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HOSM observer for a class of non-minimum phase causal nonlinear MIMO systems

S.Baev * **Y.Shtessel** ** **C.Edwards** ***

* *Department of Electrical and Computer Engineering,The University of Alabama in Huntsville, Huntsville, Alabama 35899 USA,(e-mail: baevs@eng.uah.edu)*
** *Department of Electrical and Computer Engineering, The University of Alabama in Huntsville, Huntsville, Alabama 35899 USA, (e-mail: shtessel@eng.uah.edu)*
*** *Department of Engineering, University of Leicester, Leicester, LE1 7RH, UK (e-mail: ce14@le.ac.uk)*

Abstract: A higher order sliding mode observer is proposed for asymptotic identification of the full state vector and the vector of unknown inputs for MIMO nonlinear causal systems with unstable internal dynamics. The problem is addressed via consistent application of exact higher order sliding mode (HOSM) differentiators in conjunction with the method of stable system center (SSC). A numerical example illustrates the performance of the proposed algorithm.

1. INTRODUCTION

Observation of the full state and unknown inputs in causal nonminimum-phase nonlinear systems is a challenging, real-life control problem. Classical observation algorithms such as Kalman filtering cannot always be used successfully in the case of complex nonlinear systems in which the dynamics are forced by unknown inputs. A method of identification for linear, time-variant, nonminimum phase systems is proposed in [9]. The use of hierarchical super-twisting observation and identification algorithms for a class of *linear* systems with unknown inputs is presented in [8]. A higher order sliding mode observer for exact estimation of observable states and asymptotic estimation of the unobservable ones in MIMO nonlinear systems with unknown inputs and *stable* internal dynamics is proposed in [7]. Implicit observation of the unstable internal states via the method of *stable system center* is addressed in works [2, 3, 13]. In spite of the fact that the internal dynamics are assumed to be measured, the estimation of the unique bounded profile for the internal states is generated for the purpose of output tracking control for a class of causal nonlinear nonminimum-phase MIMO systems.

The higher order sliding mode observation algorithm introduced in this paper, relaxes the requirements of the existing published methods: the linear plant model as in [8] and the stable internal dynamics as in [7]. The internal states are assumed to be forced by an unknown dynamical process of given order. The characteristic polynomial of that process is identified online via a higher order sliding mode parameter observer. The coefficients of the identified characteristic polynomial are then used in the method of stable system center [13] to find the bounded profile which asymptotically converges to the solution of the unstable internal dynamics differential equation. The identification of the unknown input is achieved by employing the exact higher order sliding mode differentiator from [12] and the estimate of the internal state.

The contribution of this work is in the consistent application of a HOSM approach to the observation problem in a class of nonminimum-phase causal nonlinear dynamic systems and can be summarized as follows:

(1) a novel approach of implicitly numerically solving the unstable differential equation is used for asymptotic estimation of the full state in nonminimum phase causal nonlinear systems;
(2) Asymptotic identification of the unknown input driving the system motion is achieved by using HOSM observation algorithms.

2. PRELIMINARIES

Consider the (locally) stable MIMO nonlinear system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\,\phi(t) \\ \mathbf{y} = \mathbf{h}(\mathbf{x}) \end{cases} \tag{1}$$

where $\mathbf{f}(\mathbf{x}) \in \Re^n$, $\mathbf{h}(\mathbf{x}) = [h_1(\mathbf{x}), h_2(x), \ldots, h_m(\mathbf{x})]^T \in \Re^m$, $\mathbf{G}(\mathbf{x}) = [g_1(\mathbf{x}), g_2(x), \ldots, g_m(\mathbf{x})]^T \in \Re^{n\times m}$, $g_i(\mathbf{x}) \in \Re^n$ for $i = \overline{1,m}$, $\mathbf{x} \in \Re^n$, $\mathbf{y}, \phi \in \Re^m$, $n \geq m$.

Remark 1. The vector functions $\mathbf{f}(\mathbf{x}), g_i(\mathbf{x}), \mathbf{h}(\mathbf{x})$ are assumed to be smooth and bounded. The stability of system (1) is local in vicinity of some equilibrium point.

The following set of conditions are assumed to hold [7]:

A1. The system (1) has a vector relative degree $r = [r_1, r_2, \ldots, r_m]$ with $r < n$, i.e.

$$\begin{array}{l} L_{g_j} L_f^k h_i(\mathbf{x}) = 0, \quad j = \overline{1,m},\ 0 \leq k < r_i - 1,\ i = \overline{1,m} \\ L_{g_j} L_f^{r_i-1} h_i(\mathbf{x}) \neq 0, \quad \text{for at least one } j \text{ from } \overline{1,m} \end{array} \tag{2}$$

A2. The $m\times m$ matrix

$$\mathbf{D}(\mathbf{x}) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(\mathbf{x}) & \ldots & L_{g_m} L_f^{r_1-1} h_1(\mathbf{x}) \\ L_{g_1} L_f^{r_2-1} h_2(\mathbf{x}) & \ldots & L_{g_m} L_f^{r_2-1} h_2(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(\mathbf{x}) & \ldots & L_{g_m} L_f^{r_m-1} h_m(\mathbf{x}) \end{bmatrix} \tag{3}$$

is nonsingular

A3. The distribution $\Gamma = \text{span}\{g_1, g_2, \ldots, g_m\}$ is involutive.

