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Conditions for which MPC fails to converge to the correct target

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Abstract: This paper considers the efficacy of disturbance models for ensuring offset free tracking and optimum steady-state target selection within linear model predictive control (MPC). Previously published methods for steady-state target determination can address model error, disturbances, and output target changes when the desired steady state is unconstrained, but may fail when there are active constraints. This paper focuses on scenarios where the most desirable target is unreachable, thus some constraints are active in steady state. Examples are given showing that the resulting 'feasible steady-state target' can converge to a point which is not as close as possible to the true target. These failures have not been widely discussed in the literature. From the closed-loop behavior, hypotheses are put forward as necessary conditions for offset-free control. These hypotheses are then investigated through the use of Karush-Kuhn-Tucker (KKT) conditions of optimality.

Keywords: Model-based control; Control theory; Constraints; Tracking control

1. INTRODUCTION

Model predictive control (MPC) refers to a control technique that makes use of the predicted evolution of a plant in determining an open-loop optimal set of future control trajectories. Usually the control law is computed using an optimization program which is solved on-line at every time-step. MPC has been employed widely in the process control industry, its popularity being largely attributed to its ability to consider both current and future input/state/output constraints in the problem formulation and to handle multi-variable systems systematically.

There have been a number of theoretical advancements in MPC over the last few decades and thus a typical MPC formulation now incorporates the following aspects:

(1) A state-space model;
(2) An optimal disturbance/state estimator;
(3) A steady-state target optimizer (SSTO) for managing unachievable setpoints, controlled variable (CV) prioritizing and non-square systems;
(4) The dual-mode paradigm, with infinite horizons and invariant set membership for recursive feasibility and nominal stability guarantees;
(5) The closed-loop paradigm for good numerical conditioning in predictions.

1.1 Integral action and offset free control in MPC

One important aspect within MPC is the incorporation of integral action to facilitate offset-free control of controlled variables despite the effects of measured and unmeasured disturbances and model uncertainty. Offset-free linear MPC can be achieved by incorporating an appropriate disturbance model into the MPC formulation, and the disturbance estimate can be used to adjust steady-state input/state targets appropriately; this is a standard linear optimal control arrangement (Kwakernaak and Sivan [1972]). At the next level in the control hierarchy, a SSTO is used to determine the optimum feasible steady-state targets for the MPC dynamic optimizer; the SSTO will also incorporate the disturbance estimate into its calculation. Thus, the general MPC arrangement is shown in figure 1.

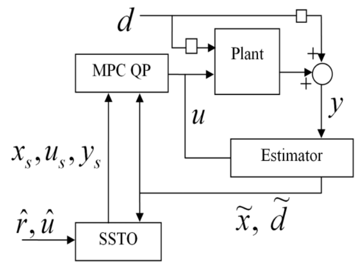

Fig. 1. Architecture diagram.

The estimator that is present for state-feedback to the optimizer (aspect (2)) is augmented with a number of state and/or output disturbances that represent both unmodeled disturbances and parametric uncertainty. The nature of the estimator/dynamic controller/SSTO system as a whole is to drive the plant output to the desired setpoint. In Meadows et al. [1994] and later in more detail in Muske and Badgwell [2002] (Theorem 4) conditions were derived for this MPC/estimator/SSTO arrangement to guarantee offset-free control at steady state. The following conditions for offset-free control are presented by Muske and Badgwell [2002] for a structured disturbance model, and Pannocchia and Rawlings [2003] for an unstructured disturbance model:

CONDITIONS

(1) the closed-loop system reaches a steady state;
(2) the closed-loop system is asymptotically stable;
(3) the process model is stabilizable and detectable;
(4) the number of disturbance states is equal to the number of outputs;
(5) the augmented system is detectable;
(6) no inequality constraints are active at steady state.

1.2 Does integral action always imply offset free control?

The main focus of this paper is to investigate the relaxation of condition 6, as for many processes the optimal steady state is at the border of one or more constraints. For instance, active steady-state constraints were considered in Rao and Rawlings [1999] where it was shown that constraints can become active at steady state as a result of choosing an infeasible target. It was also shown how the action of a disturbance with active steady-state constraints can cause offset in the controlled variables.

