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Nonlinear H_∞ control and the Hamilton-Jacobi-Isaacs equation

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Abstract: This paper considers two aspects of the nonlinear H_∞ control problem: the use of weighting functions for performance and robustness improvement, as in the linear case, and the development of a Galerkin approximation method for the solution of the Hamilton-Jacobi-Isaacs Equation (HJIE) that arises in the output feedback case. Design of nonlinear H_∞ controllers obtained by Taylor approximation and by the proposed Galerkin approximation method applied to a magnetic levitation system are presented.

Keywords: H-infinity control; Robust control; Robust estimators; Nonlinear control systems; Partial differential equations.

1. INTRODUCTION

The development of a systematic analysis of the nonlinear equivalent of the H_∞ control problem was initiated by the important contributions of Ball and Helton (1989), Başar and Bernhard (1990), and van der Schaft (1991). Although the H_∞ norm is defined as a norm on transfer matrices, when translated into the time domain, it is nothing else than the L_2-induced norm (from the input time-functions to the output time-functions for initial state zero). van der Schaft (1992) showed that the solutions of the problem in question in the case of state feedback can be determined from the solution of a Hamilton-Jacobi equation, the Hamilton-Jacobi-Isaacs Equation (HJIE) or Inequality, which is the nonlinear version of the Riccati equation considered in the H_∞ control problem for linear systems. The nonlinear H_∞ control problem via output feedback was considered, for example, by Isidori and Astolfi (1992), Ball et al. (1993), and Isidori and Kang (1995). The nonlinear output feedback H_∞ controllers have separation structures and necessary and sufficient conditions for the H_∞ control problem to have solutions involving two HJIE's, associated with the design of a state-feedback and an output-injection gain, respectively.

Although the formulation of the nonlinear theory of H_∞ control has been well developed, solving the HJIE remains a challenge and is the major bottleneck for the practical application of the theory (Beard and McLain, 1998; Aliyu, 2003; Abu-Khalaf et al., 2004). The HJIE is a first-order, nonlinear partial differential equation not solved analytically in the general case and usually very difficult to be solved for specific nonlinear systems. Thus, several numerical methods have been proposed for its solution. Starting with the work of Lukes (1969), who proposed a polynomial approximation approach based on Taylor series, many others authors have proposed similar approaches to the solution of the problem. Huang and Lin (1995), for example, find a smooth solution to the HJIE by solving for the Taylor series expansion coefficients in a efficient and organized manner. Beard and McLain (1998) combine successive approximation and Galerkin approximation methods to derive a novel algorithm that produces stabilizing *state feedback* control laws with well-defined stability regions.

This paper has two purposes. Firstly, it is shown that dynamic weighting functions can be used to improve the performance and robustness of the nonlinear H_∞ controller such as in the design of H_∞ controllers for linear plants. In the literature only static weighting functions have been explored. Secondly, the Galerkin successive approximation method is used to find approximate solutions for the nonlinear H_∞ control problem in the *output feedback* case. The results are applied to a magnetic levitation system.

2. NONLINEAR H_∞ OUTPUT FEEDBACK CONTROLLER

Consider the following affine nonlinear state-space system

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)w + g_2(x)u, \\ z &= h_1(x) + k_{11}w + k_{12}u, \\ y &= h_2(x) + k_{21}(x)w, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^r$ is the exogenous input which includes disturbance (to be rejected) and/or references (to be tracked), $z \in \mathbb{R}^s$ is the penalty variable and $y \in \mathbb{R}^p$ is the measured variables.

The mapping $f(x)$, $g_1(x)$, $g_2(x)$, $h_1(x)$, $h_2(x)$, $k_{11}(x)$, $k_{12}(x)$ and $k_{21}(x)$ are smooth mappings (i.e., mappings of class C^k for some sufficiently large k) defined in a neighborhood of the origin in $\mathbb{R}^n$. It is also assumed that $f(0) = 0$, $h_1(0) = 0$, and $h_2(0) = 0$.

The control action to (1) is provided by a controller, which processes the measured variable y and generates the appropriate control input u by

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y), \\ u &= \theta(\xi), \end{aligned} \tag{2}$$

where ξ is defined on a neighborhood Ξ of the origin in $\mathbb{R}^\nu$ and $\eta : \Xi \times \mathbb{R}^p \to \mathbb{R}^\nu$, $\theta : \Xi \to \mathbb{R}^m$ are C^k functions (for some $k \geq 1$), satisfying $\eta(0, 0) = 0$ and $\theta(0) = 0$.