The system in (1) together with assumptions **A1-A3** will be referred to as (1+) in the rest of the paper.

3. PROBLEM FORMULATION

The problem considered in the paper is one of designing a HOSM observer for system (1+) with only the measurement $\mathbf{y}$ available. The observer is supposed to generate estimates $\hat{\mathbf{x}}$ and $\hat{\phi}$ which asymptotically converge to the true state vector $\mathbf{x}$ and input vector ϕ respectively, i.e.

$$\lim_{t\to\infty} \|\hat{\mathbf{x}} - \mathbf{x}\| = 0, \quad \lim_{t\to\infty} \|\hat{\phi} - \phi\| = 0, \tag{4}$$

4. COORDINATE TRANSFORMATION

The system given by (1+) can be transformed to the new basis $\{\xi, \eta\}$ which is defined as follows:

$$\xi_i := \begin{bmatrix} y_i \\ \dot{y}_i \\ \vdots \\ y_i^{(r_i-1)} \end{bmatrix} = \begin{bmatrix} h_i(\mathbf{x}) \\ L_f h_i(\mathbf{x}) \\ \vdots \\ L_f^{r_i-1} h_i(\mathbf{x}) \end{bmatrix} \in \Re^{r_i}, \quad i = \overline{1, m} \tag{5}$$

$$\xi := \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix} \in \Re^r, \quad \eta := \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-r} \end{bmatrix} = \in \Re^{n-r} \tag{6}$$

The new state $\{\xi, \eta\}$ consists of the I/O component ξ and the internal dynamics component η. According to [10], it is always possible to find $n - r$ functions

$$\psi_i : \Re^n \to \Re, \ \eta_i = \psi_i(\mathbf{x}) \quad i = \overline{1, n - r}$$

such that the mapping

$$\begin{aligned} \mathbb{T}(\mathbf{x}) = [\ & h_1(\mathbf{x}), L_f h_1(\mathbf{x}), \ldots, L_f^{r_1-1} h_1(x) \\ & h_2(\mathbf{x}), L_f h_2(\mathbf{x}), \ldots, L_f^{r_2-1} h_2(x), \\ & \ldots \\ & h_m(\mathbf{x}), L_f h_m(\mathbf{x}), \ldots, L_f^{r_m-1} h_m(x), \\ & \psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \ldots, \psi_{n-r}(\mathbf{x})] \end{aligned} \tag{7}$$

is a local diffeomorphism in a neighborhood of any point $\mathbf{x}$ which belongs to the system trajectory, i.e.

$$\mathbf{x} = \mathbb{T}^{-1}(\xi, \eta) \tag{8}$$

Finally, the system given by (1+) can be rewritten in I/O form together with the internal dynamics [10]:

$$\dot{\xi}_i = \Lambda_i \, \xi_i + \Delta_i(\xi, \eta) + \Psi_i(\xi, \eta, \phi(t)), \quad i = \overline{1, m} \tag{9}$$

$$\dot{\eta} = \mathbf{Q} \, \eta + \theta(\xi, \eta) \tag{10}$$

where $Q \in \Re^{(n-r)\times(n-r)}$ is nonsingular and

$$\Lambda_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \quad \Delta_i(\xi, \eta) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ L_f^{r_i} h_i(\mathbb{T}^{-1}(\xi, \eta)) \end{bmatrix}$$

$$\Psi_i(\xi, \eta, \phi(t)) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sum_{j=1}^{m} L_{g_j} L_f^{r_i-1} h_i(\mathbb{T}^{-1}(\xi, \eta)) \, \phi_j(t) \end{bmatrix} \tag{11}$$

The *internal dynamics* η *are assumed to be unstable.*

Remark 2. Such a presentation of the internal dynamics equation in (9) is dictated by the identification algorithm that will be employed. There are no constraints on selection of the matrix $\mathbf{Q}$ except one of nonsingularity.

5. THE HOSM OBSERVER DESIGN

The proposed observer consists of two components: the *state observer* and the *unknown input observer.* Both of them are presented in the two following sections.