However, the main problem highlighted in Rao and Rawlings [1999] and Pannocchia et al. [2003] was that the Maximal Controlled Admissible Set (MCAS) may not be finitely determined when the origin is on the boundary of the feasible region. Consequently the MCAS was not known and recursive feasibility could not be guaranteed. This paper is concerned with a different aspect of the MPC configuration given in figure 1. In this paper, we are interested in the case when the solution of the SSTO is at one or more active constraints. The conditions for offset-free control presented previously no longer hold and the target determined by the SSTO may not be optimal in a least squares sense.

Model accuracy requirements for MPC with disturbance correction are discussed in Forbes and Marlin [1994], considering each expected plant operating point separately for a band of possible parameter values. This paper, however, pursues a more general, analytic approach.

Robustification of an LP based SSTO has been considered in Kassmann et al. [2000]. In this work the SSTO was modified by contracting constraints due to ellipsoidal bounded uncertainty such that the computed steady-state target is feasible for the real plant. Backing away from constraints may be conservative, and not allow attainment of the true optimum, so this paper considers the situations where deterministic MPC gives acceptable performance.

Remark 1. Another important consideration is the impact of the violation of conditions (2) and (6) simultaneously for systems with input constraints. The inequality constraints can actually stabilize the controller such that the closed-loop system settles at a steady state on the perimeter of the feasible region. A proper understanding of this scenario may determine the situations where the plant actually settles at the desired constrained operating point, and thus enlarge the set of plant models that result in the steady state attained being as close as possible to the target in a least-squares sense.

This paper considers the case of linear systems with input constraints. Section 2 gives the mathematical description of a state-space MPC algorithm necessary for discussing further its properties. Section 3 introduces a simple example to show steady-state offset from the ideal constrained solution is easily encountered with unreachable setpoints. Section 4 takes the principles of section 3 as inspiration for determining conditions for offset-free tracking with active constraints. The problem is tackled systematically, by firstly considering single-input-single-output (SISO) systems, and then multiple-input-multiple output (MIMO) systems. Section 5 concludes with a summary of what has been achieved, and the future research direction.

2. BACKGROUND

This section gives the MPC mathematical background necessary to discuss the main issues in this paper.

2.1 Modeling, feedback and predictions

Consider the following discrete time system with unstructured disturbance model:

$$x_{k+1} = Ax_k + Bu_k + B_d d_k + w_k \tag{1}$$

$$y_k = Cx_k + C_d d_k + v_k,\ z_k = Hy_k \tag{2}$$

where $u \in \mathbb{R}^m, x \in \mathbb{R}^n, y \in \mathbb{R}^l, z \in \mathbb{R}^p$, and d_k is a disturbance vector, $d \in \mathbb{R}^{n_d}$. Following the guidelines of Pannocchia and Rawlings [2003] for the particular disturbance model displayed in (1), an estimator is designed (through choice of B_d, C_d and noise weighting matrices R_v and Q_w) based on the system model augmented by disturbance states:

$$\begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{d}_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ \tilde{d}_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k \tag{3}$$

If (HC, A) is detectable, and B_d and C_d are chosen such that the augmented system (3) is detectable, then a stable linear estimator exists. System (3) is detectable if:

$$\text{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + n_d \tag{4}$$

Use of estimates $\tilde{x}$, $\tilde{d}$ are assumed henceforth.

Let the 'predicted' control law (Rossiter et al. [1998]) for sample times k be:

$$(u_k - u_s) = \begin{cases} -K(\tilde{x}_k - x_s) + c_k & k \in [0, n_c - 1] \\ -K(\tilde{x}_k - x_s) & k \geq n_c \end{cases} \tag{5}$$

where c_k are the d.o.f. (or control perturbations) available for constraint handling and $u_s,\ x_s$ are the expected steady state input/state required to give offset-free tracking in the steady state. In order to determine $x_s,\ u_s$, a separate SSTO is usually performed, such as that in Muske and Rawlings [1993]:

$$J_s(x_s, u_s, y_s) = \|r - y_s\|^2_{Q_s} + \|\hat{u}_s - u_s\|^2_{R_s} \tag{6}$$

