Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Singularly Perturbed Derivative Coupling-Filter

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Abstract: In some control and observations problems, it may be convenient, at least from the analysis point of view, to use non proper systems. However, as far as their implementation is concerned, proper approximations have to be designed. In this paper, we show how exponential approximations can be rather easily designed. We consider the MIMO case and study the relative stability of the approximation. As a by product, we show that the set of proper time-invariant linear systems is dense in the set of regular linear descriptor systems.

NOTATION

- s is the Laplace complex variable. $\jmath$ is the complex number $\sqrt{-1}$. Let $x = a + \jmath b \in \mathbb{C}$, $|x|$ stands for its modulus and $\arg x$ stands for the principal value of the argument of x. $\underline{/F(\jmath\omega)}$ stands for the phase-angle of the complex function $F(\jmath\omega)$ with $\omega \geq 0$, namely: *(i)* $\underline{/F(0)} = \arg F(0)$, *(ii)* $\underline{/F(\jmath\omega)} = \arg F(\jmath\omega) + 2k\pi$ with $k \in \mathbb{Z}$, and *(iii)* $\underline{/F(\jmath\omega)}$ is a continuous function of ω.

- $\|\cdot\|_2$ stands for the Euclidean norm. We write: $f(\varepsilon) = \mathcal{O}(\varphi(\varepsilon))$ when there exist $\varepsilon^* > 0$ and $K > 0$ such that $|f(\varepsilon)| \leq K\varphi(\varepsilon)$ for $\varepsilon \in (0,\ \varepsilon^*)$ and $\varphi(\varepsilon) > 0$. $g + \mathcal{O}(\varphi(\varepsilon))$ means: $g + f(\varepsilon)$ with $f(\varepsilon) = \mathcal{O}(\varphi(\varepsilon))$ $\mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ is the space of infinitely differentiable functions $v : \mathbb{R}^+ \to \mathbb{R}^m$ and $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m)$ stands for the locally integrable functions $v : \mathbb{R}^+ \to \mathbb{R}^m$, namely $\int_{t_1}^{t_2} \|v(t)\|_2 \, \mathrm{d}t < \infty$ for all $t_1, t_2 \in \mathbb{R}^+$.

- $\underline{\chi}_k^i$ denotes a $k \times 1$ vector whose i-th component is 1 and the others are zero (in the case that $i > k$ all its components are taken equal to zero). I_k denotes a $k \times k$ identity matrix, or simply I when the size does not have to be explicitly indicated. $BDM\{X_1, ..., X_k\}$ denotes a block diagonal matrix whose diagonal blocks are the matrices $X_1, \ldots, X_k$. $\mathrm{T}_u\{v^T\}$ denotes an upper triangular Toeplitz matrix with first row v^T. $\mathrm{T}_\ell\{v\}$ denotes a lower triangular Toeplitz matrix with first column v. Given $a, b \in \mathbb{Z}$, $a \leq b$, $[|a, b|]$ denotes the ordered set $\{z \in \mathbb{Z} \mid a \leq z \leq b\}$.

- Let $t \in \mathbb{N}$ and $\mathcal{S}_t = \{x_1, \ldots, x_t\} \subset \mathbb{R}$, for each $r \in \mathbb{N}$ such that $r \leq t$, $\mathfrak{C}_r^t(x_{i_m})$ stands for the addition of all non-repeated products of r factors taken from $\mathcal{S}_t$, *e.g.* $\mathfrak{C}_3^4(x_{i_m}) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$ and $\mathfrak{C}_1^t(x_{i_m}) = \sum_{i=1}^t x_i$; it is also written: $\mathfrak{C}_r^t(x_{i_m}) = \sum_{i_1<\cdots<i_r} \prod_{j=1}^r (x_{i_m})$. We define: $\mathfrak{C}_0^t(x_{i_m}) = 1$. $\mathfrak{C}_r^{\underline{t}}(x_{i_m})$ stands for the addition of all non-repeated products of r factors taken from $\mathcal{S}_t$ but excluding x_t, *e.g.* $\mathfrak{C}_3^{\underline{4}}(x_{i_m}) = x_1x_2x_3$ and $\mathfrak{C}_1^{\underline{t}}(x_{i_m}) = \sum_{i=1}^{t-1} x_i$. $\mathfrak{C}_r^{\overline{t}}(x_{i_m})$ stands for the addition of all non-repeated products of r factors taken from $\mathcal{S}_t$ which always include x_t, *e.g.* $\mathfrak{C}_3^{\overline{4}}(x_{i_m}) = x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$ and $\mathfrak{C}_1^{\overline{t}}(x_{i_m}) = x_t$. Note that for all $r \in [|1, t|]$ and $a \in \mathbb{R}$ hold:

$$\begin{array}{l} \mathfrak{C}_r^t(x_{i_m}) = \mathfrak{C}_r^{\underline{t}}(x_{i_m}) + \mathfrak{C}_r^{\overline{t}}(x_{i_m}) \ , \quad x_{t+1}\mathfrak{C}_r^t(x_{i_m}) = \mathfrak{C}_{r+1}^{\overline{t+1}}(x_{i_m}) \ , \\ \mathfrak{C}_r^t(x_{i_m}) = \mathfrak{C}_r^{\underline{t+1}}(x_{i_m}) \ , \quad \mathfrak{C}_r^t(ax_{i_m}) = a^r\mathfrak{C}_r^t(x_{i_m}) \end{array} \tag{1}$$

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1. INTRODUCTION

In this paper we are interested in finding a linear time-invariant proper exponential approximation which gives a solution of the following problem:

Problem 1. Given a linear time-invariant non-proper system, Σ^c, with realization:

$$N\dot{w}(t) = w(t) - \Gamma v(t) \ , \quad u(t) = \Theta w(t) \tag{2}$$

where $v \in \mathbb{R}^m$, $u \in \mathbb{R}^n$, and $w \in \mathbb{R}^{\hat{n}+n}$ are the input, the output and the descriptor variable, respectively. N is a nilpotent matrix with index of nilpotency $\overline{\kappa} + 1$ and Γ is a map such that the matrix $[N\ \Gamma]$ is epic. We assume that v is bounded and that the first $\overline{\kappa}$ derivatives of v exist and are bounded Lipschitz continuous time functions, namely:

H1. $\|v(t)\|_2 \leq K_0$, for all $t \geq 0$.
H2. $\|v(t_1) - v(t_2)\|_2 \leq L_0 |t_1 - t_2|$, $\left\|\mathrm{d}^i v(t_1)/\mathrm{d}t^i - \mathrm{d}^i v(t_2)/\mathrm{d}t^i\right\|_2 \leq L_i |t_1 - t_2|$, for all $t_1, t_2 > 0$, $i \in [|0, \overline{\kappa}|]$.

with K_0 and L_i $(i \in [|0, \overline{\kappa}|])$ some positive constants.

Find a linear time-invariant strictly proper filter, Σ_ε^f: $\dot{\zeta}(t) = A(\varepsilon)\zeta(t) + B(\varepsilon)u(t)$, $y(t) = C(\varepsilon)\zeta(t)$, and find positive constants β and ε, such that:

- Q1) $\exists\ K > 0$ such that $\lim_{\varepsilon \to 0} \|y(t) - u(t)\|_2 \leq K\mathrm{e}^{-\beta t}$, $\forall\ t > 0$.
- Q2) $\exists\ \varepsilon^* > 0$ such that Σ_ε^f is internally stable $\forall\ \varepsilon \in (0,\ \varepsilon^*)$.
- Q3) The overall system, $\Sigma_\varepsilon^f \circ \Sigma^c$ is externally equivalent [2] to a linear time-invariant proper system.

In other words, we are looking for a proper filter, Σ_ε^f, which makes proper the overall system, $\Sigma_\varepsilon^f \circ \Sigma^c$, and which output, $y(t)$, exponentially tends to the non proper behaviour of Σ^c. The interest is to finally synthesize the overall proper system $\Sigma_\varepsilon^f \circ \Sigma^c$ as a proper approximation of Σ^c.

In (Méndez *et al* 2007) we have tackled Problem 1 for the SISO case using the solid Singularly Perturbation

[2] As introduced by Willems (1983) two models are *externally equivalent*, if the corresponding sets of all possible trajectories for their external behaviour (in our case the input–output trajectories) are the same.

Theory (Kokotović 1999) as the analysis tool. In this paper we consider the MIMO case, studying also the relative stability of the approximation. As a by product, we find that the proper time-invariant linear systems are dense in the regular descriptor systems. For approximations by means of time-variant linear systems and non-linear systems the readers can see Ibrir (1999) and Levant (1998).

In Section 2 we show the principal results found in (Méndez *et al* 2007). In Section 3 we study the relative stability of the approximation for the SISO case. In Section 4 we tackle the MIMO case and in Section 5 we conclude. All the technical results are proved in the Appendix.