5.1 The State Observer

The new state $\{\xi, \eta\}$ consists of two components (the I/O form and the internal states) which will be identified separately. The first component estimate $\hat{\xi}$ is identified in an *exact* way by using the HOSM differentiator [11, 12]. The second component estimate $\hat{\eta}$ is identified *asymptotically* using the *extended method of stable system center* (ESSC) [1]

The Output State Estimation. The i^{th} output state vector along with its derivatives can be estimated in a finite time by using the exact HOSM differentiator of r_i^{th}-order:

$$\begin{cases} \dot{z}_{i,0} = \nu_{i,0} \\ \nu_{i,0} = -\lambda_0 \, |z_{i,0} - y_i|^{r_i/(r_i+1)} \, \text{sign}(z_{i,0} - y_i) + z_{i,1} \\ \quad \vdots \\ \dot{z}_{i,j} = \nu_{i,j}, \quad j = \overline{1, r_i - 1} \\ \nu_{i,j} = -\lambda_j \, |z_{i,j} - \nu_{i,j-1}|^{(r_i-j)/(r_i-j+1)} \times \\ \qquad \times \text{sign}(z_{i,j} - \nu_{i,j-1}) + z_{i,j+1}, \\ \quad \vdots \\ \dot{z}_{i,r_i} = -\lambda_{r_i} \, \text{sign}(z_{i,r_i} - \nu_{i,r_i-1}) \end{cases} \tag{12}$$

with $\hat{\xi}_{i,j} = z_{i,j-1}$ and $\hat{\dot{\xi}}_{i,j} = z_{i,j}$ for $j = \overline{1, r_i}$

Combine all the components together:

$$\hat{\xi}_i = \{\hat{\xi}_{i,1}, \hat{\xi}_{i,2}, \ldots, \hat{\xi}_{i,r_i}\}^T, \quad \hat{\dot{\xi}}_i = \{\hat{\dot{\xi}}_{i,1}, \hat{\dot{\xi}}_{i,2}, \ldots, \hat{\dot{\xi}}_{i,r_i}\}^T. \tag{13}$$

The Internal State Estimation. Since the *internal dynamics are assumed to be unstable* (unlike the case considered in [7]) it is not possible to estimate them through numerical integration of (10). Instead, the extended method of *stable system center* (ESSC) is employed. The original SSC method [13] (which is used as a basis for ESSC) assumes that the internal dynamics differential equation (10) is forced by the *causal* term $\theta(\xi, \eta)$ which appears to be a solution of a differential equation with *known characteristic polynomial.*

5.1.2.1. The SSC Method. The method of stable system center numerically solves the differential equation

$$\dot{\eta} = \mathbf{Q} \, \eta + \theta(\cdot) \tag{14}$$

without explicit integration, where $\eta \in \Re^m$, $\mathbf{Q} \in \Re^{m\times m}$ is non-Hurwitz, and $\theta \in \Re^m$ is a causal forcing term available for measurement.

[1] The ESSC method is introduced in [2, 3, 13] without explicit entitling the latter.

There is an assumption that the forcing term $\theta(\cdot)$ is a *causal* signal which coincides with the solution of a known linear differential equation with know characteristic polynomial:

$$P_k(\lambda) = \left|\lambda I - \tilde{A}\right| = \lambda^k + p_{k-1}\lambda^{k-1} + \ldots + p_1\lambda + p_0 \qquad (15)$$

The SSC method gives a procedure to design a *linear filter* which generates an estimate $\hat{\eta}$ of the solution η_c in (14). The estimate asymptotically converges to η_c. The forcing term $\theta(\cdot)$ is supposed to be the input of the filter. The following theorem presents the procedure:

Theorem 1. Given the unstable differential equation (14) driven by a causal signal $\theta(\cdot)$, which is available for measurement, and the following set of conditions [13]:

- the matrix $\mathbf{Q}$ in (14) is nonsingular;
- the internal dynamics forcing term $\theta(\cdot)$ can be piecewise modeled as the output of a dynamical process with known characteristic polynomial (15),

then the internal state estimate $\hat{\eta}\in\Re^m$ can be generated by a matrix differential equation:

$$\begin{array}{l}\hat{\eta}^{(k)} + c_{k-1}\hat{\eta}^{(k-1)} + \ldots + c_1\dot{\hat{\eta}} + c_0\hat{\eta} = \\ -\left(P_{k-1}\theta^{(k-1)} + \ldots + P_1\dot{\theta} + P_0\theta\right)\end{array} \qquad (16)$$

where the numbers $c_{k-1},\ldots,c_1,c_0$ are chosen to provide a desirable eigenvalue placement of convergence $\hat{\eta}\to\eta_c$ and the matrices $P_{k-1},\ldots,P_1,P_0\in\Re^{m\times m}$ are given by:

$$\begin{array}{ll} P_{k-1} & = \left(I + c_{k-1}\mathbf{Q}^{-1} + \ldots + c_0\mathbf{Q}^{-k}\right)\times \\ & \times\left(I + p_{k-1}\mathbf{Q}^{-1} + \ldots + p_0\mathbf{Q}^{-k}\right)^{-1} - I \\ P_{k-2} & = c_{k-2}\mathbf{Q}^{-1} + \ldots + c_0\mathbf{Q}^{-(k-1)} - (P_{k-1}+I)\times \\ & \times\left(p_{k-2}\mathbf{Q}^{-1} + \ldots + p_0\mathbf{Q}^{-(k-1)}\right) \\ \vdots & \\ P_1 & = c_1\mathbf{Q}^{-1} + c_0\mathbf{Q}^{-2} - (P_{k-1}+I)(p_1\mathbf{Q}^{-1} + p_0\mathbf{Q}^{-2}) \\ P_0 & = c_0\mathbf{Q}^{-1} - (P_{k-1}+I)\,p_0\mathbf{Q}^{-1}\end{array} \qquad (17)$$

where $p_{k-1},\ldots,p_1,p_0$ are the coefficients of the characteristic polynomial (15);