The purposes of the controller are: to achieve closed-loop stability and to attenuate the influence of the exogenous input w on the penalty variable z. A controller which locally asymptotically stabilizes the equilibrium $(x, \xi) = (0, 0)$ of the closed-loop system is said to be an admissible controller. The requirement of disturbance attenuation is characterized in the following manner: given a real number $\gamma > 0$, it is said that the exogenous signals are locally attenuated by γ if there exists a neighborhood U of the point $(x, \xi) = (0, 0)$ so that for every $T > 0$ and for every piecewise continuous function $w : [0, T] \to \mathbb{R}^r$ for which the state trajectory of the closed-loop system (1), (2) starting from the initial state $(x(0), \xi(0)) = (0, 0)$ remains in U for all $t \in [0, T]$, the response $z : [0, T] \to \mathbb{R}^s$ of (1), (2) satisfies

$$\int_0^T z^T(s)z(s)ds \leq \gamma^2 \int_0^T w^T(s)w(s)ds.$$

The nonlinear H_∞ control problem using output feedback consists in finding an admissible controller yielding local attenuation of the exogenous input. In order to describe the solution of the nonlinear H_∞ control problem using output feedback, a notion of detectability is necessary.

Definition 1. Suppose $f(0) = 0$ and $h(0) = 0$. The pair $\{f, h\}$ is said to be locally detectable if there exists a neighborhood U of the point $x = 0$ so that, if $x(t)$ is any integral curve of $\dot{x} = f(x)$ satisfying $x(0) \in U$, then $h(x(t))$ is defined for all $t \geq 0$ and $h(x(t)) = 0$ for all $t \geq 0$ implies $\lim_{t\to\infty} x(t) = 0$.

Theorem 2. (Isidori and Astolfi (1992)). Consider system (1) and suppose the following

H1) The pair (f, h_1) is locally detectable.

H2) There exists a smooth positive definite function $V(x)$, locally defined in a neighborhood of the origin in $\mathbb{R}^n$, which satisfies the HJIE

$$V_x f + h_1^T h_1 + \gamma^2 \alpha_1^T \alpha_1 - \alpha_2^T \alpha_2 = 0, \tag{3}$$

where

$$\alpha_1 = \frac{1}{2\gamma^2} g_1^T V_x^T, \quad \alpha_2 = -\frac{1}{2} g_2^T V_x^T. \tag{4}$$

H3) There exists an $n \times p$ matrix G, so that the equilibrium $\xi = 0$ of the system

$$\dot{\xi} = f(\xi) + g_1(\xi)\alpha_1(\xi) - Gh_2(\xi) \tag{5}$$

is locally asymptotically stable.

H4) There exists a smooth positive semidefinite function $W(x, \xi)$, locally defined in a neighborhood of the origin $\mathbb{R}^n \times \mathbb{R}^n$ and such that $W(0, \xi) > 0$ for each $\xi \neq 0$, which solves the HJIE

$$(W_x \ W_\xi) f_e + h_e^T h_e + \gamma^2 \Phi^T \Phi = 0, \tag{6}$$

where

$$\begin{aligned} f_e &= \begin{pmatrix} f(x) + g_1(x)\alpha_1(x) + g_2(x)\alpha_2(x) \\ f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi) \\ +G(h_2(x) - h_2(\xi)) \end{pmatrix}, \\ h_e &= \alpha_2(\xi) - \alpha_2(x), \\ \Phi &= \frac{1}{2\gamma^2}(W_x g_1(x) + W_\xi G k_{21}(x))^T. \end{aligned} \tag{7}$$

Then, the problem of nonlinear H_∞ control is solved by the output feedback

$$\begin{aligned} \dot{\xi} &= f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi) + G(y - h_2(\xi)), \\ u &= \alpha_2(\xi). \end{aligned} \tag{8}$$

Remark 3. In order to simplify the analysis and to provide a reasonable expression of the controller, it is assumed that the coefficient matrices which characterize the plant (1) satisfy the following assumptions, which are the nonlinear versions of the standing assumptions considered, for example, by Doyle et al. (1989)

$$\begin{aligned} &k_{11}(x) = 0, \quad h_1^T(x)k_{12}(x) = 0, \ k_{12}^T(x)k_{12}(x) = I, \\ &k_{21}(x)g_1^T(x) = 0, \ k_{21}^T(x)k_{21}(x) = I. \end{aligned} \tag{9}$$

Existence of a solution to (3) and (6) is shown to exist in the conditions of the following proposition.