$$\begin{aligned} J_s^*(x_s^*, u_s^*, y_s^*) &= \min_{x_s, u_s} J_s \\ \text{s. t.} \begin{bmatrix} (I-A) & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} &= \begin{bmatrix} B_d\tilde{d} \\ y_s - HC_d \end{bmatrix} \\ \begin{bmatrix} 0 & A_u \\ A_x & 0 \\ A_y HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} - \begin{bmatrix} b_u \\ b_x \\ b_y - A_y HC_d\tilde{d} \end{bmatrix} &\leq 0 \end{aligned} \tag{7}$$

For non-square systems, the degrees of freedom within the choice of steady state can be concisely re-parameterized in terms of a variable t (Shead and Rossiter [2007]), which spans the space of all (x_s, u_s) fulfilling:

$$\underbrace{[(I-A)\ \ -B]}_{E}\begin{bmatrix} x_s \\ u_s \end{bmatrix} = B_d\tilde{d}, \quad \begin{bmatrix} x_s \\ u_s \end{bmatrix} = E^{\dagger}B_d\tilde{d} + Nt \quad (8)$$

where $N = \text{null}(E)$, $t \in \mathbb{R}^{n_t}$. Substituting in for x_s, u_s from (8) into (5), the control law can be expressed as follows:

$$u_k = \begin{cases} -Kx_k + L_t t_k + L_d\tilde{d}_k + c_k, & k \in [0, n_c] \\ -Kx_k + L_t t_k + L_d\tilde{d}_k, & k \geq n_c \end{cases} \quad (9)$$

Finally, the vectors of predictions $\underset{\rightarrow}{x}, \underset{\rightarrow}{y}, \underset{\rightarrow}{u}$ corresponding to simulating (1,9) can be written in concise form, e.g.:

$$\underset{\rightarrow}{x} = P_x x_k + P_t t_k + P_d d_k + P_c \underset{\rightarrow}{c}_k \quad (10)$$

for suitable $P_x, P_t, P_d, P_c, L_t, L_d$.

2.2 Constraint handling and target optimization

Let the constraints be linearly time-invariant, i.e.

$$\begin{aligned} &\text{diag}(A_x)\underset{\rightarrow}{x} <= \text{col}(b_x);\ \text{diag}(A_u)\underset{\rightarrow}{u} <= \text{col}(b_u); \\ &\text{diag}(A_z H)\underset{\rightarrow}{y} <= \text{col}(b_z) \end{aligned} \quad (11)$$

Consequently the MCAS can be defined by substituting the predictions into (11) and takes the form:

$$\begin{aligned} \text{MCAS} = \{x : \exists\{\underset{\rightarrow}{c} \text{ or } (\underset{\rightarrow}{c},\ t)\ \} \\ \text{s. t.:}\ \ M_x x + M_c\underset{\rightarrow}{c} + M_t t + M_d\tilde{d} \leq p\} \end{aligned} \quad (12)$$

Remark 2. Many MPC algorithms (e.g. Rao and Rawlings [1999]) exclude the MCAS in the SSTO, and soft constraints may then be necessary in the dynamic MPC optimization for abrupt setpoint changes. Alternatively, the same objective in (x_s, u_s, y_s) in (6, 7) can be re-parameterized in terms of a weighted distance in the variable t, and included as part of the MPC optimization rather than through a separate SSTO. As a consequence of integrating both optimizations, the implicit outcome of the SSTO, that is the (x_s, u_s, y_s) to be used, will allow a lower predicted performance cost where this possible.

A typical dynamic optimization is the following QP, where the steady-state constraints in (7) are adequately represented by the MCAS defined in (12):

$$\begin{aligned} J_c^*(\underset{\rightarrow}{c}^*, (t^*)) = \min_{\underset{\rightarrow}{c}, t} \|\underset{\rightarrow}{c}\|^2_{W_D} + (\lambda\|t - \hat{t}\|^2_S) \\ \text{s.t.}\ \ (x, \underset{\rightarrow}{c}, t, d) \in \text{MCAS} \end{aligned} \quad (13)$$

Definitions of W_D, S and $\hat{t}$ have been omitted to save space (see Shead and Rossiter [2007] for details).