Proof. The proof is given in [13]

5.1.2.2. The ESSC Method. The ESSC method [2, 3] relaxes the requirement of knowing the characteristic polynomial coefficients. In this case the *differential equation* which describes the dynamics of the forcing term $\theta(\cdot)$ is assumed to be *unknown* but with *given order*. Now the corresponding characteristic polynomial is identified online using the HOSM parameter observer from [1, 2]. Details of the algorithm for the identification of the characteristic polynomial is given in the Appendix section A.

The ESSC method generates a bounded estimate $\hat{\eta}$ of the *ideal internal dynamics* (IID) η_c – the existing bounded solution of (10). The estimate $\hat{\eta}$ converges to η_c asymptotically as time increases, i.e.

$$\lim_{t\to\infty}\|\hat{\eta} - \eta_c\| = 0.$$

Remark 3. The ESSC method assumes the matrix $\mathbf{Q}$ to be non-Hurwitz, but it works for any nonsingular matrix $\mathbf{Q}$ which satisfies (10).

The ESSC method consists of two successive procedures:

- the identification of the characteristic polynomial coefficients;
- the use of the SSC method to design the linear filter which generates $\hat{\eta}$ on the basis of $\theta(\hat{\xi},\hat{\eta})$. (The estimate $\hat{\xi}$ is assumed to be available: see 5.1.1)

The Recovery of the Original State $\mathbf{x}$. Having the estimates $\hat{\xi}$ and $\hat{\eta}$ available, the original state estimate can be found from the inverse coordinate transformation:

$$\hat{\mathbf{x}} = \mathbb{T}^{-1}(\hat{\xi},\hat{\eta}) \qquad (18)$$

In spite of the fact of finite time convergence of $\hat{\xi}$ to ξ, the full state vector estimate $\hat{\mathbf{x}}$ will converges to $\mathbf{x}$ only asymptotically because of asymptotic convergence of the internal state estimate $\hat{\eta}$, i.e.

$$\lim_{t\to\infty}\|\hat{\mathbf{x}} - \mathbf{x}\| = 0.$$

5.2 The Unknown Input Observer

Having estimated the following quantities:

- the internal states $\eta_i\in\Re$ for $i=\overline{1,n-r}$;
- the I/O states $\xi_j\in\Re^{r_j}$ for $j=\overline{1,m}$;
- the derivatives of output states $\dot{\xi}_k\in\Re^{r_k}$ for $k=\overline{1,m}$

the estimate $\hat{\phi}$ of the unknown input ϕ in (1+) can be calculated as follows:

$$\hat{\phi} = \mathbf{D}^{-1}(\hat{\mathbf{x}})\left(\begin{bmatrix}\hat{\dot{\xi}}_{1,r_1}\\ \hat{\dot{\xi}}_{2,r_2}\\ \vdots\\ \hat{\dot{\xi}}_{m,r_m}\end{bmatrix} - \begin{bmatrix}L_f^{r_1}h_1(\hat{\mathbf{x}})\\ L_f^{r_2}h_2(\hat{\mathbf{x}}))\\ \vdots\\ L_f^{r_m}h_m(\hat{\mathbf{x}}))\end{bmatrix}\right) \qquad (19)$$

The estimate $\hat{\phi}$ converges to ϕ asymptotically since the full state vector estimate has also asymptotic convergence (see 5.1 for details).

6. EXAMPLE

The performance of the proposed observer is illustrated in this section on a numerical example. The example does not emphasize the property of the method to handle complex nonlinear systems, instead, it shows the ability of the method to cope with online changes in the input signals properties.

Consider a 4^{th} order linear system already in the form (9),(10):

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{\eta}_1\\ \dot{\eta}_2\end{bmatrix} = \left[\begin{array}{rr|rr} 1.9109 & -3.39 & -0.4793 & -0.0339\\ 1.23 & -5.324 & -2.97 & 0.17\\ \hline -7.92 & 4.79 & 0 & 1\\ 0.57 & -9.11 & -0.15 & 0.8\end{array}\right]\begin{bmatrix}y_1\\ y_2\\ \eta_1\\ \eta_2\end{bmatrix} + \begin{bmatrix}\phi_1(t)\\ \phi_2(t)\\ 0\\ 0\end{bmatrix} \qquad (20)$$

