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Observation and Separation Principle for the Chattering-Free Control of Multi-Input Systems with Simplex Sliding Mode ⋆

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Abstract: In this paper a method to simultaneously achieving the two main objective of smoothing the control while maintaining ideally infinite frequency regimes is particularized to the simplex sliding mode method. The present paper proposes a simplex sliding mode control logic based on the time derivative of the control vector. This practice, beside the standard chattering reduction effect, has the important consequence of making possible the control of a rather wide class of uncertain systems nonlinear in the control. In the uncertain case the increment of the relative degree implicit in the proposed approach requires the introduction of second order sliding mode observer. A separation theorem is proven and sufficient conditions for the finite time convergence of both the estimation and tracking errors are found in terms of further requirements on the control amplitude.

1. INTRODUCTION

This paper considers the simplex sliding mode control, Bajda and Izosimov [1985], and introduces further developments of this technique with respect to previous works, Bartolini et al. [1999], Bartolini et al. [2004]. The simplex method presents some advantages compared with standard discontinuous control strategies for multi-input uncertain systems. In this paper a method to simultaneously achieving the two main objective of smoothing the control while maintaining ideally infinite frequency regimes is particularized to the simplex sliding mode method. It is followed the well known approach, Bartolini and Pydynowski [1996], which considers an augmented order system having as input the time derivative of the actual control vector. The present paper proposes a simplex sliding mode control logic based on the time derivative of the control vector. This practice, beside the standard chattering reduction effect, has the important consequence of making possible the control of a rather wide class of uncertain systems nonlinear in the control. In the uncertain case the increment of the relative degree implicit in the proposed approach requires the introduction of second order sliding mode observer. A separation theorem is proven and sufficient conditions for the finite time convergence of both the estimation and tracking errors are found in terms of further requirements on the control amplitude.

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2. PROBLEM STATEMENT

Let us consider the control system

$$\dot{x} = f(t, x, u), \quad t \geq 0 \tag{1}$$

with the control vector $u \in R^K$, the state variable $x \in R^N$ and with control constraint

$$u \in U. \tag{2}$$

We are given a fixed sliding manifold

$$s(t, x) = 0 \tag{3}$$

with $s(t, x) \in R^M$, which fulfills prescribed control aims. We want to control the state variables $x(t)$, $t \geq 0$, of the control system (1) in order to guarantee the sliding property

$$s[t, x(t)] = 0 \tag{4}$$

for every t sufficiently large.

Let us assume that $N \geq M$, Ω is an open set of R^N and

$$s = s(t, x) : [0, +\infty) \times \Omega \to R^M$$

is differentiable almost everywhere in $(0, +\infty)$ as a function of t for each x. For each t, $s(t, \cdot) \in C^1(\Omega, R^M)$ and the $M \times N$ Jacobian matrix

$$\frac{\partial s}{\partial x}(t, x) \quad \text{has maximum rank } M \tag{5}$$

for almost every t and $x \in \Omega$. Moreover the control region

$$U \text{ is a nonempty closed set in } R^K. \tag{6}$$

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The dynamics f in (1)

$$f : [0, +\infty) \times \Omega \times U \to R^N$$

is a Carathéodory mapping.

3. THE SIMPLEX CONTROL WITH CHATTERING REDUCTION FOR NONLINEAR SYSTEMS

Consider the control system (1) with control constraint (2) and sliding manifold (3). In order to reduce chattering due to the discontinuous nature of the control u, we shall apply a fixed simplex control logic (see the Appendix) to the augmented control system

$$\dot{x} = f(t, x, u), \quad \dot{u} = v, \quad t \geq 0, \tag{7}$$

with augmented state variable $y = (x', u')' \in R^{N+K}$, control variable $v \in R^K$, and dynamics $g(t, y, v) = (f(t, y), v)'$. If the new control v is discontinuous, then the vector u turns out to be (absolutely) continuous.