2. ANTECEDENTS

In (Méndez *et al* 2007), we have tackled Problem 1 for the SISO case. For this, we have considered the singularly perturbed derivative coupling filter (SPDC–Filter):

$$\begin{aligned} \dot{x}(t) &= -\beta x(t) - \varepsilon^{\kappa+1}(\underline{\chi}_\kappa^1)^T z(t) , \\ \varepsilon\dot{z}(t) &= \underline{\chi}_\kappa^\kappa x(t) - \big(M_\kappa - U_\kappa\big) z(t) + \underline{\chi}_\kappa^\kappa u(t) , \\ y(t) &= (\underline{\chi}_\kappa^1)^T z(t) \end{aligned} \qquad (3)$$

where $x \in \mathbb{R}^1$, $z \in \mathbb{R}^\kappa$, $u \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$. $\kappa \in \mathbb{Z}^+$ and β, $\varepsilon \in \mathbb{R}^+$. $U_\kappa = \mathrm{T}_u\{(\underline{\chi}_\kappa^2)^T\}$ and M_κ is defined as [3]:

$$\begin{aligned} M_\kappa &= BDM\{M_1, ..., M_{\kappa/2}\} \text{ if } \kappa \text{ is even , and} \\ M_\kappa &= BDM\{M_1, ..., M_{(\kappa-1)/2}, 1\} \text{ if } \kappa \text{ is odd} \end{aligned} \qquad (4)$$

$$\begin{aligned} M_i &= (\sin\theta_i)\mathrm{I}_2 + \mathrm{T}_\ell\{(\cos^2\theta_i)\underline{\chi}_2^2\} , \quad \theta_1 = \pi/(2\kappa) \\ \theta_{i+1} &= \theta_i + \Delta\theta , \quad \Delta\theta = \pi/\kappa , \quad i \in [|1, \kappa-1|] \end{aligned} \qquad (5)$$

The slow part of (3) is composed by a first order system with an eigenvalue in $-\beta$ and the fast part is composed by a normalized low pass Butterworth filter. The transfer function of the fast part is:

$$(\underline{\chi}_\kappa^1)^T \Big(\mathrm{s}\mathrm{I}_\kappa + (M_\kappa - U_\kappa)\Big)^{-1} \underline{\chi}_\kappa^\kappa = 1\big/\Delta_B(\mathrm{s}) \qquad (6)$$

$$\Delta_B(\mathrm{s}) = (\mathrm{s}+1)^{\sigma_o} \prod_{i=1}^{\sigma_\kappa} \big((\mathrm{s}+\sin\theta_i)^2 + \cos^2\theta_i\big) \qquad (7)$$

If κ is odd: $\sigma_o = 1$ & $\sigma_\kappa = (\kappa-1)/2$, else: $\sigma_o = 0$ & $\sigma_\kappa = \kappa/2$.

Following Lemmas 2.1, 2.2, Theorem 3.1 and Theorem 5.1 of Kokotović (1999) we have the three key results:

Lemma 2. (Méndez *et al* (2007)). Let $\varepsilon_1(\kappa)$ be defined as:

$$\varepsilon_1(\kappa) = \left(\kappa^{\frac{3}{2}}\left((\beta+1) + \kappa^{\frac{1}{2}} + 2\sqrt{(\beta+1)\kappa^{1/2}}\right)\right)^{-1} \qquad (8)$$

If $0 < \varepsilon < \min\{1, \varepsilon_1\}$ then the SPDC–Filter (3) can be expressed as the two-time-scale model:

$$\begin{aligned} \dot{\xi}(t) &= -\big(\beta - \varepsilon^{\kappa+1}(\underline{\chi}_\kappa^1)^T L(\varepsilon)\big)\xi(t) - H(\varepsilon)\underline{\chi}_\kappa^\kappa u(t) \\ \varepsilon\dot{\eta}(t) &= -\Big((M_\kappa - U_\kappa) + \varepsilon^{\kappa+2} L(\varepsilon)(\underline{\chi}_\kappa^1)^T\Big)\eta(t) + \underline{\chi}_\kappa^\kappa u(t) \\ y(t) &= (\underline{\chi}_\kappa^1)^T L(\varepsilon)\xi(t) + (\underline{\chi}_\kappa^1)^T\big(1 - \varepsilon L(\epsilon)H(\varepsilon)\big)\eta(t) \end{aligned} \qquad (9)$$

where $L(\varepsilon) = \big(\mathrm{I}_\kappa + \varepsilon\beta(M_\kappa - U_\kappa)^{-1}\big) L(0) + \mathcal{O}(\varepsilon^2)$ and $H(\varepsilon) = \varepsilon^{\kappa+1}H(0) + \mathcal{O}(\varepsilon^{\kappa+2})$. If κ is odd then: $L(0) = -\big[1 \ \sin\theta_1 \ \cdots \ 1 \ \sin\theta_{\frac{\kappa-1}{2}} \ 1\big]^T$ and $H(0) = \big[\sin\theta_1 \ 1 \ \cdots \ \sin\theta_{\frac{\kappa-1}{2}} \ 1 \ 1\big]$, else: $L(0) = -\big[1 \ \sin\theta_1 \ \cdots \ 1 \ \sin\theta_{\frac{\kappa}{2}}\big]^T$ and $H(0) = \big[\sin\theta_1 \ 1 \ \cdots \ \sin\theta_{\frac{\kappa}{2}} \ 1\big]$.

Theorem 3. (Méndez *et al* (2007)). If $0 < \varepsilon < \min\Big\{1, \varepsilon_1(\kappa), \big(\sin\pi/(2\kappa)\big)^{1/(\kappa+1)}\Big\}$ there then exists $\varepsilon_0^* > 0$, such that the two-time-scale model (9) (and so the SPDC–Filter (3)) is Hurwitz stable.

[3] If $\kappa = 1$, then: $U_\kappa = \mathbf{0}$ and $M_\kappa = 1$.

Theorem 4. (Méndez *et al* (2007)). Let us assume that $u(t)$ is a bounded Lipschitz continuous function. Namely, for all $t_1, t_2 \geq 0$ there exist $\overline{K}_0$, $\overline{L}_0 \in \mathbb{R}^+$ such that:

$$|u(t_1)| \leq \overline{K}_0 \ \text{ and } \ |u(t_1) - u(t_2)| \leq \overline{L}_0\,|t_1 - t_2| \qquad (10)$$

Let $\varepsilon \in \mathbb{R}^+$ and $p, q \in \mathbb{Z}^+$ such that $1/p + 1/q = 1$. If:

$$\varepsilon < \min\left\{1, \ \varepsilon_1(\kappa), \ \tfrac{\beta}{2}, \ \left(\tfrac{\sin(\pi/(2\kappa))}{\sqrt{2}}\right)^{1/(\kappa+1)}, \ \overline{K}_0^{-1}, \ \overline{L}_0^{-q}\right\} \qquad (11)$$

there then exists $\varepsilon^* \in (0, \varepsilon_0^*)$, such that for all $\varepsilon \in (0, \varepsilon^*)$, the output of the SPDC–Filter (3) is approximated by $y(t) = u(t) + \mathrm{e}^{-(\beta+\varepsilon^{\kappa+1})t}x(0) + \mathcal{O}(\varepsilon^{1/p}) \ \ \forall\, t > \max\{t^*, 0\}$, where $t^* = \mathcal{O}((\varepsilon/(\sin\theta_1 - \sqrt{2}\varepsilon^{\kappa+1}))\ln(\overline{K}_0/\varepsilon^{1/p}))$.

3. RELATIVE STABILITY OF THE SPDC–FILTER

In Theorem 4, we have given conditions over the positive parameter ε which guarantee a certain precision of the approximation of the SPDC–Filter output. In this Section we show that the relative stability of the SPDC–Filter (3) depends on the choice of the positive parameter β. For this, we study the phase and gain margins of its characteristic equation F_B (recall (5), (6) and (7)):

$$\begin{aligned} &F_B(\jmath\omega) = 1 + K_\kappa G_\kappa(\jmath\omega) , \quad K_\kappa = \varepsilon^{\kappa+1}\big/\beta , \\ &G_\kappa^{-1}(\jmath\omega) = (1 + \jmath\omega/\beta)\Delta_B(\jmath\varepsilon\omega) , \\ &\Delta_B(\jmath\varepsilon\omega) = (1 + \jmath\varepsilon\omega)^{\sigma_o}\prod_{i=1}^{\sigma_\kappa}\Big((1 - (\varepsilon\omega)^2) + \jmath(2\varepsilon\omega\sin\theta_{i,\kappa})\Big) , \end{aligned} \qquad (12)$$

If κ is odd: $\sigma_o = 1$ & $\sigma_\kappa = (\kappa-1)/2$, else: $\sigma_o = 0$ & $\sigma_\kappa = \kappa/2$; and $\theta_{i,\kappa} = \pi/(2\kappa) + (i-1)\pi/\kappa$, $i \in [|1, \kappa|]$.