The plant has two measurable outputs y_1 and y_2, two unknown inputs ϕ_1 and ϕ_2 and two internal states η_1 and η_2. The measurement system outputs are the only signals available for use in the scheme. The only knowledge about the system inputs is that each of them has a harmonic characteristic with some particular magnitude and frequency which are assumed to be unknown inside the algorithm. [2]

[2] Since the system is linear and there are 2 harmonic inputs that force its motion, it is assumed that a 4^{th} order exogenous system describes the motion of the forcing term $\theta(\cdot)$ in (10)

The 4×4 linear system state matrix in (20) is Hurwitz and results in a stable system motion. The eigenvalues are placed according to a 4^{th} order Butterworth distribution with $\omega = 1$. However, the sub-matrix which is responsible for the internal dynamics (i.e. $\left[\begin{smallmatrix} 0 & 1 \\ -0.15 & 0.8 \end{smallmatrix}\right]$) is picked to have unstable eigenvalues $\{0.3, 0.5\}$ which gives a nonminimum-phase property to the system.

Immediately after the simulation begins, the algorithm implicitly identifies all the necessary parameters of the input signals (i.e. the corresponding characteristic polynomial) and starts the estimation process. In order to illustrate the power of algorithm, the parameters of the input signals have been changed in the middle of the simulation. The new frequencies and the new magnitudes destroy the estimation process, but the algorithm reacts to the break in the sliding mode in the differentiator which occurs, and re-starts the adaptation process (identification of the new characteristic polynomial). As soon as the new properties of the system are taken into account – the algorithm starts generating the correct estimates for the internal states and the unknown inputs.

In the simulation the 'unknown' inputs are generated according to the following description:

$$\phi_1(t) = \begin{cases} 1.5\,\sin(6.93\,t), & t<25 \\ 2.17\,\sin(4.19\,t), & t\geq 25 \end{cases}$$
$$\phi_2(t) = \begin{cases} 0.34\,\sin(2.57\,t), & t<25 \\ 0.95\,\sin(6.35\,t), & t\geq 25 \end{cases}$$

The simulation results are shown in Figs. 1-4 .

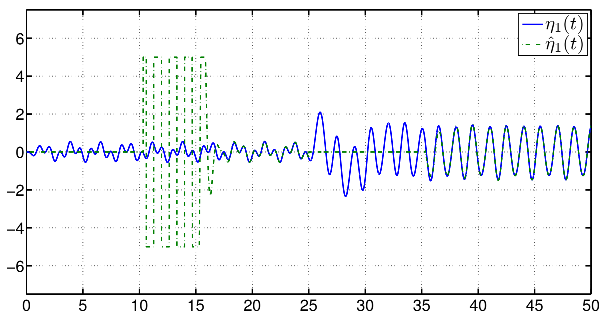

Fig. 1. The estimation of the 1^{st} internal state

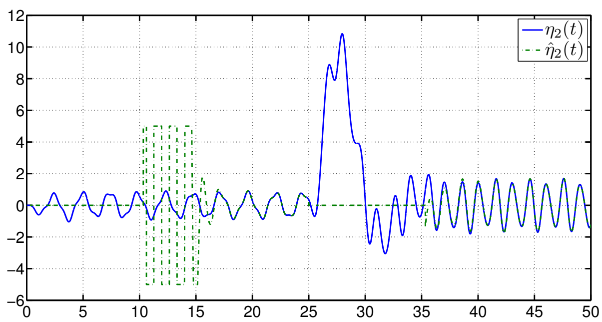

Fig. 2. The estimation of the 2^{nd} internal state

The ability of the proposed algorithm to adapt to the changes in the unknown input properties is effectively

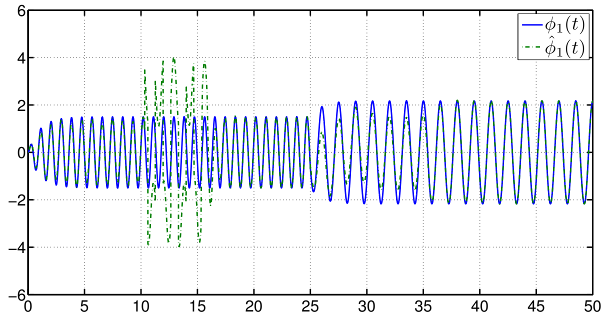

Fig. 3. The estimation of the 1^{st} input

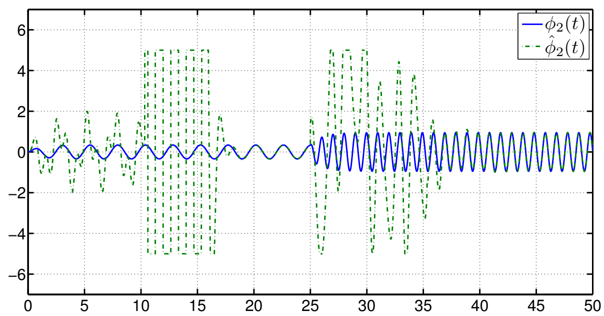

Fig. 4. The estimation of the 2^{nd} input

illustrated in the above plots. The *internal state estimates* along with the *unknown input estimates* asymptotically converge to the true profiles. The estimation process is temporally destroyed in the middle of the simulation because of the changes in the properties of the input signals. However the algorithm adapts to the different conditions and starts generating the correct estimates again.