Proposition 4. (Isidori and Astolfi (1992)). Let system (1) linearized around the origin, given by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w. \end{aligned} \tag{10}$$

Assume that the linear system (10) satisfies

L1) The pair (A, B_1) is stabilizable.

L2) The pair (A, C_1) is detectable.

L3) There exists a positive definite symmetric solution X of the Riccati equation

$$A^T X + XA + C_1^T C_1 - XB_2 B_2^T X + \frac{1}{\gamma^2} XB_1 B_1^T X = 0. \tag{11}$$

L4) There exists a positive definite symmetric solution Y of the Riccati equation

$$YA^T + AY + B_1 B_1^T - YC_2^T C_2 Y + \frac{1}{\gamma^2} YC_1^T C_1 Y = 0. \tag{12}$$

L5) $\rho(XY) < \gamma^2$.

(that is, there exists a solution for the linear H_∞ control problem via output feedback). Then hypotheses H1)–H4) of theorem 2 hold and the nonlinear H_∞ control problem via output feedback is solvable with

$$\begin{aligned} G &= ZC_2^T, \quad V(x) = x^T X x, \\ W(x, \xi) &= \gamma^2 (x - \xi)^T Z^{-1} (x - \xi), \\ Z &= Y \left(I - \frac{1}{\gamma^2} XY \right)^{-1}. \end{aligned}$$

3. THE WEIGHTED NONLINEAR H_∞ CONTROL PROBLEM

The penalty variable z comprises any output whose L_2 norm is desired to be minimized. In the linear H_∞ con-

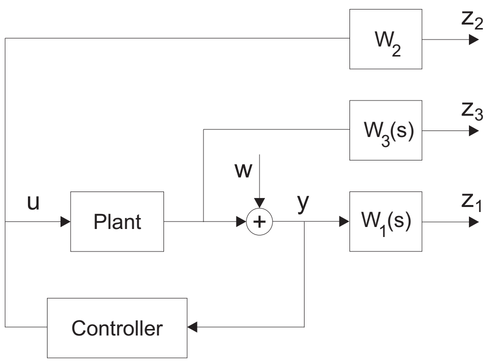


Fig. 1. Closed loop system including weighting functions

troller design, the penalty variable can be weighted as required, either through static weighting, frequency (linear) weighting, nonlinear weighting function or any combination of these. The penalty variable can include tracking error, actuator effort, sensitivity specifications via norm bounds, robustness specifications via norm bounds and frequency response criteria. The selection of weighting functions for linear design problem is detailed in (Zhou and Doyle, 1998). On the other hand, in the nonlinear H_∞ controller design, only static weighting functions have been explored (e.g., Sinha and Pechev, 2004). In the present paper it is shown that dynamic weighting functions can also be used in the nonlinear H_∞ controller design, with similar effects to the linear case.

Consider a nonlinear system modeled by equations of the form

$$\begin{aligned} \dot{x}_p &= f_p(x_p) + g_{p1}(x_p)w + g_{p2}(x_p)u, \\ y &= h_{p2}(x_p) + k_{p21}(x_p)w. \end{aligned} \tag{13}$$

Consider that linear dynamic weighting functions $W_1(s)$ and $W_3(s)$ with state space description

$$\begin{aligned} \dot{x}_{w1} &= A_{w1}x_{w1} + B_{w1}(h_{p2}(x_p) + k_{p21}(x_p)w), \\ z_1 &= C_{w1}x_{w1}, \\ \dot{x}_{w3} &= A_{w3}x_{w3} + B_{w3}h_{p2}(x_p), \\ z_3 &= C_{w3}x_{w3} + D_{w3}h_{p2}(x_p), \end{aligned} \tag{14}$$

are included in the design as in Fig. 1. In addition, consider that

$$z_2 = W_2 u \tag{15}$$

weights the control effort through the constant gain W_2, for simplicity. Combining (13), (14) and (15), the system can be expressed as (1) with