Assume that f, s are both of class C^2 everywhere. Then for almost every t

$$\dot{s} = s_t(t, x) + s_x(t, x) f(t, x, u). \tag{8}$$

We fix a constant $M \times M$ matrix

$$\Lambda = \mathrm{diag}(\lambda_i), \quad \lambda_i > 0, \quad i = 1, \ldots, M,$$

and consider

$$\sigma(t, x, u) = \dot{s} + \Lambda s. \tag{9}$$

We assume that σ is available to the controller for feedback purposes.

Then for every t

$$\dot{\sigma}(t, x, u) = A(t, x, u) + B(t, x, u) v, \tag{10}$$

where A, B are continuous and

$$\begin{array}{l} A = \sigma_t + \sigma_x f, \\ B = \sigma_u = s_x f_u. \end{array} \tag{11}$$

We want to control the state variables $y(t) = (x'(t), u'(t))'$, $t \geq 0$, of the augmented system in order to guarantee the sliding property

$$\sigma[t, x(t), u(t)] = 0 \tag{12}$$

for every t sufficiently large. If $\sigma = 0$, the original sliding output $s[t, x(t)] \to 0$ as $t \to +\infty$ exponentially fast. This means that $s[t, x(t)]$ is arbitrarily close to 0 for t sufficiently large. If suitable first order approximability properties are fulfilled by the given control system (1) (see Bartolini and Zolezzi [1986], Bartolini and Zolezzi [1993], Zolezzi [2005] and Levaggi and Villa [to appear]), then $x(t)$ is arbitrarily close to the ideal sliding state on the bounded time intervals, thus x fulfills approximately the control aims guaranteed by the choice of the sliding manifold $s[t, x] = 0$, by using the original continuous control variable u. In this way chattering is counteracted in the given control system.

To this aim we employ a fixed simplex control algorithm, as follows.

The Anti Chattering Simplex Control Algorithm

Fix $M + 1$ vectors

$$v_1, \ldots, v_{M+1} \in R^M \tag{13}$$

such that for some constants $a, b > 0$, $c \neq 0$ we have

$$v_i' v_h \leq -c^2 |v_i||v_h| \quad \text{if } i \neq h, \text{ and } 0 < a \leq |v_i| \leq b. \tag{14}$$

Then R^M is partitioned in $M+1$ cones Q_h, $h = 1, \ldots, M+1$,

$$Q_h = \mathrm{cone}(v_i : \ i = 1, \ldots, M+1, \ i \neq h), \tag{15}$$

with pairwise disjoint interiors.

Given (t, x, u), $t \geq 0$, let h be the least index such that

$$\sigma(t, x, u) \in Q_h.$$

The control algorithm is defined by the discontinuous switching logic

$$\text{if} \quad \sigma(t, x, u) \in Q_h \quad \text{then} \quad v_*(t, x, u) = v_h. \tag{16}$$

Then we take into account the augmented control system

$$\dot{y} = g(t, y, v_*) \tag{17}$$

in the Filippov sense (see Filippov [1988]).

Consider any state $y = (x', u')'$ of the augmented system (17) corresponding to v_* in the Filippov sense on $[0, +\infty)$, and write

$$\theta = 2\sigma' B^{-1} A - \sigma' B^{-1} \dot{B} \sigma - \sigma' B^{-1} B_u \dot{u} B^{-1} \sigma, \tag{18}$$

where $\dot{B} = B_t + B_x f + B_u \dot{u}$.

A constant $L > 0$ is fixed. We want to control those states $y = y(t)$, $t \geq 0$, of the control system (7) such that

$$|y(t)| \leq L, \qquad t \geq 0, \tag{19}$$

in order to guarantee the sliding property (12). We posit the following assumptions:

for every $t \geq 0$, $\frac{\partial \sigma}{\partial x}$ is of maximum rank for almost every x;

for every $t \geq 0$ and $|x| + |u| \leq L$, we have that

$$B(t, x, u) \quad \text{is positive definite,} \tag{20}$$

where B is defined in (11);

a constant C_1 is available to the controller such that

$$C_1 \geq 2\left|B^{-1} A\right| + \left|\sigma B^{-1} \dot{B} B^{-1}\right| + b\left|\sigma B^{-1} B_u\right|\left|B^{-1}\right|. \tag{21}$$

We remark that the existence of C_1 follows by smoothness of f and σ provided both do not depend on t.

