}(t) + B_{K2}dw_{K2}(t) \\ &\quad + B_{K3}dy(t); \\ du(t) &= C_K\xi(t)dt + dw_{K1}(t) \end{aligned} \qquad (6)$$

where $\xi(t) = [\,\xi_1(t) \ \ldots \ \xi_{n_K}(t)\,]^T$ is a vector of self-adjoint controller variables of the same dimension as $x(t)$ (i.e., the controller is of the same order as the plant), B_{K2} is a square matrix of the same dimension as A_K, and B_{K1} has the same number of columns as there are rows of C_K. The noises $w_{Ki}(t)$, $i = 1, 2$, are a vector of non-commutative Wiener processes (in vacuum states) with non-zero Ito products and which are independent of $w(t)$. We assume that $\xi(0)\xi(0)^T - (\xi(0)\xi(0)^T)^T = \Theta_K$.

Assume further that $x(0)\xi(0)^T - (\xi(0)x(0)^T)^T = 0$, i.e., the plant and controller are initially decoupled. The *closed loop system* is obtained by the identification $\beta_u(t) \equiv C_K\xi(t)$ and $\tilde{u}(t) \equiv w_{K1}(t)$, and interconnecting (4) and (6) to give

$$\begin{aligned} d\eta(t) &= \mathcal{A}\eta(t)dt + \mathcal{B}dw_{cl}(t) \\ z(t) &= \mathcal{C}\eta(t) \end{aligned} \qquad (7)$$

where $\eta(t) = [\,x(t)^T \ \xi(t)^T\,]^T$,

$$w_{cl}(t) = \begin{bmatrix} w(t) \\ w_{K1}(t) \\ w_{K2}(t) \end{bmatrix}; \quad \mathcal{A} = \begin{bmatrix} A & BC_K \\ B_{K3}C & A_K \end{bmatrix};$$

$$\mathcal{B} = \begin{bmatrix} B_w & B & 0_{2\times 2} \\ B_{K3}D_w & B_{K1} & B_{K2} \end{bmatrix}; \quad \mathcal{C} = [\,C_z \ \ D_zC_K\,];$$

With (7) we associate a quadratic performance index.

$$J(t_f) = \int_0^{t_f} \langle z^T(t)z(t) \rangle \, dt. \qquad (8)$$

We shall proceed to derive an explicit expression for this index; see also Shaiju et al. [2007]. To this end, define the symmetrized covariance matrix $P(t)$ by:

$$P(t) = \frac{1}{2}\langle \eta(t)\eta^T(t) + (\eta(t)\eta^T(t))^T \rangle. \qquad (9)$$

Using the quantum Ito rule, we have

$$\begin{aligned} dP(t) &= \frac{1}{2}(\langle d\eta(t)\,\eta^T(t)\rangle + \langle (d\eta(t)\,\eta^T(t))^T\rangle + \\ &\quad \langle \eta(t)\,d\eta^T(t)\rangle + \langle (\eta(t)\,d\eta^T(t))^T\rangle + \\ &\quad (\mathcal{B}F_{w_{cl}}\mathcal{B}^T + (\mathcal{B}F_{w_{cl}}\mathcal{B}^T)^T)\ dt) \\ &= (\mathcal{A}P(t) + P(t)\mathcal{A}^T + \frac{1}{2}\mathcal{B}(F_{w_{cl}} + F_{w_{cl}}^T)\mathcal{B}^T)\ dt, \\ &= (\mathcal{A}P(t) + P(t)\mathcal{A}^T + \mathcal{B}\mathcal{B}^T)\ dt, \end{aligned}$$

where the last equality follows from our convention that all noises are canonical (hence $\frac{1}{2}(F_{w_{cl}} + F_{w_{cl}}^T) = I$). Hence $P(\cdot)$ satisfies the differential equation

$$\dot{P}(t) = \mathcal{A}P(t) + P(t)\mathcal{A}^T + \mathcal{B}\mathcal{B}^T; \quad P(0) = P_0. \tag{10}$$

We now have, using the symmetry of $\mathcal{C}^T\mathcal{C}$ and P,

$$\begin{aligned} \langle z^T z\rangle &= \langle \eta^T\mathcal{C}^T\mathcal{C}\eta\rangle = \langle \mathrm{Tr}(\eta^T\mathcal{C}^T\mathcal{C}\eta)\rangle \\ &= \frac{1}{2}\langle \mathrm{Tr}(\mathcal{C}^T\mathcal{C}[\eta\eta^T + (\eta\eta^T)^T])\rangle = \mathrm{Tr}(\mathcal{C}^T\mathcal{C}P). \end{aligned}$$