Let us recall that the magnitude Bode diagram of the Butterworth filter is maximally flat at the origin and moreover: $|\Delta_B(\jmath\varepsilon\omega)| = \sqrt{1 + (\varepsilon\omega)^{2\kappa}}$, which implies:

$$\begin{aligned} &\Big|G_\kappa(\sqrt{\beta/\varepsilon})\Big| = \sqrt{\varepsilon\beta}\Big/\sqrt{1 + (\varepsilon\beta) + (\varepsilon\beta)^\kappa + (\varepsilon\beta)^{\kappa+1}} , \\ &|G_\kappa(0)| = 1 , \quad \partial\,|G_\kappa(\jmath\omega)|\big/\partial\omega < 0 \ \ \forall\ \omega > 0 \\ &|\Delta_B(\jmath)| = k_o\prod_{i=1}^{\sigma_\kappa}(2\sin\theta_{i,\kappa}) = \sqrt{2} \\ &\text{If } \kappa \text{ is odd: } k_o = \sqrt{2} \ \&\ \sigma_\kappa = (\kappa-1)/2, \text{ else: } k_o = 1 \ \&\ \sigma_\kappa = \kappa/2 \end{aligned} \qquad (13)$$

We assume, in this Section, that inequality (11) of Theorem 4 holds. Since $\varepsilon < 1$ and $\varepsilon < \beta/2$, we get: $K_\kappa < 1$. Thus, the phase margin of (12), $\mathcal{P}h_{\mathcal{M}}(K_\kappa G_\kappa(\jmath\omega))$, is lower bounded by $\mathcal{P}h_{\mathcal{M}}(G_\kappa(\jmath\omega)) = 180^\circ - \underline{/G_\kappa(0)} = +180^\circ$. So, we only need to study the gain margin of (12), $\mathcal{G}_{\mathcal{M}}(K_\kappa G_\kappa(\jmath\omega))$. Namely, we study the behaviour of the phase-angle of $G_\kappa(\jmath\omega)$:

$$\begin{aligned} &\text{For } \kappa \text{ odd: } \ \underline{/G_\kappa(\jmath\omega)} = -\phi_\beta(\omega) - \phi_\varepsilon(\omega) - \textstyle\sum_{i=1}^{(\kappa-1)/2}\phi_{i,\kappa}(\omega) \\ &\text{For } \kappa \text{ even: } \ \underline{/G_\kappa(\jmath\omega)} = -\phi_\beta(\omega) - \textstyle\sum_{i=1}^{\kappa}\phi_{i,\kappa}(\omega) \\ &\phi_\beta(\omega) = \arg\big(1 + \jmath(\omega/\beta)\big) , \quad \phi_\varepsilon(\omega) = \arg\big(1 + \jmath(\varepsilon\omega)\big) , \\ &\phi_{i,\kappa}(\omega) = \arg\Big(\big(1 - (\varepsilon\omega)^2\big) + \jmath\big(2\varepsilon\omega\sin\theta_{i,\kappa}\big)\Big) \end{aligned} \qquad (14)$$

3.1 Case $\kappa \leq 3$

For $\kappa = 1$, we get: $\mathcal{G}_{\mathcal{M}}(K_1G_1(\jmath\omega)) = K_1^{-1}\left|G_1^{-1}(\jmath\infty)\right| = +\infty$.

For $\kappa = 2$, we have: $0 \leq \phi_\beta(\omega) < \pi/2$ for all $\omega \in [0, +\infty)$ and $0 \leq \phi_{1,1}(\omega) + \phi_{1,1}(\omega) \leq \pi/2$ for all $\omega \in [0, \varepsilon^{-1}]$. then:

$$\mathcal{G}_{\mathcal{M}}(K_2G_2(\jmath\omega)) \geq K_2^{-1}\left|G_2^{-1}(\jmath\varepsilon^{-1})\right| = (1/\varepsilon)^{2+1}\varepsilon^{-1}\sqrt{2 + 2(\varepsilon\beta)^2}$$

Let us now consider the case $\kappa = 3$. For this, let us first note that (8) and (11) imply that: $\varepsilon\beta < \varepsilon/\varepsilon_1 < 1$. And (13) implies that: $2\sin\theta_{1,3} = 1$. Let $\bar{\omega}_c$ be the geometric mean of the two corner frequencies, β and $1/\varepsilon$, namely $\log(\bar{\omega}_c) = \frac{1}{2}\big(\log(\beta) + \log(1/\varepsilon)\big)$. Thus, for $\omega \in [0, \bar{\omega}_c]$ we get:

$0 \le \omega/\beta \le 1/\sqrt{\varepsilon\beta}$, $0 \le \varepsilon\omega \le \sqrt{\varepsilon\beta}/1$, and $0 < 1 - (\varepsilon\beta) \le 1 - (\varepsilon\omega)^2 \le 1$, which imply (see (14)): $0 \le \phi_\beta(\omega) + \phi_\varepsilon(\omega) \le \pi/2$ and $0 \le \phi_{1,3}(\omega) < \pi/2$, for all $\omega \in [0,\ \bar{\omega}]$. Then: $0 \le -\underline{/G_3(\jmath\omega)} < \pi$, for all $\omega \in [0,\ \sqrt{\beta/\varepsilon}]$. Therefore (recall (13)):

$$\begin{aligned}\mathcal{G}_{\mathcal{M}}(K_3G_3(\jmath\omega)) &\ge K_3^{-1}\left|G_3^{-1}(\jmath\sqrt{\beta/\varepsilon})\right| \\ &= (1/\varepsilon)^{3+1}\sqrt{\beta/\varepsilon}\sqrt{1+(\varepsilon\beta)+(\varepsilon\beta)^3+(\varepsilon\beta)^{3+1}}\end{aligned}$$

3.2 Case $\kappa > 3$

We have already shown, that:

$$0 \le \phi_\beta(\omega) + \phi_\varepsilon(\omega) \le \pi/2\ , \quad \text{for all } \omega \in [0,\ \sqrt{\beta/\varepsilon}] \tag{15}$$

So, we only have to test the behaviour of $\sum_{i=1}^{\eta}\phi_{i,\kappa}(\omega)$, with $\eta = (\kappa-1)/2$ for κ odd and $\eta = \kappa/2$ for κ even. We need the following two technical Lemmas:

Lemma 5. Let $S_\eta = \{x_1, \ldots, x_\eta\}$, $x_i = \sin\theta_{i,\kappa}$. Then:

$$\begin{aligned}&\tan\Big(\textstyle\sum_{i=1}^{\eta}\phi_{i,\kappa}(\omega)\Big) = \varphi_\eta(\omega)\big/\varrho_\eta(\omega)\ ,\\ &\varphi_\eta(\omega) = \textstyle\sum_{i=1}^{\sigma_n}(-1)^{i+1}(2\varepsilon\omega)^{2i-1}\big(1-(\varepsilon\omega)^2\big)^{\eta-(2i-1)}\mathfrak{C}_{2i-1}^{\eta}(x_{i_m})\\ &\varrho_\eta(\omega) = \\ &\qquad \big(1-(\varepsilon\omega)^2\big)^{\eta} + \textstyle\sum_{i=1}^{\sigma_d}(-1)^{i}(2\varepsilon\omega)^{2i}\big(1-(\varepsilon\omega)^2\big)^{\eta-2i}\mathfrak{C}_{2i}^{\eta}(x_{i_m})\end{aligned} \tag{16}$$

when η is even, set: $\sigma_n = \sigma_d = \eta/2$, and when η is odd, set: $\sigma_n = (\eta+1)/2$ and $\sigma_d = (\eta-1)/2$.

Lemma 6. Let $\eta \in \mathbb{N}\setminus\{1\}$, $\mathcal{S}_\eta = \{x_1,\ \ldots, x_\eta\} \subset \mathbb{R}^+$,

$$\begin{aligned}\aleph_{\ell,\eta}(\omega) = &\ (2\varepsilon\omega)^{\ell}\big(1-(\varepsilon\omega)^2\big)^{\eta-\ell}\mathfrak{C}_{\ell}^{\eta}(x_{i_m})\\ &-(2\varepsilon\omega)^{\ell+2}\big(1-(\varepsilon\omega)^2\big)^{\eta-\ell-2}\mathfrak{C}_{\ell+2}^{\eta}(x_{i_m})\end{aligned} \tag{17}$$

where $\ell \in [|0, \eta-2|]$. If:

$$\omega \le \varepsilon^{-1}\left(\sqrt{1+\mathfrak{C}_2^{\eta}(x_{i_m})} + \sqrt{\mathfrak{C}_2^{\eta}(x_{i_m})}\right)^{-1} \tag{18}$$

Then:

$$\aleph_{\ell,\eta}(\omega) > 0 \tag{19}$$

We then have the third principal result:

Theorem 7. Let $\kappa > 3$. If:

$$\begin{aligned}\beta \le (1/\varepsilon)\Bigg(&\sqrt{1+\textstyle\sum_{i=1}^{\eta-1}\sum_{j=i+1}^{\eta}\sin\theta_{i,\kappa}\sin\theta_{j,\kappa}}\\ &+\sqrt{\textstyle\sum_{i=1}^{\eta-1}\sum_{j=i+1}^{\eta}\sin\theta_{i,\kappa}\sin\theta_{j,\kappa}}\ \Bigg)^{-2}\end{aligned} \tag{20}$$

where $\eta = (\kappa-1)/2$ for κ odd and $\eta = \kappa/2$ for κ even. Then:

$$0 \le -\underline{/G_\kappa(\jmath\omega)} < \pi\ , \quad \text{for all } \omega \in [0,\ \sqrt{\beta/\varepsilon}] \tag{21}$$

Moreover:

$$\begin{aligned}\mathcal{G}_{\mathcal{M}}(K_\kappa G_\kappa(\jmath\omega)) &\ge K_\kappa^{-1}\left|G_\kappa^{-1}(\jmath\sqrt{\beta/\varepsilon})\right| \\ &= (1/\varepsilon)^{\kappa+1}\sqrt{\beta/\varepsilon}\sqrt{1+(\varepsilon\beta)+(\varepsilon\beta)^{\kappa}+(\varepsilon\beta)^{\kappa+1}}\end{aligned} \tag{22}$$

Proof. Since $\kappa > 3$, then: $\eta \in \mathbb{N}\setminus\{1\}$. Let $S_\eta = \{x_1, \ldots, x_\eta\}$, and $x_i = \sin\theta_{i,\kappa}$. From (16) and (17), we have that $\varphi_\eta(\omega)$ and $\varrho_\eta(\omega)$ can be also expressed as follows:

$$\varphi_\eta(\omega) = \textstyle\sum_{i=1}^{\bar{\sigma}_n}\aleph_{(4i+1),\eta}(\omega) + \psi_{\varphi_\eta}(\omega)\ , \tag{23}$$

where if σ_n is even: $\bar{\sigma}_n = \sigma_n/2 - 1$ and $\psi_{\varphi_\eta}(\omega) \equiv 0$, else: $\bar{\sigma}_n = (\sigma_n - 3)/2 - 1$ and $\psi_{\varphi_\eta}(\omega) = (2\varepsilon\omega)^{2\sigma_n-1}\big(1-(\varepsilon\omega)^2\big)^{\eta-(2\sigma_n-1)}\mathfrak{C}_{2\sigma_n-1}^{\eta}(x_{i_m})$.

$$\varrho_\eta(\omega) = \textstyle\sum_{i=1}^{\bar{\sigma}_d}\aleph_{4i,\eta}(\omega) + \psi_{\varrho_\eta}(\omega)\ , \tag{24}$$

where if σ_d is odd: $\bar{\sigma}_n = (\sigma_d - 1)/2$ and $\psi_{\varrho_\eta}(\omega) \equiv 0$, else $\bar{\sigma}_d = \sigma_d/2 - 1$ and $\psi_{\varrho_\eta}(\omega) = (2\varepsilon\omega)^{2\sigma_d}\big(1-(\varepsilon\omega)^2\big)^{\eta-2\sigma_d}\mathfrak{C}_{2\sigma_d}^{\eta}(x_{i_m})$.