$$x = \begin{bmatrix} x_p \\ x_{w1} \\ x_{w3} \end{bmatrix}, \quad f(x) = \begin{bmatrix} f_p(x_p) \\ A_{w1}x_{w1} + B_{w1}h_{p21}(x_p) \\ A_{w3}x_{w3} + D_{w3}h_{p2}(x_p) \end{bmatrix},$$

$$g_1(x) = \begin{bmatrix} g_{p1}(x_p) \\ B_{w1}k_{p21}(x_p) \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} g_{p2}(x_p) \\ 0 \\ 0 \end{bmatrix},$$

$$h_1(x) = \begin{bmatrix} C_{w1}x_{w1} \\ 0 \\ C_{w3}x_{w3} + D_{w3}h_{p2}(x_p) \end{bmatrix}, \quad k_{11}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$k_{12}(x) = \begin{bmatrix} 0 \\ W_2 \\ 0 \end{bmatrix}, \quad h_2(x) = h_{p2}(x_p), \quad k_{21}(x) = k_{p21}(x_p).$$

Suppose that there is a linear H_∞ controller to the linear system obtained from the linearization of system (13). Then, under the conditions of proposition 4, there will be a nonlinear H_∞ controller for the system (13). In other words, if the Riccati equations (11) and (12) are solvable for the linearized system, then the HJIEs (3) and (6) are solvable in a neighborhood of the origin in $\mathbb{R}^n$.

4. GALERKIN SUCCESSIVE APPROXIMATION FOR NONLINEAR H_∞ CONTROL PROBLEM VIA OUTPUT FEEDBACK

4.1 Successive approximation

The HJIE (6) can be rewritten as

$$K(x, W_x^T, w_{**}, y_*) - H(x, V_x^T, w_*, u_*) = 0, \tag{16}$$

where

$$\begin{aligned} K(x, W_x^T, w_{**}, y_*) &= W_x f + W_x g_1 w_{**} - y_*^T h_2 - y_*^T k_{21} w \\ &\quad + h_1^T h_1 - \gamma^2 w_{**}^T w_{**} \\ w_{**} &= \frac{1}{2\gamma^2} g_1^T W_x - k_{21}^T h_2 \end{aligned}$$

and $H(x, V_x^T, w_*, u_*)$ is the left-hand side of (3). The basic idea of successive approximation is to compute W_* and w_{**} iteratively, as shown below.

Algorithm 1. Let w be an exogenous input with stability region Ω.

For $i = 0$ to ∞
Solve for $w^{(i)}$ from
$W_x f + W_x g_1 w^{(i)} - y^T h_2 - y^T k_{21} w^{(i)} + h_1^T h_1$
$-\gamma^2 \|w^{(i)}\|^2 - H_*$
Update the exogenous input
$w^{(i+1)} = \frac{1}{2\gamma^2} g_1^T W_x - k_{21}^T h_2$
End.

The essence of algorithm 1 is to reduce the HJIE to an infinite sequence of linear partial differential equations. Since these equations are difficult to be solved analytically, in the next section a Galerkin approximation method will be used to construct an approximate solution to the HJIE.

4.2 Galerkin approximation method

Let a partial differential equation $\mathcal{A}(V) = 0$ with boundary conditions $V(0) = 0$. Galerkin method assumes a complete set of basis functions $\{\phi_j\}_{j=1}^{\infty}$, so that $\phi_j(0) = 0, \forall j$ and $V(x) = \sum_{j=1}^{\infty} c_j \phi_j(x)$, where the sum is assumed to converge pointwise in some set Ω. An approximation to V is formed by truncating the series to $V_N(x) = \sum_{j=1}^{N} c_j \phi_j(x)$. The coefficients c_j are obtained by solving the algebraic equations

$$\int_\Omega \mathcal{A}(V_N(x))\phi_l(x)dx = 0, \quad l = 1, \dots, N. \tag{17}$$

For a more rigorous and complete treatment see, for example, (Fletcher, 1984).