Hence the index (8) can be expressed as

$$J(t_f) = \int_0^{t_f} \mathrm{Tr}(\mathcal{C}^T\mathcal{C}P(t))\ dt \tag{11}$$

where $P(t)$ solves (10). We will focus our attention on the infinite horizon case where we allow $t_f \uparrow \infty$. Assuming that $\mathcal{A}$ is asymptotically stable, standard results on Lyapunov equations give us $\lim_{t\to\infty} P(t) = P$, where P is the unique symmetric positive definite solution of the Lyapunov equation:

$$\mathcal{A}P + P\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = 0. \tag{12}$$

Furthermore, by standard methods of analysis we have

$$\limsup_{t_f\to\infty} \frac{1}{t_f}\int_0^{t_f} \langle z^T(t)z(t)\rangle dt = \mathrm{Tr}(\mathcal{C}^T\mathcal{C}P) = \mathrm{Tr}(\mathcal{C}P\mathcal{C}^T).$$

Let $\mathrm{diag}_m(J)$ denote a block diagonal $2m\times 2m$ matrix with m J matrices on the diagonal block and let n_i denote the dimension of w_{Ki} for $i = 1, 2, 3$. We may now formulate our cost bounded quantum LQG control problem:

Problem 1. Given a fixed choice of Θ_K and a cost bound parameter $\gamma > 0$, find controller matrices A_K, B_{K1}, B_{K2}, B_{K3} and C_K such that

- F1. There exists a symmetric matrix $P > 0$ satisfying (12).
- F2. $J_\infty = \mathrm{Tr}(\mathcal{C}P\mathcal{C}^T) < \gamma$.
- F3. The controller (6) is physically realizable. That is, it satisfies the conditions of [James et al., 2007, Theorem 3.4] with the identification $A \equiv A_K$, $B = [\,B_{K1}\ B_{K2}\ B_{K3}\,]$, $C \equiv C_K$, $D \equiv [\,I_{n_u\times n_u}\ 0\,]$, and $w \equiv [\,w_{K1}^T\ w_{K2}^T\ y^T\,]^T$, leading to the constraints:

$$\begin{aligned} &A_K\Theta_K + \Theta_K A_K^T + B_{K1}\,\mathrm{diag}_{\frac{n_1}{2}}(J)B_{K1}^T + \\ &B_{K2}\,\mathrm{diag}_{\frac{n_2}{2}}(J)B_{K2}^T + B_{K3}\mathrm{diag}_{\frac{n_3}{2}}(J)B_{K3}^T = 0 \end{aligned} \tag{13}$$

$$B_{K1} = \Theta_K C_K^T\,\mathrm{diag}_{n_u/2}(J). \tag{14}$$

In the above problem Θ_K is a fixed but freely specified real skew symmetric matrix that determines the type of controller sought (see also Remark 2 below). For example, if Θ_K is canonical then the controller will be fully quantum. Our formulation of the LQG problem *differs* from previous formulations of the quantum LQG, such as given in Edwards and Belavkin [2005] and Doherty and Jacobs [1999]. The important difference is that in the earlier works the controller is classical whereas in our formulation we seek a controller which may possibly be another quantum system (depending on how Θ_K is defined) which generates an optical field to drive the quantum plant. What is new here are the additional constraints (13) and (14) that must also be satisfied by the controller to be physically realizable (for details, see [James et al., 2007, Section III]. This is natural since for applications the controller should represent a physical system. Constraint (13) is a non-convex, non-linear equality constraint on the controller matrices A_K, B_{K1}, B_{K2}, B_{K3} and C_K that presents a formidable challenge in the controller design.

Remark 2. It is important to note that it is not essential to fix $\Theta_K = \mathrm{diag}_{n/2}(J)$. Instead, it may also be fixed to be $\Theta_K^S = S\,\mathrm{diag}_{n/2}(J)S^T$ for any real invertible matrix S. Indeed, it can be immediately checked that if A_K^S, $B_{Ki}^S\,(i = 1, 2, 3)$, C_K^S solves Problem 1 for $\Theta_K = \Theta_K^S$ then $A_K = S^{-1}A_K^S S$, $B_{Ki} = S^{-1}B_{Ki}^S$, $(i = 1, 2, 3)$, $C_K = C_K^S S$ solves Problem 1 for $\Theta_K = J$. This additional freedom will be useful for numerical attempts at solving Problem 1.