From (16): $\varphi_\eta(0) = 0$ and $\varrho_\eta(0) = 1$. Then: $\sum_{i=1}^{\eta}\phi_{i,\kappa}(0) = 0$. From (23), (24), (20), and Lemma 6, we get (note that (20) also implies $\beta\varepsilon < 1$): $\varphi_\eta(\omega) > 0$ and $\varrho_\eta(\omega) > 0$, for all $\omega \in \big[0,\ \sqrt{\beta/\varepsilon}\big]$. Then: $0 \le \sum_{i=1}^{\eta}\phi_{i,\kappa}(\omega) < \pi/2$, for all $\omega \in \big[0,\ \sqrt{\beta/\varepsilon}\big]$, which together with (15) imply (21).

(22) directly follows from (21) and (13). □

4. PROPER APPROXIMATION

We are now in position for solving Problem 1 in the MIMO case. For this, let us assume that the non proper compensator (2) is completely observable. Then, its Kronecker canonical form has only n row minimal indices blocks of sizes $(\kappa_i+1)\times(\kappa_i+1)$, $i \in [|1,n|]$, such that $\kappa_i > 0$, $\sum_{i=1}^{n}\kappa_i = \hat{n}$, and $\max\{\kappa_1,\ldots,\kappa_n\} = \overline{\kappa}$ (Gantmacher 1959). Carrying system (2) to its Kronecker canonical form, we get:

$$\begin{aligned}&N = BDM\{N_1,\ldots,N_n\}\ ,\quad N_i = \mathrm{T}_\ell\{\underline{\chi}^2_{(\kappa_i+1)}\}\ ,\\ &\Theta = BDM\big\{\underline{\theta}_1^T,\ldots,\underline{\theta}_n^T\big\}\ ,\quad \underline{\theta}_i = \underline{\chi}^{(\kappa_i+1)}_{(\kappa_i+1)}\ ,\\ &\Gamma = \big[\,\Gamma_1^T\big|\cdots\big|\Gamma_n^T\,\big]^T\ ,\quad \Gamma_i = \big[\,\underline{\gamma}_{i,0}\big|\underline{\gamma}_{i,1}\big|\cdots\big|\underline{\gamma}_{i,\kappa_i}\,\big]^T\ ,\\ &\text{with } \underline{\gamma}_{i,j} \in \mathbb{R}^m \text{ and } \underline{\gamma}_{i,0} \ne 0\ ,\quad j \in [|1,\kappa_i|]\ ,\quad i \in [|1,n|]\end{aligned} \tag{25}$$

Defining the following matrices:

$$\begin{aligned}&\widehat{\Gamma}_i = \big[\,\underline{\hat{\gamma}}_{1,i}\big|\cdots\big|\underline{\hat{\gamma}}_{n,i}\,\big]^T\ ,\quad i \in [|0,\overline{\kappa}|]\ ,\quad \text{where}\\ &\underline{\hat{\gamma}}_{j,i} = \begin{cases}\underline{\gamma}_{j,(\kappa_j-\overline{\kappa}+i)} & \text{for } i \in [|\overline{\kappa}-\kappa_j,\overline{\kappa}|]\\ 0 & \text{otherwise}\end{cases}\ ,\quad j \in [|1,n|]\end{aligned} \tag{26}$$

we get from (25) and (2):

$$u(t) = \left(\widehat{\Gamma}_0\tfrac{\mathrm{d}^{\overline{\kappa}}}{\mathrm{d}t^{\overline{\kappa}}} + \widehat{\Gamma}_1\tfrac{\mathrm{d}^{\overline{\kappa}-1}}{\mathrm{d}t^{\overline{\kappa}-1}} + \cdots + \widehat{\Gamma}_{\overline{\kappa}-1}\tfrac{\mathrm{d}}{\mathrm{d}t} + \widehat{\Gamma}_{\overline{\kappa}}\right)v(t) \tag{27}$$

with $\widehat{\Gamma}_0 \ne 0$. Let $K_{0,j}$, $L_{0,j}$, $L_{1,j}$, $\ldots$, $L_{\bar{\eta},j}$, the Lipschitz positive constants of the m components, v_j, of v; namely: $\|v_j(t)\|_2 \le K_{0,j}$, $\|v_j(t_1) - v_j(t_2)\|_2 \le L_{0,j}|t_1 - t_2|$, $\big\|\mathrm{d}^iv_j(t_1)/\mathrm{d}t^i - \mathrm{d}^iv_j(t_2)/\mathrm{d}t^i\big\|_2 \le L_{i,j}|t_1 - t_2|$, for all $t \ge 0$, $t_1, t_2 > 0$, $j \in [|1,m|]$, $i \in [|0,\overline{\kappa}|]$. In accordance with (27) and (10), let us define the Lipschitz positive constants:

$$\overline{K}_0 = \textstyle\sum_{j=1}^{m}\left(\big\|\widehat{\Gamma}_{\overline{\kappa}}\underline{\chi}_m^j\big\|_2 K_{0,j} + \sum_{i=0}^{\overline{\kappa}-1}\big\|\widehat{\Gamma}_i\underline{\chi}_m^j\big\|_2 L_{(\overline{\kappa}-1-i),j}\right),$$

$$\overline{L}_0 = \textstyle\sum_{j=1}^{m}\sum_{i=0}^{\overline{\kappa}}\big\|\widehat{\Gamma}_i\underline{\chi}_m^j\big\|_2 L_{(\overline{\kappa}-i),j}.$$

Then: $|u(t_1)| \le \overline{K}_0$ and $|u(t_1) - u(t_2)| \le \overline{L}_0|t_1 - t_2|\ \forall\ t_1,\ t_2 \ge 0$.

4.1 Convergence

Problem 1 is solved by the fourth principal result:

Theorem 8. Let the Multivariable SPDC–Filter, Σ_ε^f,

$$\begin{aligned}\dot{x}(t) &= -\beta x(t) - \varepsilon K_\varepsilon C_o z(t)\ ,\\ \varepsilon\dot{z}(t) &= B_o x(t) + A_o z(t) + B_o u(t)\ ,\quad y(t) = C_o z(t)\end{aligned} \tag{28}$$

where: $x \in \mathbb{R}^n$, $z \in \mathbb{R}^{\hat{n}}$, $u \in \mathbb{R}^n$, and $y \in \mathbb{R}^n$, and:

$$\begin{aligned}&A_o = BDM\{A_1,\ldots,A_n\}\ ,\ B_o = BDM\big\{\underline{b}_1,\ldots,\underline{b}_n\big\}\ ,\\ &K_\varepsilon = BDM\{\varepsilon^{\kappa_1},\ldots,\varepsilon^{\kappa_n}\}\ ,\ C_o = BDM\big\{\underline{c}_1^T,\ldots,\underline{c}_n^T\big\}\end{aligned} \tag{29}$$

$$A_i = (-M_{\kappa_i} + U_{\kappa_i}),\ \underline{b}_i = \underline{\chi}^{\kappa_i}_{\kappa_i},\ \underline{c}_i = \underline{\chi}^{1}_{\kappa_i},\quad \text{with } i \in [|1,n|] \tag{30}$$

Let $\varepsilon \in \mathbb{R}^+$ and $p, q \in \mathbb{Z}^+$ such that $1/p + 1/q = 1$. If:

$$\varepsilon < \min\left\{1,\ \varepsilon_1(\bar{\kappa}),\ \tfrac{\beta}{2},\ \left(\tfrac{\sin(\pi/(2\overline{\kappa}))}{\sqrt{2}}\right)^{1/(\overline{\kappa}+1)},\ \overline{K}_0^{-1},\ \overline{L}_0^{-q}\right\} \tag{31}$$

there then exists $\bar{\varepsilon}^* > 0$ such that, for all $\varepsilon \in (0,\ \bar{\varepsilon}^*)$, the Multivariable SPDC–Filter, Σ_ε^f, is internally stable and:

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon\in(0,\ \bar{\varepsilon}^*)}}\left\|y(t) - \left(\textstyle\sum_{i=1}^{\overline{\kappa}}\widehat{\Gamma}_{\overline{\kappa}-i}\dfrac{\mathrm{d}^i}{\mathrm{d}t^i} + \widehat{\Gamma}_{\overline{\kappa}}\right)v(t)\right\|_2 \le \|x(0)\|_2\,\mathrm{e}^{-\beta t}\ , \tag{32}$$

for all $t > 0$. Moreover, the composite overall system, $\Sigma_\varepsilon^f \circ \Sigma^c$, is externally equivalent to the Singularly Perturbed Proper System:

$$\begin{aligned} \dot{x}(t) &= -\beta x(t) - \varepsilon K_\varepsilon C_o \bar{z}(t) - \varepsilon \overline{\Gamma}_0 v(t) \ , \\ \varepsilon \dot{\bar{z}}(t) &= B_o x(t) + A_o \bar{z}(t) + \left(A_o \overline{R}_p + B_o \Theta \right) \Gamma v(t) \ , \\ y(t) &= C_o \bar{z}(t) + K_\varepsilon^{-1} \overline{\Gamma}_0 v(t) \end{aligned} \tag{33}$$

where:

$$\begin{aligned} \overline{\Gamma}_0 &= \left[\underline{\gamma}_{1,0} \middle| \cdots \middle| \underline{\gamma}_{n,0} \right]^T , \quad \overline{R}_p = \left[\overline{R}_{p_1} \middle| \cdots \middle| \overline{R}_{p_n} \right] , \\ \overline{R}_{p_i} &= \left[(1/\varepsilon^{\kappa_i - 1}) A_o^{\kappa_i - 1} \underline{\hat{b}}_{p_i} \middle| \cdots \middle| \underline{\hat{b}}_{p_i} \middle| 0 \right] , \ i \in [|1, n|] \end{aligned} \tag{34}$$

$$\begin{aligned} \underline{\hat{b}}_{p_i} &= \left[\underline{\tilde{b}}_{i,1}^T \ \cdots \ \underline{\tilde{b}}_{i,n}^T \right]^T \in \mathbb{R}^{\hat{n}} \ , \quad \underline{\tilde{b}}_{i,j} \in \mathbb{R}^{\kappa_j} \ , \\ \underline{\tilde{b}}_{\substack{i,j \\ j \neq i}} &= 0 \ , \quad \underline{\tilde{b}}_{i,i} = (1/\varepsilon) \underline{b}_i \ , \quad i \in [|1, n|] \end{aligned} \tag{35}$$

with: $\bar{z}(t) = z(t) - \overline{R}_p w(t)$.

Proof.

Given the block diagonal structure of (28)–(30), we get from (31) and Theorem 3 that the Multivariable SPDC–Filter, Σ_ε^f, is Hurwitz stable for all $\varepsilon \in (0, \bar{\varepsilon}^*)$. And from (31), (27), and Theorem 4, we get (32).

From (2) and (28), $\Sigma_\varepsilon^f \circ \Sigma^c$ is:

$$\begin{aligned} \left[\begin{array}{c|c} \mathrm{I}_{n+\hat{n}} & 0 \\ \hline 0 & N \end{array} \right] \dot{x}_{fc}(t) &= \left[\begin{array}{c|c} A_p & B_p \\ \hline 0 & \mathrm{I}_{n+\hat{n}} \end{array} \right] x_{fc}(t) + \left[\begin{array}{c} 0 \\ \hline -\Gamma \end{array} \right] v(t) \ , \\ y(t) &= \left[C_p \middle| 0 \right] x_{fc}(t) \end{aligned} \tag{36}$$

$$\begin{aligned} A_p &= \begin{bmatrix} -\beta \mathrm{I}_n & -\varepsilon K_\varepsilon C_o \\ (1/\varepsilon) B_o & (1/\varepsilon) A_o \end{bmatrix} , \ B_p = \begin{bmatrix} 0 \\ (1/\varepsilon) B_o \Theta \end{bmatrix} , \\ C_p &= \begin{bmatrix} 0 & C_o \end{bmatrix} \end{aligned} \tag{37}$$

with: $x_{fc} = \left[x^T \ z^T \middle| w^T \right]^T$. Taking into account the particular forms of Σ^c and Σ^f, we get from (25), (29) and (37):

$$B_p = \left[B_{p_1} \middle| \cdots \middle| B_{p_n} \right] , \ B_{p_i} = \left[0 \middle| \cdots \middle| 0 \middle| \underline{b}_{p_i} \right] , \ \underline{b}_{p_i} = \left[0 \ \underline{\hat{b}}_{p_i}^T \right]^T \tag{38}$$

where: $B_{p_i} \in \mathbb{R}^{(n+\hat{n}) \times (\kappa_i + 1)}$, $\underline{b}_{p_i} \in \mathbb{R}^{n+\hat{n}}$, and $i \in [|1, n|]$.

We need the following Lemma:

Lemma 9. Let us define the matrices Q_p and R_p as follows:

$$\begin{aligned} Q_p &= -(A_p R_p + B_p) \ , \quad R_p = \left[R_{p_1} \middle| R_{p_2} \middle| \cdots \middle| R_{p_n} \right] , \\ R_{p_i} &= \left[A_p^{\kappa_i - 1} \underline{b}_{p_i} \middle| \cdots \middle| A_p \underline{b}_{p_i} \middle| \underline{b}_{p_i} \middle| 0 \right] , \quad i \in [|1, n|] \end{aligned} \tag{39}$$

Then for $i \in [|1, n|]$:

$$R_p + Q_p N = 0 \tag{40}$$

$$\begin{aligned} &\text{For } \kappa_i > 1 : \ C_p A_p^{k_i - 1} \underline{b}_{p_i} = (1/\varepsilon^{\kappa_i}) \underline{\chi}_n^i \ , \\ &\text{For } \kappa_i > 2 : \ C_p A_p^j \underline{b}_{p_i} = 0 \ , \quad j \in [|1, \kappa_i - 2|] \ , \\ &\text{For } \kappa_i = 1 : \ C_p \underline{b}_{p_i} = (1/\varepsilon^{\kappa_i}) \underline{\chi}_n^i \ , \end{aligned} \tag{41}$$

$$C_p R_p = BDM \left\{ \underline{\nu}_1^T, \ldots, \underline{\nu}_n^T \right\} \ , \quad \underline{\nu}_i = (1/\varepsilon^{\kappa_i}) \underline{\chi}_{(\kappa_i + 1)}^1 \tag{42}$$

Let us apply the following change of variable $\xi_{fc} = \left[\bar{\xi}_{fc}^T \ \tilde{\xi}_{fc}^T \right]^T = \begin{bmatrix} \mathrm{I}_{n+\hat{n}} & -R_p \\ 0 & \mathrm{I}_{n+\hat{n}} \end{bmatrix} x_{fc}$ and let us premultiply (36) by $\begin{bmatrix} \mathrm{I}_{n+\hat{n}} & Q_p \\ 0 & \mathrm{I}_{n+\hat{n}} \end{bmatrix}$. Then, we get from Lemma 9:

$$\begin{aligned} \begin{bmatrix} \mathrm{I}_{n+\hat{n}} & 0 \\ 0 & N \end{bmatrix} \dot{\xi}_{fc}(t) &= \begin{bmatrix} A_p & 0 \\ 0 & \mathrm{I}_{n+\hat{n}} \end{bmatrix} \xi_{fc}(t) + \begin{bmatrix} -Q_p \Gamma \\ -\Gamma \end{bmatrix} v(t) \ , \\ y(t) &= \left[C_p \ (C_p R_p) \right] \xi_{fc}(t) \end{aligned} \tag{43}$$

Now, since: $-\left(\mathrm{I} + \sum_{i=1}^{\overline{\kappa}} N^i \mathrm{d}^i / \mathrm{d}t^i \right) (N \mathrm{d}/\mathrm{d}t - \mathrm{I}) = \mathrm{I}$, we get: $y(t) = C_p \bar{\xi}_{fc}(t) + (C_p R_p) \left(\mathrm{I} + \sum_{i=1}^{\overline{\kappa}} N^i \mathrm{d}^i / \mathrm{d}t^i \right) \Gamma v(t)$. But, from (25) and (42) we realize that:

$$\begin{aligned} y(t) &= C_p \bar{\xi}_{fc}(t) \\ &\quad + BDM \left\{ \frac{1}{\varepsilon^{\kappa_1}} \left(\underline{\chi}_{\kappa_1 + 1}^1 \right)^T, \ldots, \frac{1}{\varepsilon^{\kappa_n}} \left(\underline{\chi}_{\kappa_n + 1}^1 \right)^T \right\} \Gamma v(t) \\ &= C_p \bar{\xi}_{fc}(t) + K_\varepsilon^{-1} BDM \left\{ \underline{\gamma}_{1,0}^T, \ldots, \underline{\gamma}_{n,0}^T \right\} v(t) \end{aligned}$$

which together with (43), (39) and (34.a) imply that the composite overall system (36) is externally equivalent to:

$$\begin{aligned} \dot{\bar{\xi}}_{fc}(t) &= A_p \bar{\xi}_{fc}(t) + (A_p R_p + B_p) \Gamma v(t) \ , \\ y(t) &= C_p \bar{\xi}_{fc}(t) + K_\varepsilon^{-1} \overline{\Gamma}_0 v(t) \end{aligned}$$

Let us note that (30), (4) and (5) imply: $(\underline{\chi}_{\kappa_i}^1)^T A_i^{j-1} \underline{\chi}_{\kappa_i}^{\kappa_i} = 0$, $j \in [|1, \kappa_i - 1|]$. Then for $i \in [|1, n|]$ and $j \in [|1, \kappa_i|]$:

$$A_p^{j-1} \underline{b}_{p_i} = \left[\frac{-(1/\varepsilon) K_\varepsilon C_o A_o^{j-2} \underline{\hat{b}}_{p_i}}{(1/\varepsilon^k) A_o^{j-1} \underline{\hat{b}}_{p_i}} \right] = \left[\frac{0}{(1/\varepsilon^k) A_o^{j-1} \underline{\hat{b}}_{p_i}} \right]$$