7. CONCLUSIONS

A full state and unknown input identification problem for a class of non-minimum phase causal nonlinear MIMO systems has been studied:

- first a reversible system transformation to a canonical form involving the I/O and internal dynamics has been undertaken;
- the I/0 states ξ are exactly estimated in finite time using an exact HOSM differentiator;
- The internal state (internal dynamics) η are asymptotically estimated by using the extended method of stable system center and so the full state vector is asymptotically recovered by using the inverse coordinate transformation;
- The unknown input is asymptotically recovered using the inverse coordinate transformation and state vector estimate.

REFERENCES

[1] Baev S, Shkolnikov I, Shtessel Y, Poznyak A. Parameter identification of non-linear system using traditional and high order sliding modes. *Proceedings of ACC'06* Minneapolis, Minnesota, USA, 2006.

[2] Baev S, Shtessel Y, Shkolnikov I. Nonminimum-phase output tracking in causal systems using higher order sliding modes. *Proceedings of ACC'07* New York, NY, USA, 2007.
[3] Baev S, Shtessel Y, Shkolnikov I. HOSM driven output tracking in the nonminimum-phase causal nonlinear systems. *Proceedings of CDC'07* New Orleans, LA, USA, 2007.
[4] Boiko I, Fridman L, Castellanos MI. Analysis of second-order sliding-mode algorithms in the frequency domain. *IEEE Transactions on Automatic Control* 2004; **49**:946-950.
[5] Devasia S, Chen D, Paden B. Nonlinear inversion-based output tracking. *IEEE Transactions on Automatic Control* 1996; **41**:930-942.
[6] Edwards C, Spurgeon S. *Sliding Mode Control*. Taylor & Francis: Bristol, PA, 1998.
[7] Fridman L, Shtessel Y, Edwards C, Yan XG. State Estimation and Input Reconstruction in Nonlinear Systems via Higher Order Sliding Mode Observer. *Proceedings of ACC'07* New York, NY, USA, 2007
[8] Ferreira A, Bejarano F, Fridman L. Output Hierarchical Super Twisting Observation Based Robust Feedback Control: With or Without Chattering? *Proceedings of CDC'07* New Orleans, LA, USA, 2007 (not published).
[9] Giannakis GB, Mendel JM. Identification of nonminimum phase systems using higher order statistics *IEEE Transactions on Acoustics, Speech, and Signal Processing* 1989; **37**(3):360-377.
[10] Isidori A, Byrnes CI. Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control* 1990;**35**:131-140.
[11] Levant A. Robust exact differentiation via sliding mode technique. *Automatica* 1998; **34**(3):379-384.
[12] Levant A. Higher order sliding modes, differentiation and output feedback control. *International Journal of Control* 2003; **26**(9/10):924-942.
[13] Shkolnikov I, Shtessel Y. Tracking in a class of nonminimum-phase systems with nonlinear internal dynamics *via* sliding mode control using method of system center. *Automatica* 2002; **38**:837-842.
[14] Shkolnikov IA, Shtessel YB. Aircraft nonminimum phase control in dynamic sliding manifolds. *AIAA Journal on Guidance, Control, and Dynamics* 2001; **24**(3):566-572.
[15] Shtessel Y, Poznyak A. Parameter identification of affine time varying systems using traditional and high order sliding modes. *Proceedings of ACC'05* Portland, Oregon, USA, 2005.
[16] V.Utkin, J.Gulder, and M.Shijun, *Sliding Mode Control in Electromechanical Systems*, 2nd Edition, Taylor and Francis, 1999

Appendix A. IDENTIFICATION OF CHARACTERISTIC POLYNOMIAL

Consider an *unknown linear dynamical process* of given order k.

$$\begin{cases} \dot{\tau} = A\,\tau \\ \theta = C\,\tau \end{cases} \tag{A.1}$$

where $\tau \in \Re^k$ is the state and $\theta \in \Re^m$ is the output which is available for measurement. The matrices $A \in \Re^{k\times k}$ and $C \in \Re^{m\times k}$ are unknown but they are assumed to satisfy the *observability condition*:

$$\begin{array}{l} \mathrm{rank}(M) = k,\ M = \{M_1^T, \ldots, M_k^T\}^T \in \Re^{k\,m\times k}, \\ M_i = C\,A^{i-1} \in \Re^{m\times k},\ \ i = \overline{1,k} \end{array} \tag{A.2}$$

Consider the characteristic polynomial which corresponds to the process dynamics:

$$P_k(\lambda) = |A - \lambda\, I| = \lambda^k + p_{k-1}\,\lambda^{k-1} + \ldots + p_1\,\lambda + p_0 \tag{A.3}$$

with *unknown coefficients* p_i which are to be identified.