4. REFORMULATION OF QUANTUM LQG PROBLEM INTO A RANK CONSTRAINED LMI PROBLEM

We shall now discuss how to transform the quantum LQG problem into a rank constrained LMI problem that is amenable to numerical methods. To best illustrate the idea we opt to restrict our attention to the case where Θ_K is canonical. Moreover, to facilitate easy and explicit exposition of the matrix lifting and linearization technique, we shall take for a “canonical” example, a plant and controller of order n (recall that in our setup we are looking for a controller which is of the same order as the plant) with $n_y = n_u = n$ and B_{K1}, B_{K2}, B_{K3}, C_K are all of dimension $n \times n$. Nonetheless, the matrix lifting principle described for this canonical case can in principle for Θ_K degenerate canonical, but the lifting will be too complicated to describe in general. Furthermore, the transformation is *not unique* and for efficiency the choice of suitable lifting variables should be considered on a case by case basis to exploit any special structure that may be present in the problem.

Consider a n-th order plant (4) with $n_y = n_u = n$ and a n-th order controller (6) with $n_{w_{K1}} = n_{w_{K2}} = n$ (hence $B_{K1}, B_{K2} \in \mathbb{R}^{n\times n}$). Then P will be a symmetric matrix of dimension $2n \times 2n$. The first step is to transform the constraints (12) and $J_\infty < \gamma$ into an LMI constraint. To do this we exploit a non-linear change of variables given in [Scherer et al., 1997, Eq.(35)], but to do this we first need to suitably redefine our plant and controller equations while leaving the closed-loop equations unaltered. To this end, let us redefine our plant as:

$$\begin{aligned} dx(t) &= Ax(t)dt + B\beta_u(t) + B'_{w'}dw'(t); \quad x(0) = x; \\ dy'(t) &= C'x(t)dt + D'_{w'}dw'(t); \\ z(t) &= C_z x(t) + D_z\beta_u(t), \end{aligned} \tag{15}$$

with $w' = [\,w^T\ w_{K1}^T\ w_{K2}^T\,]^T$, $B'_{w'} = [\,B_w\ B\ 0_{n\times n}\,]$, $C' = [\,0_{n\times n}\ 0_{n\times n}\ C^T\,]^T$ and

$$D'_{w'} = \begin{bmatrix} 0_{n\times n_w} & I_{n\times n} & 0_{n\times n} \\ 0_{n\times n_w} & 0_{n\times n_w} & I_{n\times n} \\ D_w & 0_{n\times n} & 0_{n\times n} \end{bmatrix}.$$

We also redefine our controller equations as:

$$\begin{aligned} d\xi(t) &= A_K\xi(t)dt + B_K dy'(t); \\ \beta_u(t) &= C_K\xi(t), \end{aligned} \tag{16}$$

with $B_K = [\, B_{K1}\ B_{K2}\ B_{K3}\,]$. It is easily seen that interconnecting (15) and (16) gives the closed-loop equation (7). Now we are in the setup of Scherer et al. [1997] with $D_K = 0$ in [Scherer et al., 1997, Eq.(2)].

It is shown in Scherer et al. [1997] that by introducing auxiliary variables $N, M, \mathbf{X}, \mathbf{Y}, Q \in \mathbb{R}^{n\times n}$, with $\mathbf{X}, \mathbf{Y}, Q$ symmetric, and applying the following non-linear change of variables (see [Scherer et al., 1997, Section IV-B] with $\hat{\mathbf{D}} = D_K = 0$):

$$\mathbf{A} = NA_KM^T + NB_KC'\mathbf{X} + \mathbf{Y}BC_KM^T + \mathbf{Y}A\mathbf{X}; \tag{17}$$

$$\mathbf{B} = NB_K; \tag{18}$$

$$\mathbf{C} = C_KM^T, \tag{19}$$

the constraints (12) and $J_\infty < \gamma$ can be rewritten as the LMI constraint [Scherer et al., 1997, Eq.(5)]:

$$\begin{bmatrix} A\mathbf{X} + \mathbf{X}A^T + B\mathbf{C} + (B\mathbf{C})^T & \mathbf{A}^T + A & B'_{w'} \\ \mathbf{A} + A^T & A^T\mathbf{Y} + \mathbf{Y}A + \mathbf{B}C' + (\mathbf{B}C')^T & \mathbf{Y}B'_{w'} + \mathbf{B}D'_{w'} \\ (B'_{w'})^T & (\mathbf{Y}B'_{w'} + \mathbf{B}D'_{w'})^T & -I \end{bmatrix} < 0 \tag{20}$$

$$\begin{bmatrix} \mathbf{X} & I & (C_z\mathbf{X} + D_z\mathbf{C})^T \\ I & \mathbf{Y} & C_z^T \\ C_z\mathbf{X} + D_z\mathbf{C} & C_z & Q \end{bmatrix} > 0 \tag{21}$$

$$\mathrm{tr}(Q) < \gamma. \tag{22}$$

Since the controller is of the same order as the plant, the matrices N and M can be freely chosen to be any pair of invertible matrices satisfying $MN^T = I - \mathbf{XY}$.

Once matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{X}, \mathbf{Y}, Q$ satisfying the LMIs (22) and matrices N and M satisfying the conditions of the last paragraph have been found, the original controller matrices A_K, B_K, C_K can be reconstructed as in [Scherer et al., 1997, Eq.(40)]:

$$C_K = \mathbf{C} \tag{23}$$

$$B_K = N^{-1}\mathbf{B} \tag{24}$$

$$A_K = N^{-1}(\mathbf{A} - NB_KC'\mathbf{X} - \mathbf{Y}B\mathbf{C}M^T - \mathbf{Y}A\mathbf{X})M^{-T}. \tag{25}$$

For simplicity in what will follow, we choose to set M and N as: $M = I_{n\times n}$ and $N = I - \mathbf{YX}$. However, this assumption is not strictly required and a general treatment without it can be found in Nurdin et al. [2007]. Multiplying the left and right hand sides of (13) with N and N^T, respectively, and introducing new variables $\breve{N} = N\Theta_K$, $\breve{A}_K = NA_K$ and $\breve{B}_{Ki} = NB_{Ki}$, $i = 1, 2, 3$, (13) and (14) can be expressed as:

$$\begin{aligned} &(-\mathbf{A} + (\breve{B}_{K3}C + \mathbf{Y}A)\mathbf{X} + \mathbf{Y}B\mathbf{C})\breve{N}^T \\ &+\ \breve{N}(\mathbf{A} - (\breve{B}_{K3}C + \mathbf{Y}A)\mathbf{X} - \mathbf{Y}B\mathbf{C})^T \\ &+ \sum_{i=1}^{3} \breve{B}_{Ki}\,\mathrm{diag}_{n/2}(J)\breve{B}_{Ki}^T = 0; \end{aligned} \tag{26}$$

$$\breve{B}_{K1} = \breve{N}\mathbf{C}^T\,\mathrm{diag}_{n/2}(J). \tag{27}$$

Let us pause to note that constraints (26) and (27) are *polynomial matrix equality constraints* in the parameters $(\mathbf{A}, \breve{B}_{K1}, \breve{B}_{K2}, \breve{B}_{K3}, \mathbf{X}, \mathbf{Y}, \breve{N})$. By this we mean that they are equality constraints in a matrix-valued multivariate polynomial with matrix-valued variables. We now proceed to "linearize" (26) and (27) by introducing appropriate matrix lifting variables and the associated equality constraints, and transforming them into an LMI with a rank n constraint. The 14 matrix lifting variables $W_1, W_2, \ldots, W_{14} \in \mathbb{R}^{n\times n}$ are as follows: $W_i = \breve{B}_{Ki}\,\mathrm{diag}_{\frac{n}{2}}(J)$, $i = 1, 2, 3$, $W_4 = \mathbf{Y}B$, $W_5 = \breve{B}_{K3}C + \mathbf{Y}A$, $W_6 = \breve{N}\mathbf{C}^T$, $W_7 = \breve{N}\mathbf{X}$, $W_8 = \mathbf{A}\breve{N}^T$, $W_9 = \mathbf{YX}$, $W_{10} = W_4W_6^T$, $W_{11} = W_5W_7^T$, $W_{12} = W_1\breve{B}_{K1}^T$, $W_{13} = W_2\breve{B}_{K2}^T$ and $W_{14} = W_3\breve{B}_{K3}^T$. Now, let Z be a $23n \times 23n$ symmetric matrix,