Hence: $R_{p_i} = \left[\frac{0}{\overline{R}_{p_i}} \right]$. Also (30), (4) and (5) imply: $(\underline{\chi}_{\kappa_i}^1)^T A_i^{\kappa_i - 1} \underline{\chi}_{\kappa_i}^{\kappa_i} = 1$, hence: $C_o \overline{R}_p \Gamma = K_\varepsilon^{-1} \overline{\Gamma}_0$, namely: $A_p R_p \Gamma = \left[\frac{-\varepsilon K_\varepsilon C_o \overline{R}_p \Gamma}{(1/\varepsilon) A_o \overline{R}_p \Gamma} \right] = \left[\frac{-\varepsilon \overline{\Gamma}_0}{(1/\varepsilon) A_o \overline{R}_p \Gamma} \right]$. □

4.2 Stability margin

Let us now explore the stability margin of the open-loop Multivariable SPDC–Filter (28). For this, let us find the characteristic functions, $\ell_i(\mathrm{s})$, of the return-ratio matrix, $L_{rr}(\mathrm{s})$ (see MacFarlane and Postlethwaite (1977) for details). The open-loop system of (28) is:

$$\begin{aligned} \begin{bmatrix} \dot{z}(t) \\ \dot{x}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} (1/\varepsilon) A_o & 0 \\ \varepsilon K_\varepsilon C_o & -\beta \mathrm{I}_n \end{bmatrix}}_{A_{OL}} \begin{bmatrix} z(t) \\ x(t) \end{bmatrix} + \underbrace{\begin{bmatrix} (1/\varepsilon) B_o \\ 0 \end{bmatrix}}_{B_{OL}} u_1(t) \ , \\ y_2(t) &= \underbrace{\begin{bmatrix} 0 & \mathrm{I}_n \end{bmatrix}}_{C_{OL}} \left[z^T(t) \ x^T(t) \right]^T \end{aligned} \tag{44}$$

where $y_2(t) = x(t)$ and when closing the loop $u_1(t) = u(t) - y_2(t)$. The determinant of the return-ratio matrix is: $\det(L_{rr}(\mathrm{s})) = \det \left(C_{OL} (\mathrm{sI}_{(\hat{n}+n)} - A_{OL})^{-1} B_{OL} \right) = \prod_{i=1}^n K_{\kappa_i} G_{\kappa_i}(\mathrm{s})$, where (recall (6) and (7)):

$$\begin{aligned} K_{\kappa_i} &= \varepsilon^{\kappa_i + 1} / \beta \ , \quad G_{\kappa_i}^{-1}(\mathrm{s}) = (\mathrm{s}/\beta + 1) \Delta_{B,i}(\varepsilon \mathrm{s}) \ , \\ \Delta_{B,i}(\mathrm{s}) &= (\mathrm{s} + 1) \prod_{j=1}^{\frac{(\kappa - 1)}{2}} \left((\mathrm{s} + \sin \theta_{j,\kappa_i})^2 + \cos^2 \theta_{j,\kappa_i} \right) \end{aligned} \tag{45}$$

with: $i \in [|1, n|]$. Hence, the n characteristic functions are:

$$\ell_i(\mathrm{s}) = K_{\kappa_i} G_{\kappa_i}(\mathrm{s}) \ , \quad i \in [|1, n|] \tag{46}$$

The stability margin is given by the fifth principal result:

Theorem 10. if in addition to conditions of Theorem 8:

$$\beta \leq (1/\varepsilon) \rho^* \tag{47}$$

$$\rho^* = 1 \ , \quad \text{for } \overline{\kappa} = 3 \tag{48}$$

$$\begin{aligned} \rho^* = \Bigg(&\sqrt{1 + \sum_{i=1}^{(\overline{\kappa}-3)/2} \sum_{j=i+1}^{(\overline{\kappa}-1)/2} \sin \theta_{i,\overline{\kappa}} \sin \theta_{j,\overline{\kappa}}} \\ &+ \sqrt{\sum_{i=1}^{(\overline{\kappa}-3)/2} \sum_{j=i+1}^{(\overline{\kappa}-1)/2} \sin \theta_{i,\overline{\kappa}} \sin \theta_{j,\overline{\kappa}}} \ \Bigg)^{-2} , \quad \text{for } \overline{\kappa} > 3 \end{aligned} \tag{49}$$

Then the gain margins of the characteristic functions (46) and (45) are lower bounded by:

$$\begin{aligned} \mathcal{G}_{\mathcal{M}}(\ell_i(\jmath\omega)) &\geq K_{\kappa_i}^{-1} \left| G_{\kappa_i}^{-1}(\jmath \sqrt{\beta/\varepsilon}) \right| \\ &\geq (1/\varepsilon)^{\underline{\kappa}+1} \sqrt{\beta/\varepsilon} \sqrt{1 + (\varepsilon\beta) + (\varepsilon\beta)^{\overline{\kappa}} + (\varepsilon\beta)^{\overline{\kappa}+1}} \end{aligned}$$

where $\underline{\kappa} = \min\{\kappa_1, \ldots, \kappa_n\}$ and $i \in [|1, n|]$.

Proof.

From (47), (48) and Theorem 7, we get: $0 \le -\underline{/\ell_i(\jmath\omega)} < \pi$, $i \in [|1, n|]$, $\forall\ \omega \in [0, \sqrt{\beta/\varepsilon}]$. And thus, $\mathcal{G}_{\mathcal{M}}(\ell_i(\jmath\omega)) \ge K_{\kappa_i}^{-1} \left|G_{\kappa_i}^{-1}(\jmath\sqrt{\beta/\varepsilon})\right| = (1/\varepsilon)^{\kappa_i+1}\sqrt{\beta/\varepsilon}\sqrt{1+(\varepsilon\beta)+(\varepsilon\beta)^{\kappa_i}+(\varepsilon\beta)^{\kappa_i+1}}$ $\ge (1/\varepsilon)^{\underline{\kappa}+1}\sqrt{\beta/\varepsilon}\sqrt{1+(\varepsilon\beta)+(\varepsilon\beta)^{\overline{\kappa}}+(\varepsilon\beta)^{\overline{\kappa}+1}}$, $\forall\, i \in [|1, n|]$. □

4.3 Proper Systems Density

The following Corollary states that the set of proper systems is dense in the set of regular generalized systems. Namely every non-proper system can be approximated arbitrarily close (ε-near) by a proper system, moreover this approximation can be done while keeping an exponential convergence (β-dynamics) and insuring some relative stability (gain margin).

Corollary 11. Let the conditions of Theorems 8 and 10 hold. Let $\beta = (1/\varepsilon)a\rho^*$, where a is a sufficiently small positive constant, $0 < a \le 1$, such that the behaviour of the Multivariable SPDC–Filter (28) remains in a two-time-scale, for all $\varepsilon \in (0, \bar{\varepsilon}^*)$. Then, for every[4] $u \in \Sigma^c\left(\mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)\right)$ there exists a sequence $y_\varepsilon \in \Sigma_\varepsilon^f \circ \Sigma^c\left(\mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)\right)$ such that y_ε converges to u in the sense of $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m)$.

Proof. From Theorem 8 we get for all $v \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ and for all $t_1, t_2 \in \mathbb{R}^+$, with $t_1 < t_2$:

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon\in(0,\ \bar{\varepsilon}^*)}} \int_{t_1}^{t_2} \left\|\left(\Sigma_\varepsilon^f \circ \Sigma^c - \Sigma^c\right)(v(t))\right\|_2 \mathrm{d}t = \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in(0,\ \bar{\varepsilon}^*)}} \int_{t_1}^{t_2} \|y_\varepsilon(t) - u(t)\|_2 \mathrm{d}t \le \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in(0,\ \bar{\varepsilon}^*)}} \|x(0)\|_2 \int_0^\infty \mathrm{e}^{-\beta t}\mathrm{d}t = \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in(0,\ \bar{\varepsilon}^*)}} \|x(0)\|_2/\beta = \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in(0,\ \bar{\varepsilon}^*)}} \varepsilon\|x(0)\|_2/(a\rho^*) = 0. \quad \square$$

5. CONCLUSION

In this paper we have extended the SISO approximation of Méndez *et al* (2007) for solving Problem 1. Our proposition relies on the substitution of the rudimentary filter $1/(\varepsilon s + 1)^\kappa$ by a Butterworth low pass filter, which allows for a nice application of the results of (Kokotović 1999). Our solution for designing the adequate filters nicely separates the quality of the approximation, given by the fast subsystem, parameterized by the inverse of the positive constant ε, from the convergence ratio, given by the slow subsystem parameterized by the positive constant β.

The parameter ε is chosen in such a way that: (*i*) it guarantees the separation of the two time scales, the slow one and the fast one, by diagonalizing the fast and slow subsystems, (*ii*) it guarantees the stability of the SPDC–Filter (3), and (*iii*) it takes into account the functional characteristics of the bounded Lipschitz continuous signal $u(t)$ (reflected by its L_∞ norm, $\overline{K}_0$, and its Lipschitz constant, $\overline{L}_0$) for solving the Problem 1, see Theorem 8.

We have also shown that the selection of the parameter β is directly related with the relative stability of the approximation, see Theorems 7 and 10.