4.3 Output feedback

In this section an algorithm for output feedback control design is proposed. In fact, the proposed algorithm is a dual version of the state feedback algorithm originally proposed in (Beard and McLain, 1998). Assume that $w : \Omega \to \mathbb{R}^s$ is the exogenous input for the system (1) on a compact set Ω. Also assume that the set $\{\phi_w^{(j)}\}_{j=1}^{\infty}$ is a complete basis set for the domain of the HJIE (16). Then, according to (17), an approximate solution to (16) is given by $W_N(x) = \sum_{j=1}^{N} c_w^{(j)} \phi_w^{(j)}(x)$, where the coefficients satisfy the equation

$$\int_{\Omega} [K(x, W_x^T, w_{**}, y_*) - H(x, V_x^T, w_*, u_*)]\phi_l dx = 0. \quad (18)$$

Substituting $W(x)$ by $W_N(x)$ in (18) and defining $c_w = [\, c_w^{(1)} \ldots c_w^{(N)} \,]^T$, $\phi_w(x) = [\, \phi_w^{(1)} \ldots \phi_w^{(N)} \,]$ and $\nabla\phi_w(x) = [\, \partial\phi_w^{(1)}/\partial x \ldots \partial\phi_w^{(N)}/\partial x \,]$, (18), after some algebraic manipulations, can be rewritten as

$$\left(A_1 + \frac{A_2(c_w^{(n)})}{2\gamma^2}\right) c_w^{(n+1)} = \gamma^2 b_1 + b_2 + \frac{A_2(c_w^{(n)})c_w^{(n)}}{4\gamma^2} + \frac{b_3}{4\gamma^2} + \frac{b_4}{4},$$

where

$$A_1 = \int_{\Omega} \phi_w f^T \nabla\phi_w^T dx, \qquad A_2(c_w) = \sum_{j=1}^{N} c_w^{(j)} X^{(j)},$$

$$X^{(j)} = \int_{\Omega} \phi_w \frac{\partial\phi_w^{(j)}}{\partial x}^T g_1 g_1^T \nabla\phi_w^T dx,$$

$$b_1 = \int_{\Omega} \phi_w h_2^T h_2 dx, \qquad b_2 = \int_{\Omega} \phi_w f^T \nabla\phi_v^T c_v dx,$$

$$b_3 = \int_{\Omega} \phi_w c_v^T \nabla\phi_v g_1 g_1^T \nabla\phi_v^T c_v dx,$$

$$b_4 = \int_{\Omega} \phi_w c_v^T \nabla\phi_v g_2 g_2^T \nabla\phi_v^T c_v dx.$$

The coefficients c_w pull outside the integrals, and $A_2(w)$, $b_2(w)$ and $b_3(u)$ can be computed iteratively once the matrices $\{X^{(j)}\}_{j=1}^{N}$ have been calculated.

Algorithm 2. Let $W^{(0)}$ be an initial stabilizing solution to (1). Let c_v be the coefficients obtained by algorithm 1 to approximate (3). Pre compute the integrals A_1, $A_2(w)$, b_1, b_2, b_3, b_4 and $\{X^{(j)}\}_{j=1}^{N}$.

For $i = 0$ to ∞
  If $i = 0$
    $A^{(i)} = A_1 + \frac{1}{2\gamma^2} A_2(W_x^{(0)})$
    $b^{(i)} = \gamma^2 b_1 + \frac{1}{4\gamma^2} A_2(W_x^{(0)})c_w^{(0)} + b_2 + \frac{1}{4\gamma^2} b_3 - \frac{1}{4} b_4$
  Else
    $A^{(i)} = A_1 + \frac{1}{2\gamma^2} \sum_{j=1}^{N} c_w^{(i-1)}(j) X^{(j)}$
    $b^{(i)} = \gamma^2 b_1 + \frac{1}{4\gamma^2} \sum_{j=1}^{N} c_w^{(i-1)}(j) X^{(j)} c_w^{(i-1)} + b_2$
    $+ \frac{1}{4\gamma^2} b_3 - \frac{1}{4} b_4$
  End
  $c_w^{(i)} = \left[A^{(i)}\right]^{-1} b^{(i)}$
End
Extract $R_1(x)$ from $x^T R_1(x) = c_w^T \nabla\phi_w - c_v^T \nabla\phi_v$
Extract $L(x)$ from $x^T L(x) = 2\gamma^2 h_2$
Compute the output feedback with $G(x) = R_1^{-1}(x)L(x)$.