The following convergence theorem holds.

Theorem 1. Let assumptions (20) and (21) hold. Then every state $\overline{y}$, which verifies (19) and is a solution of (17) on $[0, +\infty)$ in the Filippov sense, fulfills the sliding condition (12) for every t sufficiently large, provided

$$2ac^2 > C_1. \tag{22}$$

Sketch of the Proof of Theorem 1. For every $h = 1, \ldots, M+1$, the set

$$\left\{(t,x,u) \in R^{N+K+1} : \quad \sigma(t,x,u) \in Q_h\right\}$$

is Lebesgue measurable since σ given by (9) is continuous, hence v_* is Lebesgue measurable, so that the dynamics $g = g(t,y,v_*)$ are Lebesgue measurable, thus it makes sense to consider states $y = (x,u)$ corresponding to v_* in the Filippov sense, see Filippov [1988].

Let y be any state fulfilling (19). We write for short

$$\sigma = \sigma[t, y(t)], \quad A = A[t, y(t)], \quad \text{etc.}$$

Consider

$$V(t) = \sigma' B^{-1} \sigma. \tag{23}$$

where B is given by (11).

Then for almost every $t \geq 0$

$$\dot{V}(t) = 2\sigma' B^{-1}\dot{\sigma} - \sigma' B^{-1}\dot{B}B^{-1}\sigma$$

since $\frac{d}{dt}\left(B^{-1}\right) = -B^{-1}\dot{B}B^{-1}$.

Remembering (10) and (18) we get

$$\dot{V} = 2\sigma'\dot{u} + \theta. \tag{24}$$

Then by (24)

$$\dot{V} = 2\sum_{i \neq h} \alpha_i v_i' v_h + \theta.$$

and by (14)

$$\dot{V} \leq -2c^2 \sum_{i \neq h} \alpha_i |v_i| |v_h| + |\theta|.$$

By (21) and (18)

$$|\theta| \leq C_1 |\sigma|$$

since $\dot{u} \in \text{co}\{v_1, \ldots, v_{M+1}\}$, so that

$$\dot{V} \leq -2c^2 a |\sigma| + C_1 |\sigma|. \tag{25}$$

Then we have, by (25),

$$\dot{V} \leq -k^2 |\sigma| \tag{26}$$

where $k^2 = 2c^2 a - C_1 > 0$ by assumption.

Let now q_2 be an upper bound of the eigenvalues of B^{-1} when $|x| + |u| \leq L$. Then $V \leq q_2 |\sigma|^2$ by (23), hence by (26) $\dot{V} \leq -\frac{k^2}{\sqrt{q_2}}\sqrt{V}$ for almost all $t \geq 0$.

This implies, as well known, that $V(t) = 0$ for all sufficiently large t, then $\sigma[t, x(t), u(t)] = 0$ for such t, as required. □

4. THE SIMPLEX SLIDING MODE FOR NONLINEAR SYSTEMS: SEPARATION BETWEEN ESTIMATION AND CONTROL

In previous section the proposed anti chattering methodology is based on a strategy which acts on an augmented order system by means of the time derivative of the control vector.

The method relies on the assumption that, for every state x and control u, $\sigma(t,x,u)$ is accessible to the controller for feedback purposes, for each $t \geq 0$.

Let us consider the case σ is not available to design the control, e.g. the sliding output cannot be measured, either it cannot be explicitly computed since the considered system is uncertain.

We introduce a different simplex sliding mode control strategy, which is based on an estimated sliding output $\hat{\sigma}$. First we introduce the estimation procedure for the considered system, then we propose the modified simplex switching logic. A separation principle between the estimation and the control is proved to hold, thus the control law guarantees the finite time convergence to a boundary layer of the original sliding output σ.