In Corollary 11 we have proved that the proper time-invariant linear systems are dense in the regular descriptor systems. This implies that every non-proper system can be approximated arbitrarily close (ε-near) by a proper system, moreover this closing procedure can be done while keeping exponential convergence (β-dynamics) and relative stability (gain margin). It has to be noted that the proposition of this paper is not at all a high gain approximation but a singularly perturbed approximation. Also note that for avoiding the high tansitories due to discontinuites (non derivable signals) one has to put the correct initial conditions.

The following step in our research is to show that our proposition enables to apply a certain separation property, that is to say, when applying the proper approximation, the obtained features reached with the PD output feedback, are maintained by the approximation.

[4] To assume that $v \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ is not restrictive since $\mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ is dense in $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m)$, see Theorem 2.4.10 of Polderman and Willems (1998).

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Appendix A. PROOF OF LEMMA 5

We need the following three Lemmas:

Lemma 12. Let $\eta \in \mathbb{N} \setminus \{1\}$, $x_1, ..., x_{\eta+1} \in \mathbb{R}$, then:

(1) $\sum_{i=1}^{\sigma_1}(-1)^{i+1}\mathfrak{C}_{2i-1}^{\eta}(x_{i_m}) + x_{\eta+1}\left(1 + \sum_{i=1}^{\sigma_2}(-1)^i\mathfrak{C}_{2i}^{\eta}(x_{i_m})\right) = \sum_{i=1}^{\sigma_3}(-1)^{i+1}\mathfrak{C}_{2i-1}^{\eta+1}(x_{i_m})$

(2) $\sum_{i=1}^{\sigma_4}(-1)^i\mathfrak{C}_{2i}^{\eta}(x_{i_m}) + x_{\eta+1}\left(\sum_{i=1}^{\sigma_5}(-1)^i\mathfrak{C}_{2i-1}^{\eta}(x_{i_m})\right) = \sum_{i=1}^{\sigma_6}(-1)^i\mathfrak{C}_{2i}^{\eta+1}(x_{i_m})$

when η is even, set: $\sigma_1 = \sigma_2 = \eta/2$, $\sigma_3 = (\eta+2)/2$, and $\sigma_4 = \sigma_5 = \sigma_6 = \eta/2$, and when η is odd, set: $\sigma_1 = \sigma_3 = (\eta+1)/2$, $\sigma_2 = (\eta-1)/2$, $\sigma_4 = (\eta-1)/2$, $\sigma_5 = \sigma_6 = (\eta+1)/2$.

Lemma 13. Let $\eta \in \mathbb{N} \setminus \{1\}$, $\alpha_1, ..., \alpha_\eta \in \mathbb{R}$, then:

$$\tan\left(\sum_{i=1}^{\eta}\alpha_i\right) = \frac{\left(\sum_{i=1}^{\sigma_n}(-1)^{i+1}\mathfrak{C}_{2i-1}^{\eta}\left(\tan(\alpha_{i_m})\right)\right)}{\left(1+\sum_{i=1}^{\sigma_d}(-1)^i\mathfrak{C}_{2i}^{\eta}\left(\tan(\alpha_{i_m})\right)\right)}$$

when η is even, set: $\sigma_n = \sigma_d = \eta/2$, and when η is odd, set: $\sigma_n = (\eta+1)/2$, and $\sigma_d = (\eta-1)/2$.

Lemma 14. Let $\eta \in \mathbb{N} \setminus \{1\}$, $x_1, ..., x_\eta \in \mathbb{R}$, $a, b \in \mathbb{R}^+ \cup \{0\}$, and $\alpha_i = \arg(b + \jmath(ax_i))$, then:

$$\tan\left(\sum_{i=1}^{\eta} \alpha_i\right) = \frac{\left(\sum_{i=1}^{\sigma_n} (-1)^{i+1} a^{2i-1} b^{\eta-(2i-1)} \mathfrak{C}^{\eta}_{2i-1}(x_{i_m})\right)}{\left(b^\eta + \sum_{i=1}^{\sigma_d} (-1)^i a^{2i} b^{\eta-2i} \mathfrak{C}^{\eta}_{2i}(x_{i_m})\right)}$$

when η is even, set: $\sigma_n = \sigma_d = \eta/2$, and when η is odd, set: $\sigma_n = (\eta+1)/2$ and $\sigma_d = (\eta-1)/2$.

Proof of Lemma 12 We prove the even case (recall (1)):

First item : $\sum_{i=1}^{\eta/2} (-1)^{i+1} \mathfrak{C}^{\eta}_{2i-1}(x_{i_m}) + x_{\eta+1}\left(1 + \sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\eta}_{2i}(x_{i_m})\right) = \sum_{i=1}^{\eta/2} (-1)^{i+1} \mathfrak{C}^{\eta}_{2i-1}(x_{i_m}) + x_{\eta+1} + \sum_{i=1}^{\eta/2} (-1)^i x_{\eta+1} \mathfrak{C}^{\eta}_{2i}(x_{i_m}) = \sum_{i=1}^{\eta} x_i + \sum_{i=2}^{\eta/2} (-1)^{i+1} \mathfrak{C}^{\underline{\eta+1}}_{2i-1}(x_{i_m}) + x_{\eta+1} + \sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\overline{\eta+1}}_{2i+1}(x_{i_m}) = \sum_{i=1}^{\eta+1} x_i + \sum_{i=2}^{\eta/2} (-1)^{i+1} \mathfrak{C}^{\underline{\eta+1}}_{2i-1}(x_{i_m}) + \sum_{i=1}^{\eta/2-1} (-1)^i \mathfrak{C}^{\overline{\eta+1}}_{2i+1}(x_{i_m}) + (-1)^{\eta/2} \prod_{\underline{i=1}}^{\eta+1} x_i = \sum_{i=1}^{\eta+1} x_i + \sum_{i=2}^{\eta/2} (-1)^{i+1} \mathfrak{C}^{\underline{\eta+1}}_{2i-1}(x_{i_m}) + \sum_{i=2}^{\eta/2} (-1)^{i+1} \mathfrak{C}^{\overline{\eta+1}}_{2i-1}(x_{i_m}) + (-1)^{\eta/2} \prod_{i=1}^{\eta+1} x_i = \sum_{i=1}^{\eta+1} x_i + \sum_{i=2}^{\eta/2} (-1)^{i+1}\left(\mathfrak{C}^{\underline{\eta+1}}_{2i-1}(x_{i_m}) + \mathfrak{C}^{\overline{\eta+1}}_{2i-1}(x_{i_m})\right) + \prod_{i=1}^{\eta+1} x_i = \sum_{i=1}^{\eta+1} x_i + \sum_{i=2}^{\eta/2} (-1)^{i+1} \mathfrak{C}^{\eta+1}_{2i-1}(x_{i_m}) + \prod_{i=1}^{\eta+1} x_i = \sum_{i=1}^{\eta/2+1} (-1)^{i+1} \mathfrak{C}^{\eta+1}_{2i-1}(x_{i_m})$.

Second item : $\sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\eta}_{2i}(x_{i_m}) + x_{\eta+1}\left(\sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\eta}_{2i-1}(x_{i_m})\right) = \sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\eta}_{2i}(x_{i_m}) + \sum_{i=1}^{\eta/2} (-1)^i x_{\eta+1} \mathfrak{C}^{\eta}_{2i-1}(x_{i_m}) = \sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\eta}_{2i}(x_{i_m}) + \sum_{\underline{i=1}}^{\eta/2} (-1)^i \mathfrak{C}^{\overline{\eta+1}}_{2i}(x_{i_m}) = \sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\underline{\eta+1}}_{2i}(x_{i_m}) + \sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\overline{\eta+1}}_{2i}(x_{i_m}) = \sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\eta+1}_{2i}(x_{i_m})$.

The other case is proved in the same way. □

Proof of Lemma 13 Let $x_i = \tan(\alpha_i)$. Lemma 12 implies:

$$\tan\left(\alpha_1 + \alpha_2\right) = \frac{\tan(\alpha_1) + \tan(\alpha_2)}{1 - \tan(\alpha_1)\tan(\alpha_2)} = \frac{\sum_{i=1}^{2/2} (-1)^{i+1} \mathfrak{C}^2_{2i-1}(x_{i_m})}{1 + \sum_{i=1}^{2/2} (-1)^i \mathfrak{C}^2_{2i}(x_{i_m})}$$

$$\tan\left(\alpha_1 + \alpha_2 + \alpha_3\right) = \frac{\tan(\alpha_1 + \alpha_2) + \tan(\alpha_3)}{1 - \tan(\alpha_1 + \alpha_2)\tan(\alpha_3)}$$

$$= \frac{\mathfrak{C}^2_1(x_{i_m}) + x_3\left(1 - \mathfrak{C}^2_2(x_{i_m})\right)}{1 - \mathfrak{C}^2_2(x_{i_m}) - x_3 \mathfrak{C}^2_1(x_{i_m})} = \frac{\sum_{i=1}^{(3+1)/2} (-1)^{i+1} \mathfrak{C}^3_{2i-1}(x_{i_m})}{1 + \sum_{i=1}^{(3-1)/2} (-1)^i \mathfrak{C}^3_{2i}(x_{i_m})}$$