$$\mathbf{Z}_{i,j} = [Z_{kl}]_{k=in+1,(i+1)n,l=jn+1,(j+1)n},$$
$$x = (x_1, \ldots, x_8) = (1, 2, \ldots, 8),$$

and

$$v = (v_1, \ldots, v_{14}) = (9, 10, \ldots, 22).$$

We require that Z satisfy the constraints:

$$\left.\begin{array}{cc} Z \geq 0 & \mathbf{Z}_{v_6,1} - \mathbf{Z}_{x_8,x_5} = 0 \\ \mathbf{Z}_{0,0} - I_{n\times n} = 0 & \mathbf{Z}_{v_7,1} - \mathbf{Z}_{x_8,x_6} = 0 \\ \mathbf{Z}_{1,x_6} - \mathbf{Z}_{x_6,1} = 0 & \mathbf{Z}_{v_8,1} - \mathbf{Z}_{x_1,x_8} = 0 \\ \mathbf{Z}_{1,x_7} - \mathbf{Z}_{x_7,1} = 0 & \mathbf{Z}_{v_9,1} - \mathbf{Z}_{x_7,x_6} = 0 \\ \mathbf{Z}_{v_1,1} - \mathbf{Z}_{x_2,1}\,\mathrm{diag}_{n/2}(J) = 0 & \mathbf{Z}_{v_{10},1} - \mathbf{Z}_{v_4,v_6} = 0 \\ \mathbf{Z}_{v_2,1} - \mathbf{Z}_{x_3,1}\,\mathrm{diag}_{n/2}(J) = 0 & \mathbf{Z}_{v_{11},1} - \mathbf{Z}_{v_5,v_7} = 0 \\ \mathbf{Z}_{v_3,1} - \mathbf{Z}_{x_4,1}\,\mathrm{diag}_{n/2}(J) = 0 & \mathbf{Z}_{v_{12},1} - \mathbf{Z}_{v_1,x_2} = 0 \\ \mathbf{Z}_{v_4,1} - \mathbf{Z}_{x_7,1}B = 0 & \mathbf{Z}_{v_{13},1} - \mathbf{Z}_{v_2,x_3} = 0 \\ \mathbf{Z}_{v_5,1} - \mathbf{Z}_{x_4,1}C - \mathbf{Z}_{x_7,1}A = 0 & \mathbf{Z}_{v_{14},1} - \mathbf{Z}_{v_3,x_4} = 0 \\ \mathbf{Z}_{x_8,1} - \Theta_K + \mathbf{Z}_{v_9,1}\Theta_K = 0 & \end{array}\right\} \tag{28}$$

The LMI constraints (20)-(22) can be expressed in terms of Z by replacing $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{X}$, $\mathbf{Y}$ respectively with $\mathbf{Z}_{x_1,1}, [\,\mathbf{Z}_{x_2,1}\ \mathbf{Z}_{x_3,1}\ \mathbf{Z}_{x_4,1}\,], \mathbf{Z}_{x_5,1}, \mathbf{Z}_{x_6,1}, \mathbf{Z}_{x_7,1}$, while the physical realizability constraints (26) and (27) become the following pair of linear equality constraints:

$$\begin{aligned} &-\mathbf{Z}_{v_8,1} + \mathbf{Z}_{v_8,1}^T + \mathbf{Z}_{v_{11},1} - \mathbf{Z}_{v_{11},1}^T + \mathbf{Z}_{v_{10},1} - \mathbf{Z}_{v_{10},1}^T + \\ &\mathbf{Z}_{v_{12},1} + \mathbf{Z}_{v_{13},1} + \mathbf{Z}_{v_{14},1} = 0 \end{aligned} \tag{29}$$

$$\mathbf{Z}_{x_2,1} - \mathbf{Z}_{v_6,1}\,\mathrm{diag}_{n/2}(J) = 0. \tag{30}$$