In the sliding mode literature various solutions have been proposed to the observer/differentiator problems Levant [1998], Levant [2003], Bartolini et al. [2000a], Bartolini et al. [2000b]; among these proposals, which guarantee the desired estimates, we choose, for example, the one based on the so called second order sliding mode suboptimal algorithm Bartolini et al. [1998].

The Estimation Process

We introduce a second order sliding mode observer, in order to provide an estimate of $\dot{s}$ in finite time. The second order sliding mode observer is defined as a set of M double integrators

$$\ddot{z} = w, \qquad z, w \in R^M. \tag{27}$$

The dynamics of the estimation error $e = z - s$ are then

$$\ddot{e} = w - \ddot{s} = P(t,x,u,v) + w, \tag{28}$$

where $P(t,x,u,v) = -A_1(t,x,u) - B(t,x,u)v$. We assume to know a constant bound $\overline{P}$ of the uncertain drift term, namely $|P(t,x,u,v)| < \overline{P}$.

The dynamics (28) consists of a set of second order differential equations decoupled with respect to the observer control vector w and with matching uncertainties $P(t,x,u,v)$.

The aim is that of steering to zero in finite time both the error vector e and its time derivative $\dot{e}$.

We apply the observer control vector w designed according to the following decoupled second order sliding mode control law

$$w_i = -W_i(t)\,\text{sign}\left[e_i(t) - \beta_i e^j_{i_m}(t)\right], \; i = 1, \ldots, M, \tag{29}$$

$$W_i(t) = \begin{cases} W_{i\,Max}, & e^j_{i_m}\left(e_i - \beta_i e^j_{i_m}\right) > 0, \\ \alpha_i W_{i\,Max}, & e^j_{i_m}\left(e_i - \beta_i e^j_{i_m}\right) \leq 0, \end{cases}$$

$$\beta_i \in [0,1), \quad \alpha_i > \max\left(1, \frac{(1-\beta_i)}{(1+\beta_i)}\right),$$

$$W_{i\,Max} > \max\left(2\overline{P}, \frac{2\overline{P}}{(1+\beta_i)\alpha_i + (\beta_i - 1)}\right),$$

where, $e^j_{i_m}$, $i = 1, \ldots, M$, is the extremal (either maximum or minimum) value of e_i in the time interval between two successive commutation instants t_i^j and t_i^{j+1}.

It can be proved Bartolini et al. [1998] that, despite of the uncertainties, if the control amplitudes $W_{i\,Max}$ are sufficiently high, the application of the control strategy (29) generates sequences of successive extremal points $\left\{e^j_{i_m}\right\}$. The generated sequences of the extremal points and corresponding velocities are strictly contractive, that is $\frac{|e^{j+1}_{i_m}|}{|e^j_{i_m}|} \leq q_1 < 1$ and $\frac{|\dot{e}^{j+1}_{i_m}|}{|\dot{e}^j_{i_m}|} \leq q_2 < 1$. The reaching time is the sum of a series of positive elements upper-bounded by a geometric series with ratio strictly less than one, therefore, $\sum_{j=1}^{\infty}(t_i^{j+1} - t_i^j) = T_i < \infty$. After a finite time $T = \max_{i=1,\ldots,M} T_i$ all the components of the unavailable $\dot{s}$ are estimated by $\dot{z}$

Suppose that the sliding output σ, defined by (9), is not available to the controller for feedback purposes. We define

$$\hat{\sigma}(t) = \dot{z} + \Lambda s \tag{30}$$

where $\Lambda = \operatorname{diag}(\lambda_i)$, $\lambda_i > 0$, $i = 1, \ldots, M$.

We assume that $\hat{\sigma}$ is available to the controller for feedback purposes.

We want to control the state variables $y(t) = (x'(t), u'(t))'$, $t \geq 0$, of the augmented system (7) in order to guarantee the sliding property (12) for every t sufficiently large.

In case the sliding output σ is available with a bounded error, it converges to a boundary layer of the sliding manifold $\sigma = 0$ for every t sufficiently large.