The identification procedure consists of two steps:

(1) Reducing the problem to a regressive form by using the exact HOSM differentiator [12];
(2) Applying the least-squares parameter estimation (LSPE) method [1] to identify the polynomial coefficients.

A.1 Reducing the Problem to Regressive Form

Apply the k^{th}-order exact HOSM differentiator to the j^{th} component of the output vector $\theta \in \Re^m$:

$$\begin{cases} \dot{z}_{0,j} \ = \ \nu_{0,j} \\ \nu_{0,j} \ = \ -\lambda_0\,|z_{0,j} - \theta_j|^{n/(n+1)}\,\mathrm{sign}(z_{0,j} - \theta_j) + z_{1,j} \\ \quad\vdots \\ \dot{z}_{i,j} \ = \ \nu_{i,j} \\ \nu_{i,j} \ = \ -\lambda_i\,|z_{i,j} - \nu_{i-1,j}|^{(k-i)/(k-i+1)}\times \\ \qquad\quad \times\mathrm{sign}(z_{i,j} - \nu_{i-1,j}) + z_{i+1,j}, \\ \quad\vdots \\ \dot{z}_{k,j} \ = \ -\lambda_k\,\mathrm{sign}(z_{k,j} - \nu_{k-1,j}) \end{cases} \tag{A.4}$$

where the term $z_{i,j}$ stands for i^{th} derivative of the j^{th} component of the vector θ, and the coefficients λ_i have been selected to guarantee the finite time convergence of a differentiator [12]. Combining the $z_{i,j}$ by the i^{th} index yields the following:

$$\begin{cases} Z_0 := \{z_{0,1}, z_{0,2}, \ldots, z_{0,m}\}^T = \theta = C\,\tau \\ Z_1 := \{z_{1,1}, z_{1,2}, \ldots, z_{1,m}\}^T = \dot{\theta} = C\,A\,\tau \\ \vdots \\ Z_{k-1} := \{z_{k-1,1}, z_{k-1,2}, \ldots, z_{k-1,m}\}^T = \theta^{(k-1)} = C\,A^{k-1}\,\tau \\ Z_k := \{z_{k,1}, z_{k,2}, \ldots, z_{k,m}\}^T = \theta^{(k)} = C\,A^k\,\tau \end{cases} \tag{A.5}$$

where $Z_i \in \Re^m$ corresponds to the i^{th} derivative of θ.

Introduce two auxiliary vectors:

$$Z := \{Z_0^T, Z_1^T, \ldots, Z_{k-1}^T\}^T \quad \text{and} \quad \bar{Z} := \{Z_1^T, Z_2^T, \ldots, Z_k^T\}^T \tag{A.6}$$

which are related through the time derivative $\bar{Z} \equiv \dot{Z}$.

Using (A.2),(A.5) and (A.6) introduce a linear transformation of the state vector τ:

$$Z = M\,\tau, \tag{A.7}$$

Further introduce an *arbitrary, but known* matrix $D \in \Re^{k\times k\,m}$ of rank k such that $\mathrm{rank}(DM) = k$. Premultiplying both sides of (A.7) by D, and define

$$\tilde{Z} := (D\,Z) \in \Re^k, \qquad \tilde{M} := (D\,M) \in \Re^{k\times k}$$

then since $\tilde{M}$ is *assumed to be nonsingular*

$$\tilde{Z} = \tilde{M}\,\tau \qquad \therefore \quad \tau = \tilde{M}^{-1}\,\tilde{Z}$$

Taking the derivative of both sides, the dynamics of the system which is *similar* to the (A.1) system can be derived as follows:

$$\dot{\tilde{Z}} = \tilde{M}\,\dot{\tau} = \tilde{M}\,A\,\tau = \underbrace{\tilde{M}\,A\,\tilde{M}^{-1}}_{\tilde{A}}\,\tilde{Z} = \tilde{A}\,\tilde{Z} \tag{A.8}$$

Recalling that $\tilde{Z} = D\,Z$ and $\dot{Z} = \bar{Z}$ gives

$$\dot{\tilde{Z}} = D\,\bar{Z}$$

System (A.8) can be treated as a set of k linear equations in the *regressive form*:

$$\begin{array}{ll} H = K\,S & -\ \text{equation in the regressive form;} \\ H := \dot{\tilde{Z}} = D\,\bar{Z} & -\ \text{known left-hand side vector;} \\ S := \tilde{Z} = D\,Z & -\ \text{known right-hand side vector;} \\ K := \tilde{A} & -\ \textit{unknown} \text{ matrix to be identified;} \end{array} \tag{A.9}$$

or in the scalar notation:

$$H_i = \sum_{j=1}^{k} K_{i,j}\,S_j \tag{A.10}$$

The unknown coefficients $K_{i,j}$ for $i, j = \overline{1, k}$ in scalar equations (A.10) can be identified via the least squares parameter estimation (LSPE) method presented in the sequel. As soon as the matrix $\tilde{A} \equiv K$ is estimated, its characteristic polynomial can easily be computed. Since from (A.8) the matrices $\tilde{A}$ and A are similar, the characteristic polynomials are the same.