Finally, we also require that Z satisfy a rank n constraint:

$$\mathrm{rank}(Z) \leq n. \tag{31}$$

To understand the above rank constrained LMI and its relation to our original constraints, suppose that there is a Z satisfying (28)-(31) and the LMI constraints (expressed in terms of block elements of Z). Then, since $Z \geq 0$ and is of rank at most n, we may factorize it as $Z = VV^T$, where $V \in \mathbb{R}^{23n\times n}$ and satisfies $[V_{ij}]_{i,j=1,n} = I_{n\times n}$, and by (28) we recover $\mathbf{A}$, $\breve{B}_{Ki}$ $(i = 1, 2, 3)$, $\mathbf{C}, \mathbf{X}, \mathbf{Y}, \breve{N}$ respectively as $\mathbf{Z}_{x_1,1}, \ldots, \mathbf{Z}_{x_8,1}$, and also recover $W_i =$

$\mathbf{Z}_{v_i,1}$, $i = 1, \ldots, 14$. Then $N = \check{N}\Theta_K^{-1}$ (recall that we are considering Θ_K canonical, hence it is invertible) and the controller matrices A_K, B_K, C_K are given by (23)-(25) and by construction they will satisfy (20)-(22), (13) and (14). Thus we obtain a solution to Problem 1.

An extension of the technique introduced here wherein Θ_K is no longer fixed, but is allowed to also be a free variable, is developed in Nurdin et al. [2007].

5. NUMERICAL SOLUTION OF RANK CONSTRAINED LMI PROBLEM

We have seen in the preceding section that our problem can be converted into a polynomial matrix programming (to be precise, feasibility) problem (since LMIs can themselves be viewed as polynomial matrix inequalities) and that the latter can be converted to a rank constrained problem. It is well-known that many important practical control problems can be formulated as polynomial programming problems, including reduced order robust controller design, static output feedback and gain scheduling (see Henrion and Lasserre [2006] and the references therein). They are non-convex and non-linear problems which are, in general, difficult to solve. In fact, some of these problems are known to be NP-hard Shor [1990], Lasserre [2001].

If one tries to directly attack the (scalar or matrix) polynomial programming problem then a specialized method for solving them is to employ LMI relaxation techniques based on the theory of moments and the dual theory of sum of squares (SOS) polynomials Lasserre [2001], Kojima [2003], Hol and Scherer [2004], Henrion and Lasserre [2006]. Under appropriate conditions, relaxation methods can be guaranteed to converge as the order of relaxation is increased and it can be checked whether a global optima may have been obtained at a particular relaxation. Despite its attractive features, the size of the relaxed LMI problem grows very quickly with the number of decision variables, the degree of polynomials involved and the order of relaxation, making the method impractical for problems with many decision variables.

On the other hand, if the problem is converted to a rank constrained LMI problem then there are *iterative* algorithms in the literature that try to directly search for a feasible point satisfying the set of LMIs and the rank constraint, mostly based on the idea of alternating projections (see Orsi et al. [2006] and the references cited therein). The main drawback of these algorithms is that they are difficult to analyze and are not in general guaranteed to converge from arbitrary starting points, even if a solution exists. However, since there are no relaxations involved that increase the size of the problem to be solved, they can be more attractive for solving medium and larger size polynomial programming problems. This makes them more suitable for our current problem, which can be considered to be of a substantial size if converted to a scalar polynomial programming problem.

To solve the rank constrained LMI problem formulated in the last section we shall use an algorithm by Orsi et al. [2006] which has been implemented in the freely available Matlab toolbox LMIRank Orsi [2005] and can be called via the Yalmip optimization prototyping environment Löfberg [2004]. This algorithm is based on alternating projections but, unlike previous alternating projections algorithms, has a built-in Newton step which has the potential to accelerate convergence.

Solvers for rank constrained LMI problems are not guaranteed to converge from arbitrary starting points, even if a solution exists. Therefore, it is important to have a "reasonable" starting point for these algorithms, if possible. For a given $\gamma > 0$, as a starting point for the LMIRank solver, without a particular justification, we propose to first solve the LMIs (20)-(22) to obtain $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{X}, \mathbf{Y}, Q$. Then set $N = I - \mathbf{YX}$ and compute $\breve{B}_{K1}$, $\breve{B}_{K2}$, $\breve{B}_{K3}$, $\check{N}$ and the matrix lifting variables $W_1, \ldots, W_{14}$ according to the definitions given in Section 4. Let

$$V_0 = [\, I \;\; \mathbf{A}^T \;\; \breve{B}_{K1}^T \;\; \breve{B}_{K2}^T \;\; \breve{B}_{K3}^T \;\; \mathbf{C}^T \;\; \mathbf{X}^T \;\; \mathbf{Y}^T \;\; \check{N}^T \;\; W_1^T \\ \ldots \; W_{14}^T \,]^T .$$

Then we set $Z = V_0 V_0^T$ as a heuristic starting point.