If σ is not available, we apply the following simplex control algorithm based on a modified switching logic, which relies on the availability of the sliding vector $\hat{\sigma}$.

The Simplex Control Algorithm with Switching Logic Based on Estimates

Fix $M+1$ vectors $v_1, \ldots, v_{M+1} \in U$ fulfilling (14) and consider again the cones Q_h defined by (15).

Given (t, x, u), $t \geq 0$, let h be the least index such that

$$\hat{\sigma}(t, x, u) \in Q_h,$$

then the simplex control $\tilde{v}$ is defined by

$$\tilde{v}(t, x, u) = v_h. \tag{31}$$

Having fixed a constant L, we want to guarantee the sliding property (12) for those states $\overline{y} = (x', u')'$ such that

$$|\overline{y}(t)| \leq L, \quad t \geq 0. \tag{32}$$

We assume that the sliding output vector σ is available through $\hat{\sigma}$ up to a bounded error. Namely, we assume the existence of a constant W_1 such that

$$\hat{\sigma} = \sigma + \eta, \ |\eta(t, x, u)| \leq W_1, \text{ if } t \geq 0 \text{ and } |x| + |u| \leq L. \tag{33}$$

We assume that (20) holds. Moreover we suppose that there exists a constant W_2 such that

$$|\theta^*(t, x, u)| \leq W_2, \quad \text{if} \quad t \geq 0 \quad \text{and } |x| + |u| \leq L, \tag{34}$$

where

$$\theta^* = 2B^{-1}A - B^{-1}\dot{B}B^{-1}\sigma. \tag{35}$$

Assumption (34) requires of course available bounds on the uncertain dynamics f and its partial derivatives. In the following theorem, conditions are found under which a separation holds between the estimation process (27) and (29), and the control process (31). According to the theorem, the switching logic (31) guarantees convergence in finite time of the actual sliding output σ to a boundary layer of the sliding manifold $\sigma = 0$, the size of which is arbitrarily close to (const.)W_1, i.e. proportional to the bound on the sliding output error $|\eta|$.

Theorem 2. Assume (21), (33) and (34). Then for every uncertain state fulfilling (32), the switching logic (31) implies that, for all t sufficiently large, we have

$$|\sigma| \leq 4\sqrt{\frac{q_2}{q_1}} \frac{W_1\left(ac^2 + b\right)}{2ac^2 - W_2}, \tag{36}$$

where $0 < q_1 \leq \lambda \leq q_2$ for every eigenvalue λ of $B(t, x, u)$ if $t \geq 0$ and $|x| + |u| \leq L$, provided

$$2ac^2 > W_2. \tag{37}$$

The proof of Theorem 2 requires the following lemma, which is a trivial consequence of the Comparison Lemma, Khalil [2002].

Lemma 3. Let $V = V(t) \geq 0$ be locally absolutely continuous in $[0, +\infty)$ and let k, ω be positive constants such that $\dot{V}(t) = -k\sqrt{V(t)} + \omega$ for a.e. $t \geq 0$. Then

$$\limsup_{t \to +\infty} V(t) \leq \left(\frac{\omega}{k}\right)^2 \quad \text{if} V(0) > \left(\frac{\omega}{k}\right)^2; \tag{38}$$

$$V(t) \leq \left(\frac{\omega}{k}\right)^2, \qquad t \geq 0 \quad \text{if} 0 \leq V(0) \leq \left(\frac{\omega}{k}\right)^2. \tag{39}$$