Remark 3. In practice there may be a number of related issues connected to the recursive feasibility of (13). Clearly the method used here does have a guarantee of recursive feasibility in the nominal case; for the uncertain case modifications in the target alone may be insufficient and one may have to resort to conventional techniques such as constraint softening but that is not a topic of this paper.

Hereafter we focus on the issue of whether the SSTO, embedded within the MPC optimization or taken separately (without the MCAS incorporated), is able to determine the true minimum least-squares distance to the steady-state target (based on weights in the SSTO) when nested within the feedback configuration of figure 1.

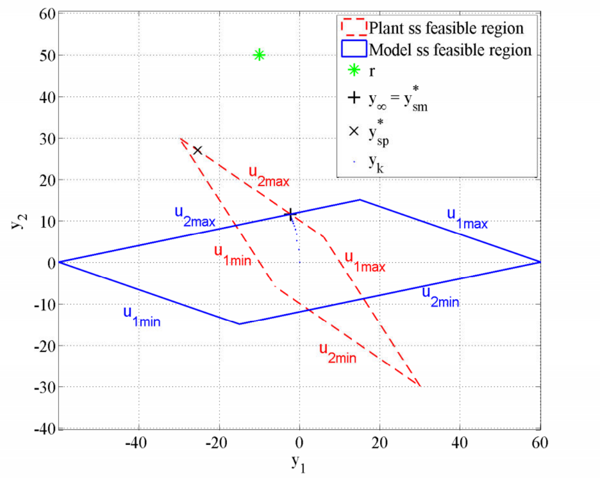

Fig. 2. Motivating example, with setpoint of (-10, 50), showing discrepancy between resulting simulated steady state and setpoint.

3. MOTIVATING EXAMPLE

The consequences of removing condition (6) given in section 1 are illustrated in this section. In this case, the SSTO can converge to a steady-state target that is suboptimal even when the other five conditions in section 1 hold. Although the process achieves the steady-state target determined by the SSTO through the integral action in MPC, the presence of active constraints can lead to *constrained offset* between the steady-state target determined by the SSTO and the optimal target with minimal deviation from the unattainable desired operating point. The conditions under which the closed-loop configuration in figure 1 converges to a suboptimal target that is in fact not as close to the infeasible desired steady state as possible are illustrated using a simple numerical example.

3.1 Numerical example

For ease of presentation, a simple two-state plant is chosen; later sections then consider higher dimensional systems. The notation $(\cdot)_p$ refers to the true process and $(\cdot)_m$ the process model; hence an amount of parameter uncertainty is assumed throughout. Here the case of input constraints only (i.e. no state/output constraints) is adequate for demonstrating the problem.

$$\begin{aligned} &A_p = \begin{bmatrix} 0.7 & -0.2 \\ -0.2 & 0.7 \end{bmatrix}, \ G_p = \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix} \\ &B_p = C_p = I,\ d_k = 0\ \forall k \end{aligned} \quad (14)$$

$$\begin{aligned} &A_m = A_p + \Delta A,\ \Delta A = \begin{bmatrix} 0.2 & -0.1 \\ 0.3 & -0.2 \end{bmatrix} \\ &A_m = \begin{bmatrix} 0.9 & -0.3 \\ 0.1 & 0.5 \end{bmatrix}, \ G_m = \begin{bmatrix} 12.5 & -7.5 \\ 2.5 & 2.5 \end{bmatrix} \\ &B_m = B,\ C_m = 2C,\ \|u\|_\infty \leq 3,\ H = I \text{ (i.e. } z = y) \end{aligned} \quad (15)$$

where G_p, G_m are the steady-state gain matrices. Controller weightings/parameters were as follows:

$$\begin{aligned} &Q = R = Q_s = I,\ R_s = 0,\ n_c = 5,\ B_d = C_d = I \\ &Q_w = R_v = 0.02I,\ \lambda = 1,\ \tilde{d} = 0,\ \tilde{x} = 0 \end{aligned} \quad (16)$$

The MCAS in this case was finitely determined according to the procedure in Gilbert and Tan [1991].