6. QUANTUM LQG CONTROL DESIGN EXAMPLE

In this section we apply the transformation and matrix lifting technique of Section 4 to compute a fully quantum LQG controller to asymptotically stabilize a marginally stable fully quantum plant. We work in the Yalmip prototyping environment and a solution was computed using LMIRank. The semidefinite program solver used for LMIRank is SeDuMi Version 1.1 Release 3 Advanced Optimization Lab, McMaster University [2006].

The quantum plant to be controlled is a physically realizable ([James et al., 2007, Section III]) fully quantum system with Hamiltonian matrix R and coupling matrix Λ given by:

$$R = \frac{1}{2}\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \quad \Lambda = \begin{bmatrix} \sqrt{\kappa_1} & 0 \\ \sqrt{\kappa_2} & 0 \\ \sqrt{\kappa_3} & 0 \end{bmatrix},$$

and its dynamics given by:

$$\begin{aligned} dx &= \begin{bmatrix} 0 & \Delta \\ -\Delta & 0 \end{bmatrix} x dt + \begin{bmatrix} 0 & 0 \\ 0 & -2\sqrt{k_1} \end{bmatrix} du + \\ &\quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\sqrt{k_2} & 0 & -2\sqrt{k_3} \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix}, \\ dy &= \begin{bmatrix} 2\sqrt{k_2} & 0 \\ 0 & 0 \end{bmatrix} x dt + dw_1 \end{aligned} \tag{32}$$

with $\Delta = 0.1$ and $k_1 = k_2 = k_3 = 10^{-2}$. Here the quantum noise fields couple only to the position operator of the harmonic oscillator, which is the typical setup sought in various schemes for quantum non-demolition continuous measurement of position. This particular choice of coupling results in a marginally stable plant with A having two mutually conjugate eigenvalues on the imaginary axis.

Let us try to asymptotically stabilize this system with another quantum system being the LQG controller. To this end we set $z = C_z x + D_z \xi$ with $C_z = [\, I_{2\times 2} \; 0_{2\times 2} \,]^T$ and $D_z = [\, 0_{2\times 2} \; I_{2\times 2} \,]^T$. Choosing $\gamma = 5.75$, the numerical procedure of Sections 4 and 5 was not found to converge for the choice $\Theta_K = J$. However, a solution was found for $\Theta_K = 0.01J$, corresponding to the choice $S = 0.1 I_{2\times 2}$ in Remark 2. After

1000 iterations of LMIRank and an application of the similarity transformation $(A_K, B_{K1}, B_{K2}, B_{K3}, C_K) \rightarrow (S^{-1}A_KS, S^{-1}B_{K1}, S^{-1}B_{K2}, S^{-1}B_{K3}, C_KS)$ of Remark 2, this yields the following physically realizable controller:

$$
\begin{aligned}
d\xi = &\begin{bmatrix} -0.9227 & 0.3506 \\ -1.2428 & 0.3918 \end{bmatrix} \xi dt + \begin{bmatrix} -0.0300 & 0.0009 \\ 0.0054 & -0.0017 \end{bmatrix} \times \\
&dw_{K1} + 10^{-11} \begin{bmatrix} 0.1537 & 0.2505 \\ 0.1762 & 0.3409 \end{bmatrix} dw_{K2} + \\
&\begin{bmatrix} 19.0699 & -3.3957 \\ 23.6198 & -4.1780 \end{bmatrix} dy \\
du(t) = &\begin{bmatrix} 0.0017 & 0.0009 \\ 0.0054 & 0.0300 \end{bmatrix} x dt + dw_{K1},
\end{aligned} \tag{33}
$$

that asymptotically stabilizes the closed-loop system. Notice that since elements of B_{K2} are very small, the term $B_{K2}w_{K2}$ may be practically discarded to simplify the controller. The closed loop LQG cost achieved by this controller is $J_\infty = 5.7503$.

7. CONCLUSIONS

In this paper we have formulated a quantum LQG problem that allows the possibility of the controller to be a quantum system. This problem is recast as a polynomial matrix programming problem that can be systematically converted to a rank constrained LMI problem. In an example, we consider the problem of stabilization of a marginally stable quantum plant and successfully computed a fully quantum cost bounded LQG controller that achieves this goal.

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