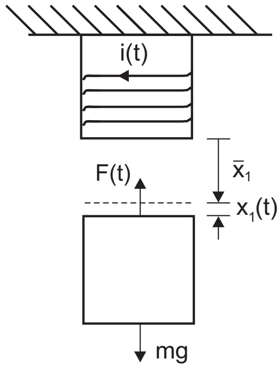


Fig. 2. Magnetic levitation system

5. NONLINEAR H_∞ CONTROL DESIGN FOR A MAGNETIC LEVITATION SYSTEM

In this section, nonlinear H_∞ controllers are designed and applied to a magnetic levitation system. In section 5.2 a nonlinear controller based on the Taylor approximation method, as proposed in (van der Schaft, 1992), is designed to illustrate the benefits produced by dynamic weighting functions. A similar design using static weighting functions is presented in (Sinha and Pechev, 2004). In section 5.3 a controller based on the proposed Galerkin approximation method is designed and compared to the Taylor approximation controller. In order to focus on the controllers, weighting functions are not employed here.

5.1 Nonlinear model for the magnetic levitation system

The magnetic levitation system considered is schematically represented in Fig. 2, where $i(t)$ indicates the current through the magnetic bearing, that produces the attraction force $F(t)$, m is the mass to be levitated and $x_1(t)$ represents the deviation from the desired levitation gap $\bar{x}_1$. Defining $u(t) = i^2(t)$ as the control action and considering w_1 and w_2 as disturbances and g the gravity force, the system is described by the following equations

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= g - g\frac{\bar{x}_1^2}{(\bar{x}_1 + x_1)^2} - \frac{k}{m}\frac{u}{(\bar{x}_1 + x_1)^2} + \frac{1}{m}w_1, \\ y &= x_1 + w_2, \end{aligned} \quad (19)$$

which may be put in the affine form (13), where

$$f_p(x) = \begin{bmatrix} x_2 \\ g - g\dfrac{\bar{x}_1^2}{(\bar{x}_1 + x_1)^2} \end{bmatrix}, \quad g_{p1}(x) = \begin{bmatrix} 0 & 0 \\ \dfrac{1}{m} & 0 \end{bmatrix},$$

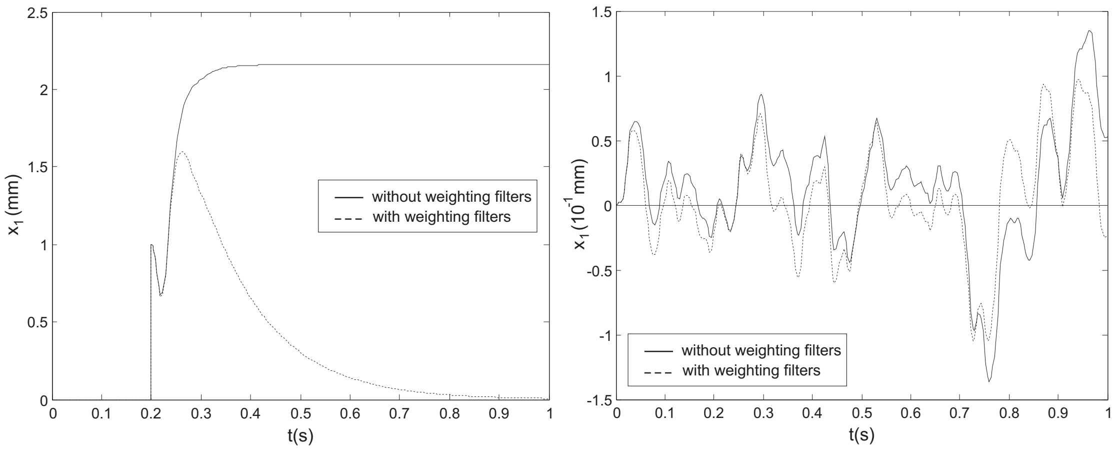
$$g_{p2}(x) = \begin{bmatrix} 0 \\ -\dfrac{k}{m}\dfrac{1}{(\bar{x}_1 + x_1)^2} \end{bmatrix}, \quad h_{p1}(x) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},$$

$$k_{p11}(x) = 0, \quad k_{p12}(x) = [\, 0 \ 0 \ 1 \,]^T,$$

$$h_{p2}(x) = x_1, \quad k_{p21} = [\, 0 \ 1 \,].$$

The adopted parameter values for the system are

$$1.6 \cdot 10^{-4} N/m, \qquad m = 0.240kg,$$
$$\bar{x}_1 = 5mm, \qquad g = 9.8m/s^2.$$