Sketch of the Proof of Theorem 2. Let $y = (x', u')'$ be any uncertain state fulfilling (32) and corresponding to $\tilde{v}$ defined by (31) in the Filippov sense on $[0, +\infty)$. Consider the corresponding σ defined by (9), fix $t \geq 0$, let $\sigma = \sigma[t, y(t)]$ and

$$V(t) = \sigma'(t) B^{-1}(t) \sigma(t). \tag{40}$$

Then by (10)

$$\dot{V} = 2\sigma' B^{-1}\dot{\sigma} - \sigma' B^{-1}\dot{B}B^{-1}\sigma = 2\sigma'\dot{u} + \sigma'\theta^*,$$

since $\frac{d}{dt}\left(B^{-1}\right) = -B^{-1}\dot{B}B^{-1}$; hence by (33)

$$\dot{V} = 2\hat{\sigma}'\dot{u} + \hat{\sigma}'\theta^* - \eta'(2\dot{u} + \theta^*). \tag{41}$$

Let $\hat{\sigma} \in \operatorname{int} Q_h$. Then

$$\hat{\sigma} = \sum_{i \neq h} \alpha_i v_i, \quad \alpha_i \geq 0$$

and by the definition of Filippov solution, see Filippov [1988], $\dot{u} = v_h$, so that by (41) and (33), (34)

$$\dot{V} \leq 2\sum_{i \neq h} \alpha_i v_i' v_h + |\hat{\sigma}| W_2 + 2bW_1 + W_1 W_2.$$

Then by (14)

$$\dot{V} \leq -|\hat{\sigma}|\left(2c^2a - W_2\right) + W_1\left(2b + W_2\right).$$

Since $|\hat{\sigma}| \geq |\sigma| - W_1$, we have then

$$\dot{V} \leq -|\sigma|\left(2c^2a - W_2\right) + 2W_1\left(c^2a + b\right). \qquad (42)$$

Of course $V \leq q_2 |\sigma|^2$, hence $|\sigma| \geq \sqrt{\frac{V}{q_2}}$ and by (42)

$$\dot{V} \leq -k\sqrt{V} + \omega,$$

where $k = \dfrac{\left(2c^2a - W_2\right)}{\sqrt{q}_2} > 0$ by assumption, $\omega = 2W_1\left(c^2a + b\right)$.

An application of Lemma 3 yields

$$|\sigma| \leq 4\sqrt{\frac{q_2}{q_1}}\frac{W_1\left(c^2a + b\right)}{2c^2a - W_2} \qquad (43)$$

for all sufficiently large t.

□

Remark The estimation process (27) and (29) guarantees that, in finite time, the estimation error η is zero. In this case, Theorem 2 assures that the actual sliding output σ converges to the sliding manifold $\sigma = 0$ in finite time.

5. CONCLUSION

In this paper a method to simultaneously achieving the two main objective of smoothing the control while maintaining ideally infinite frequency regimes has been particularized to the simplex sliding mode method. A simplex sliding mode control logic based on the time derivative of the control vector has been proposed. To validate such an approach a suitable observer has been introduced and a separation theorem has been proven. The presented methodology, beside the standard chattering reduction effect, has the important consequence of making possible the control of a rather wide class of uncertain systems nonlinear in the control. The natural field of application of the proposed method is represented by systems in which the control action is exerted in one direction only. Many examples of such actuators can be found in natural as well artificial mechanical systems.

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Appendix A. THE SIMPLEX SLIDING MODE CONTROL

The Moving Simplex Existence Assumption. The key assumption is the following. There exist constants $a > 0$, $c \neq 0$, and for every $t \geq 0$, $x \in \Omega$ and $s(t,x) \neq 0$ there exist points

$$u_1(t,x),\ u_2(t,x), \ldots,\ u_{M+1}(t,x) \quad \text{in} \quad U$$

such that every

$$u_i : [0, +\infty) \times \Omega \to U$$

is a Carathéodory function and, writing

$$g_i = g_i(t,x) = \frac{\partial s}{\partial x}(t,x) f[t, x, u_i(t,x)], \quad i = 1, \ldots, M+1$$

we have for every t and x

$$0 < a \leq |g_i| \quad \text{for every } i \qquad (A.1)$$

and

$$g_i{}' g_h \leq -c^2 |g_i||g_h| \quad \text{if } i \neq h. \qquad (A.2)$$

The vectors g_i form a simplex of vectors in R^M since they satisfy the obtuse angle condition (A.2), see Bartolini et al. [1999].