Let us suppose that the Lemma is true for $\eta \in [|2, \mu|]$, where μ, is some positive even integer then for $\bar{\eta} = \mu + 1$, we have from Lemma 12: $\tan\left(\sum_{i=1}^{\bar{\eta}} \alpha_i\right) = \frac{\tan\left(\sum_{i=1}^{\mu} \alpha_i\right) + \tan(\alpha_{\mu+1})}{1 - \tan\left(\sum_{i=1}^{\mu} \alpha_i\right)\tan(\alpha_{\mu+1})}$

$$= \frac{\sum_{i=1}^{\mu/2} (-1)^{i+1} \mathfrak{C}^{\mu}_{2i-1}(x_{i_m}) + x_{\mu+1}\left(1 + \sum_{i=1}^{\mu/2} (-1)^i \mathfrak{C}^{\mu}_{2i}(x_{i_m})\right)}{1 + \sum_{i=1}^{\mu/2} (-1)^i \mathfrak{C}^{\mu}_{2i}(x_{i_m}) + x_{\mu+1}\left(\sum_{i=1}^{\mu/2} (-1)^{i+1} \mathfrak{C}^{\mu}_{2i-1}(x_{i_m})\right)}$$

$$= \frac{\sum_{i=1}^{(\mu+2)/2} (-1)^{i+1} \mathfrak{C}^{\mu+1}_{2i-1}(x_{i_m})}{1 + \sum_{i=1}^{\mu/2} (-1)^i \mathfrak{C}^{\mu+1}_{2i}(x_{i_m})} = \frac{\sum_{i=1}^{(\bar{\eta}+1)/2} (-1)^{i+1} \mathfrak{C}^{\eta}_{2i-1}(x_{i_m})}{1 + \sum_{i=1}^{(\bar{\eta}-1)/2} (-1)^i \mathfrak{C}^{\eta}_{2i}(x_{i_m})}$$

The other case is proved in the same way. □

Proof of Lemma 14 We prove the even case. From Lemma 13 we get (recall (1)):

$$\tan\left(\sum_{i=1}^{\eta} \alpha_i\right) = \frac{\sum_{i=1}^{\eta/2} (-1)^{i+1} \mathfrak{C}^{\eta}_{2i-1}\left(\tan(\arg(b + \jmath(ax_{i_m})))\right)}{1 + \sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\eta}_{2i}\left(\tan(\arg(b + \jmath(ax_{i_m})))\right)}$$

$$= \frac{\sum_{i=1}^{\eta/2} (-1)^{i+1} \mathfrak{C}^{\eta}_{2i-1}\left((\frac{a}{b}) x_{i_m}\right)}{1 + \sum_{i=1}^{\eta/2} (-1)^i \mathfrak{C}^{\eta}_{2i}\left((\frac{a}{b}) x_{i_m}\right)} = \frac{\sum_{i=1}^{\eta/2} (-1)^{i+1} a^{2i-1} b^{\eta-(2i-1)} \mathfrak{C}^{\eta}_{2i-1}(x_{i_m})}{b^\eta + \sum_{i=1}^{\eta/2} (-1)^i a^{2i} b^{\eta-2i} \mathfrak{C}^{\eta}_{2i}(x_{i_m})}$$

The other case is proved in the same way. □

Proof of Lemma 5 Doing $a = 2\varepsilon\omega$, $b = 1 - (\varepsilon\omega)^2$, and $x_i = \sin\theta_{i,\kappa}$, we get Lemma 5 from (14) and Lemma 14. □

Appendix B. PROOF OF LEMMA 6

We need the following Lemma:

Lemma 15. Let $\eta \in \mathbb{N} \setminus \{1\}$, $\mathcal{S}_\eta = \{x_1, \ldots, x_\eta\} \subset \mathbb{R}^+$, then for each $\ell \in [|0, \eta - 2|]$ holds:

$$\left(\sqrt{1 + \mathfrak{C}^{\eta}_2(x_{i_m})} + \sqrt{\mathfrak{C}^{\eta}_2(x_{i_m})}\right)^{-1}$$

$$\leq \left(\sqrt{1 + \mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) / \mathfrak{C}^{\eta}_{\ell}(x_{i_m})} + \sqrt{\mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) / \mathfrak{C}^{\eta}_{\ell}(x_{i_m})}\right)^{-1}$$

$$= \sqrt{1 + \mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) / \mathfrak{C}^{\eta}_{\ell}(x_{i_m})} - \sqrt{\mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) / \mathfrak{C}^{\eta}_{\ell}(x_{i_m})}$$

Proof of Lemma 15 Let us note that:

$\mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) = \sum_{i_1 < \cdots < i_{\ell+2}} \prod_{j=1}^{\ell+2} (x_{i_m}) \leq \left(\sum_{i_1 < i_2} \prod_{j=1}^{2} (x_{i_m})\right) \cdot \left(\sum_{i_1 < \cdots < i_\ell} \prod_{j=1}^{\ell} (x_{i_m})\right) = \mathfrak{C}^{\eta}_2(x_{i_m})\, \mathfrak{C}^{\eta}_{\ell}(x_{i_m})$,

which implies the inequality. The equality follows from the fact: $1 = \left(\sqrt{1+a} + \sqrt{a}\right)\left(\sqrt{1+a} - \sqrt{a}\right)$ for all $a \in \mathbb{R}^+$. □

Proof of Lemma 6 Let us do: $\alpha = \varepsilon\omega$. From (18) and Lemma 15, we get: $\alpha < \sqrt{1 + \mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) / \mathfrak{C}^{\eta}_{\ell}(x_{i_m})} - \sqrt{\mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) / \mathfrak{C}^{\eta}_{\ell}(x_{i_m})}$, which implies: $\alpha^2 + 2\alpha\sqrt{\mathfrak{C}^{\eta}_{\ell+2}(x_{i_m})} / \sqrt{\mathfrak{C}^{\eta}_{\ell}(x_{i_m})} + \mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) / \mathfrak{C}^{\eta}_{\ell}(x_{i_m}) < 1 + \mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) / \mathfrak{C}^{\eta}_{\ell}(x_{i_m})$, namely: $0 < 2\alpha\sqrt{\mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) / \mathfrak{C}^{\eta}_{\ell}(x_{i_m})} < 1 - \alpha^2$. That is to say: $(2\alpha)^2 \mathfrak{C}^{\eta}_{\ell+2}(x_{i_m}) < \left(1 - \alpha^2\right)^2 \mathfrak{C}^{\eta}_{\ell}(x_{i_m})$. Multiplying the last inequality by $(2\alpha)^\ell \left(1 - \alpha^2\right)^{\eta-\ell-2}$, we get (19). □

Appendix C. PROOF OF LEMMA 9

(1) Let us first note that the Markov's parameters of each subsystem $\left\{A_i, \underline{b}_i, \underline{c}_i^T\right\}$ satisfy for $i \in [|1, n|]$:

$$\begin{array}{l} \text{If } \kappa_i = 1: \ h_{i,\kappa_i} = \underline{c}_i^T \underline{b}_i = 1, \\ \text{If } \kappa_i > 1: \ h_{i,\kappa_i} = \underline{c}_i^T A_i^{\kappa_i - 1} \underline{b}_i = 1, \\ \text{If } \kappa_i > 2: \ h_{i,j+1} = \underline{c}_i^T A_i^j \underline{b}_i = 0, \quad j \in [|0, \kappa_i - 2|] \end{array} \quad \text{(A.1)}$$

(2) *Let us next prove (40):* From (39) and (25) we get: $-Q_p N = (A_p R_p + B_p) N = \Big[\left[A_p^{\kappa_1} \underline{b}_{p_1} \middle| \cdots \middle| A_p \underline{b}_{p_1} \middle| \underline{b}_{p_1}\right] N_1 \Big| \cdots \Big| \left[A_p^{\kappa_n} \underline{b}_{p_n} \middle| \cdots \middle| A_p \underline{b}_{p_n} \middle| \underline{b}_{p_n}\right] N_n \Big] = \Big[\left[A_p^{\kappa_1 - 1} \underline{b}_{p_1} \middle| \cdots \middle| \underline{b}_{p_1} \middle| 0\right] \Big| \cdots \Big| \left[A_p^{\kappa_n - 1} \underline{b}_{p_n} \middle| \cdots \middle| \underline{b}_{p_n} \middle| 0\right] \Big] = R_p$.

(3) *Let us now show that the following statements hold:*

$$C_o A_o^{\kappa_i - 1} \hat{\underline{b}}_{p_i} = (1/\varepsilon) \underline{\chi}_n^i \ \ \& \ \ C_o A_o^j \hat{\underline{b}}_{p_i} = 0, \quad \forall \ \kappa_i > 2, \ \ j \in [|0, \kappa_i - 2|] \quad \text{(A.2)}$$

$$A_p^j \underline{b}_{p_i} = (1/\varepsilon^j) \left[0 \ \left(A_o^j \hat{\underline{b}}_{p_i}\right)^T\right]^T \quad \forall \ j \in [|0, \kappa_i - 1|] \quad \text{(A.3)}$$

Indeed from (29), (30) and (35) we get: $C_o A_o^j \hat{\underline{b}}_{p_i} = (1/\varepsilon) c_i^T A_i^j \underline{b}_i \underline{\chi}_n^i$ for all $j \in [|0, \kappa_i - 1|]$, which together with (A.1) imply (A.2). By induction, we get (A.3) from (A.2), (37.a) and (38).

(4) *Let us finally prove (41) and (42):* Indeed, from (A.3) and (37.c), we get: $C_p A_p^j \underline{b}_{p_i} = (1/\varepsilon^j) C_o A_o^j \hat{\underline{b}}_{p_i}$, which together with (A.2), imply (41). From (39.b) and (41) we get: $C_p R_{p_i} = \left[(1/\varepsilon^{\kappa_i}) \underline{\chi}_n^i \middle| 0 \middle| \ldots \middle| 0\right]$, which implies (42). □