A.2 LSPE Method

Single parameter identification. Consider a scalar linear equation:

$$h(t) = k\,s(t) \tag{A.11}$$

where k is the constant coefficient to be identified; $s(t)$ and $h(t)$ are known signals. The unknown parameter k can not be determined uniquely, since values of functions $s(t)$ and $h(t)$ do not necessarily satisfy condition (A.11) for all time moments.

Multiplying (A.11) by $s(t)$ and integrating both sides from some initial time moment t_0 to the current time gives the following:

$$k \int_{t_0}^{t} s(\tau)^2\,d\tau = \int_{t_0}^{t} s(\tau)\,h(\tau)\,d\tau$$

which yields a way to determine the unknown scalar coefficient k in real time starting from some initial time moment t_0:

$$k = \frac{\displaystyle\int_{t_0}^{t} s(\tau)\,h(\tau)\,d\tau}{\displaystyle\int_{t_0}^{t} s(\tau)^2\,d\tau} \tag{A.12}$$

Multiple parameters identification. Consider a linear equation that fits a regressive form of order k:

$$h(t) = k_1\,s_1(t) + k_2\,s_2(t) + \ldots + k_k\,s_k(t)$$

where $s_i(t)$ and $h(t)$ are known functions (with values which are measured or computed) and $\{k_1, k_2, ..., k_k\}$ is a vector of unknown constants to be identified.

In fact, there are ***k*** *unknowns and only* ***one*** *equation*, thus, this *problem can not be solved uniquely.* Since $s_i(t)$ and $h(t)$ are time functions, therefore, the following equations can be obtained:

$$\begin{cases} h(t_0) = s_1(t_0)\,k_1 + \ldots + s_k(t_0)\,k_k \\ h(t_1) = s_1(t_1)\,k_1 + \ldots + s_k(t_1)\,k_k \\ \vdots \\ h(t_{k-1}) = s_1(t_{k-1})\,k_1 + \ldots + s_k(t_{k-1})\,k_k \end{cases}$$

where $t_j = t + j \cdot \Delta$, and Δ is some reasonable constant time interval.

All of these equations can be grouped into a k-order linear algebraic system:

$$\begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,k} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ s_{k,1} & s_{k,2} & \cdots & s_{k,k} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_k \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_k \end{bmatrix} \tag{A.13}$$

where the following notation has been used:

$$s_{i,j} \equiv s_j(t_{i-1}), \quad h_i \equiv h(t_{i-1})$$

Each unknown k_i can be found, for instance, by means of Kramer's rule:

$$k_j = \frac{\begin{vmatrix} s_{1,1} & s_{1,2} & \ldots & (s_{1,j} \to h_1) & \ldots & s_{1,k} \\ s_{2,1} & s_{2,2} & \ldots & (s_{2,j} \to h_2) & \ldots & s_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{k,1} & s_{k,2} & \ldots & (s_{k,j} \to h_k) & \ldots & s_{k,k} \end{vmatrix}}{\begin{vmatrix} s_{1,1} & s_{1,2} & \ldots & s_{1,j} & \ldots & s_{1,k} \\ s_{2,1} & s_{2,2} & \ldots & s_{2,j} & \ldots & s_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{k,1} & s_{k,2} & \ldots & s_{k,j} & \ldots & s_{k,k} \end{vmatrix}} \tag{A.14}$$

Since $s_{i,j}$ and h_i are time functions, they form time dependent functions in the numerator and denominator of (A.14). Therefore, this equation can be rewritten as follows:

$$k_j = \frac{S_j(t)}{S(t)}, \qquad j = \overline{1, k}$$

Assuming $S(t) \neq 0$, the following equality $S(t)\,k_j = S_j(t)$ holds and thus can be solved by the scalar parameter identification problem considered earlier:

$$k_j = \frac{\displaystyle\int_{t_0+k\cdot\Delta}^{t} S(\tau)\,S_j(\tau)\,d\tau}{\displaystyle\int_{t_0+k\cdot\Delta}^{t} S(\tau)^2\,d\tau}, \qquad j = \overline{1, k} \tag{A.15}$$

In order to avoid division by 0, *the initial condition for the second integral (denominator) should not be equal to zero.*

A.3 Identification of Polynomial Coefficients (solving the regressive form problem)

Recalling the original problem of characteristic polynomial identification, the multi-parameter case must be applied k times (once for each scalar equation (A.10) with $i = \overline{1, k}$) to estimate the matrix $\tilde{A} = K = \{K_{i,j}\}$ for $i, j = \overline{1, k}$. Once the matrix $\tilde{A}$ estimated, it is a straightforward problem to identify its characteristic polynomial $P(\lambda)$ which coincides with the one for (A.1) because of their similarity.