Moreover we shall use the following assumptions. For every compact set $\Omega^* \subset [0, +\infty) \times \Omega$ there exists an integrable function A^* such that

$$|f[t, x, u_i(t,x)]| \leq A^*(t) \quad \text{on } \Omega^*,$$

We want to control those state variables $x(t)$ of (1) which fulfill the conditions

$$\left|\frac{\partial s}{\partial t}\left[t, x\left(t\right)\right]\right| \le W_0, \qquad \left|\frac{\partial s}{\partial x}\left[t, x\left(t\right)\right]\right| \le W, \qquad t \ge 0, \quad \text{(A.3)}$$

where W_0, W are fixed constants, possibly depending on y.

The inequalities (A.3) are fulfilled e.g. in the special case where $s(t,x) = C[x - x_d(t)] + D$ with constant matrices C, D provided $\dot{x}_d$ is bounded.

All constants a, c, W, W_0 are known to the controller.

The Simplex Control Algorithm. If there exist inputs $u_1, \dots, u_{M+1}$ as required in the assumptions and such that (A.1), (A.2) are fulfilled, then for any t and x there is an index h such that

$$s(t,x) \in Q_h,$$

with $Q_h = \sum_i \{\alpha_i g_i : \ \alpha_i \ge 0, \ i = 1, \dots, M+1, \ i \ne h\}$. The simplex control algorithm is defined by the following discontinuous switching logic:

$$\text{if} \qquad s \in Q_h \qquad \text{then} \qquad u^*(t,x) = u_h(t,x). \qquad \text{(A.4)}$$

The feedback control u^* of (A.4) expresses the *simplex control law*. It has been proved in Bartolini et al. [1999] that the algorithm (A.4) steers to zero $s\left(t,x\right)$ in finite time, under suitable conditions.

Consider system (1) subject to unknown additive uncertain perturbations of deterministic nature

$$\dot{x} = f(t,x,u) + \varphi(t,x,u), \quad t \ge 0,$$

where $\varphi : [0, +\infty) \times \Omega \times U \to R^N$ is a Carathéodory mapping and it exists a constant H_0 such that $|\varphi[t,x,u_i(t,x)]| \le H_0, \ t \ge 0, \ x \in \Omega, \ i = 1, \dots, M+1$. Theorem 1 of Bartolini et al. [2004] proves that the control algorithm (A.4) is robust to model uncertainties, provided that the minimum amplitude a of g_i satisfies

$$ac^2 > (W_0 + WH_0)\,E, \qquad \text{(A.5)}$$

where the constant E is such that

$$\sum_{i \in \mathcal{I}} \alpha_i \left|g_i\right| \le E \left|\sum_{i \in \mathcal{I}} \alpha_i g_i\right| \qquad \text{(A.6)}$$

for all $t \ge 0$, $x \in \Omega$ with $s(t,x) \ne 0$, $\alpha_i \ge 0$, $\mathcal{I}$ any subset of $\mathcal{A} = \{1, \dots, M+1\}$ with M elements, and

$$c^2 \le \min\left\{c_{ik} : \ i,k \in \mathcal{A}, \, i \ne k\right\}.$$

The *moving simplex existence assumption* requires the solution of $M+1$ non linear inequalities; this means finding, for every $t \ge 0$, $x \in \Omega$ and $s\left(t,x\right) \ne 0$, the $M+1$ points $u_i\left(t,x\right)$, such that the $M+1$ control vectors

$$g_i = g_i\left(t,x\right) = \frac{\partial s}{\partial x}\left(t,x\right) f\left[t,x,u_i\left(t,x\right)\right],$$

form a simplex of vectors in R^M satisfying (A.1) and (A.2).

The problem, in general, must be solved off-line by non trivial numerical algorithm, provided the matrix $s_x f_u$ is invertible.

Even when the dynamics f of the nominal system (1) is perfectly known, the application of the moving simplex strategy is cumbersome; the method becomes definitely unfeasible if f is uncertain.