(a) Disturbance rejection

(b) Noise atenuation

Fig. 3. Performance and robustness simulation results

5.2 Taylor approximation design including dynamic weighting functions

The weighting function W_1 is specified to produce an integral effect at low frequencies and W_3 aims at noise attenuation.

$$W_1(s) = \frac{100}{s+0.001},\ W_2(s) = 0.01,\ W_3(s) = \frac{s+0.001}{s+1000}.$$

These weighting functions are included as shown in Fig. 1, and the global system is written in the affine form as in section 3. Thus, the first order Taylor approximation to the corresponding HJIE (3) produces the following state feedback law

$$u = 5201.47x_1 + 127.44x_2 + 10099.51x_3 - 6.85x_4. \quad (20)$$

In order to get a simpler observer, W_1 and W_3 were implemented and the corresponding states, x_3 and x_4, directly measured. Thus, the first order Taylor approximation to the corresponding HJIE (6) produces the following output injection

$$G = \begin{bmatrix} 206.29 \\ 12916.01 \end{bmatrix}.$$

The lowest value of γ for which positive definite solutions were found to (11) and (12), which correspond to the first order Taylor approximations to (3) and (6), is $\gamma = 200$. Fig. 3(a) shows the integral effect produced by W_1; a unit step disturbance, applied at $t = 0.1$s, produces a steady state error when $W_1 = 1$ and zero steady state error for W_1 as specified. Fig. 3(b) shows the noise attenuation effect produced by W_3; for a white noise with standard deviation 0.1 N, added to w_1, the output variance for $W_3 = 1$ is $4.67 \cdot 10^{-5}$ mm and $4.10 \cdot 10^{-5}$ mm for W_3 as specified.

5.3 Galerkin Successive Approximation design

In this section nonlinear output feedback controller designs based on Taylor approximation and on the proposed Gallerkin approximation are presented. Weighting functions are not included here. In this case, the Taylor approximation design is given by

$$\begin{aligned} u &= 4706.12x_1 + 125.09x_2, \\ G &= \begin{bmatrix} 206.29 \\ 12916.01 \end{bmatrix}. \end{aligned} \quad (21)$$

In the Galerkin approximation design the basis functions

$$\phi_v = \begin{bmatrix} (\bar{x}_1 + x_1)^2 x_1 x_2 \\ \frac{1}{2}(\bar{x}_1 + x_1)^2 x_2^2 \end{bmatrix}, \quad \phi_w = \begin{bmatrix} x_1^3 + 10^3 x_2^2 \\ x_1^2 + 10^3 x_2^3 \end{bmatrix},$$

are proposed to the state feedback HJIE (3) and to the output feedback HJIE (6), respectively. The state feedback basis is chosen so that the control law given by (4) is linear. Using Galekin successive approximation algorithm for state feedback proposed by Beard and McLain (1998), the following control law is obtained

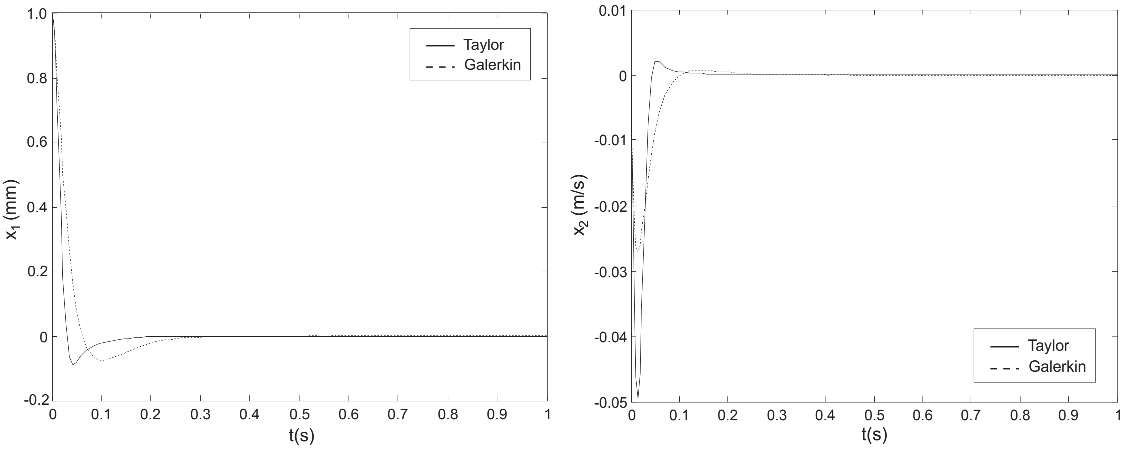
$$u = 6197.37x_1 + 162.70x_2.$$

The basis for the output feedback design is chosen so that $G(x)$ is different from zero when the observer error is zero. Thus, algorithm 2, initialized with the control law obtained by Taylor approximation (21), produces

$$G = \begin{bmatrix} 8,00 \cdot 10^4(-1,00 \cdot 10^5 - 3,95 \cdot 10^2 x_2 + 7,81x_1 \\ +7,80 \cdot 10^2 x_1^2)/(-1,50 \cdot 10^7 x_1 - 5,76 \cdot 10^4 x_1 x_2 \\ +4,87 \cdot 10^6 x_1^2 + 4,44 \cdot 10^7 x_1^3 - 2,64 \cdot 10^7 \\ -1,04 \cdot 10^5 x_2 + 3,48 \cdot 10^5 x_1^2 x_2 + 2,65 \cdot 10^9 x_1^4 \\ +2,32 \cdot 10^7 x_1^3 x_2) \\ \\ -8,00 \cdot 10^4(1,49 \cdot 10^2 + 5,95 \cdot 10^2 x_1 \\ +8,92 \cdot 10^4 x_1^2 + 3,90x_2 + 7,81 \cdot 10^2 x_1 x_2)/ \\ (-1,50 \cdot 10^7 x_1 - 5,76 \cdot 10^4 x_1 x_2 + 4,87 \cdot 10^6 x_1^2 \\ +4,44 \cdot 10^7 x_1^3 - 2,64 \cdot 10^7 - 1,04 \cdot 10^5 x_2 \\ +3,48 \cdot 10^5 x_1^2 x_2 + 2,65 \cdot 10^9 x_1^4 + 2,32 \cdot 10^7 x_1^3 x_2) \end{bmatrix}.$$

The lowest value for γ, for which algorithm 2 converges is $\gamma = 20$, indicating the possibility of a higher noise attenuation. Notwithstanding, for comparison purposes with the Taylor approximation design, $\gamma = 200$ was used.

The adopted stability domain Ω is $x_1 < |1.0 \cdot 10^{-3}|$ and $x_2 < |0.3|$. In this domain the control law used for initializing algorithm 2 stabilizes the closed loop system. It is worth observing that, while the Galerkin method does not decrease the stability domain, in the Taylor



(a) Position (b) velocity

Fig. 4. Position and velocity for output feedback for Taylor and Galerkin approximation controllers

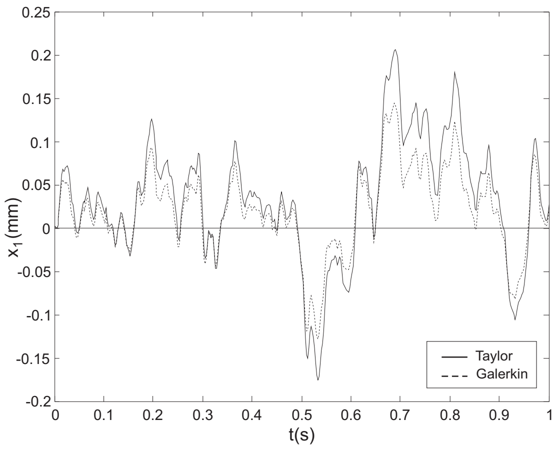


Fig. 5. Noise attenuation

approximation method the stability domain is not known a priori, either.

Fig. 4 shows the position x_1 and the velocity x_2 in the levitation transient for the Taylor and Galerkin controllers. As observed, Galerkin controller produces a smoother transient. Fig. 5 shows the higher noise attenuation produced by Galerkin controller.

6. CONCLUSION

In this paper, benefits of dynamic weighting functions are explored in nonlinear H_∞ control design, similarly to the linear case. An output feedback method based on Galerkin approximation is also proposed and shown to be more efficient than the already established Taylor approximation method in the case of a magnetic levitation system.

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