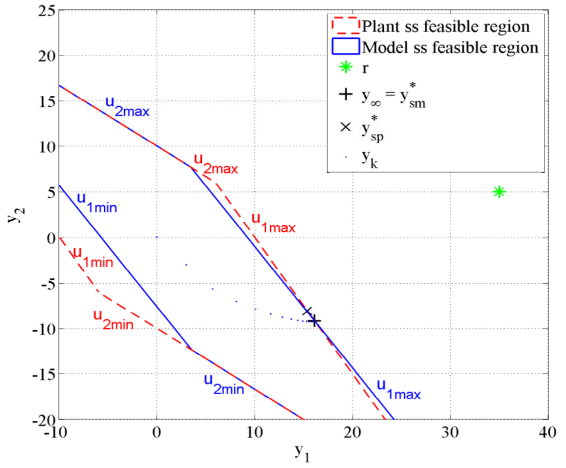

Fig. 3. Simulation depicting constrained offset even with small modeling errors.

A setpoint has been chosen that is not actually achievable [1], a common situation in industrial MPC. Simulation of the MPC controller (comprising combined dynamic and steady-state optimizer and estimator) designed for the system model, but applied to the true plant, reveals (in figure 2) a typical problem with active constraints.

Figure 2 focuses only on the achievable steady-state target regions, which is not the same as an MCAS(see Georgakis et al. [2003] for systematic analysis methodology). The reachable targets are computed by mapping the input constraints into the state space through the plant and model steady-state gains G_p, G_m. In this case it is clear that the parameter uncertainty dictates very different regions are reachable and admittedly there is substantial modeling error in this case as that facilitates a much clearer demonstration of the issue. The same problem will arise with much smaller modeling errors, as shown in figure 3, which depicts simulation results with the following changes to the previous example:

$$\Delta A = \begin{bmatrix} -0.1 & -0.1 \\ 0 & 0 \end{bmatrix},\ C = I \qquad (17)$$

Figure 2 also displays four key points: (i) the actual desired target (denoted by r) is clearly infeasible; (ii) the target y^*_{sm} arising from the SSTO is optimally close to r in a least-squares sense (i.e. the line between the solution and r is perpendicular to the u_{2max} constraint for the model), (iii) the stable estimator has converged such that $y_\infty = y^*_{sm}$, and (iv) there is therefore offset between y_∞ and the constrained true optimum, y^*_{sp}, computed using an SSTO with the true plant model. The reader may note here that the specific problem is that the SSTO will see no benefit in moving the planned steady-state value for u_1 because as far as the *model* is concerned, that would be further from the target. A change in the value of u_1 will clearly move the plant output closer to the desired infeasible target, however, u_1 has an opposite effect on the plant than on the model in this case. Since the SSTO is based on the model, it will not result in the constrained true optimum.

Remark 4. From studying the behavior of the closed-loop system in this example, if r lies *on* a single constraint of the true plant, then from the integral action of the controller $r = y^*_{sm} = y^*_{sp} = y_\infty$, but if r is unreachable with respect to the true plant, then y_∞ may deviate from the true optimum. The existence of offset between y_∞ and y^*_{sp} seems to depend on the gradient information of the active constraint(s). In this particular example, if the gradient of the mapped constraint corresponding to u_{2max} is exactly known (i.e. the rotation effect of G_m equals that of G_p), then there is zero constrained offset. Theory needs to be developed to prove that this behavior will exist in all possible scenarios.

Remark 5. In section 2 it was pointed out that a pseudo-setpoint representing the (x_s, u_s) combination (i.e. t) can be made a decision variable in the dynamic optimization, subject to the MCAS [2]. This avoids a sharp desired setpoint change resulting in infeasibility in the dynamic optimization (that would then require some form of constraint softening). Simulations revealed that the control horizon, n_c needs to be large enough to allow "creeping" within the feasible region towards the best value for y^*_{sp}. As this issue is not central, details are not discussed further.

4. CONDITIONS FOR CONSTRAINED OFFSET-FREE TRACKING WITH ACTIVE CONSTRAINTS

Motivated by the previous example, this section systematically considers the conditions necessary for constrained offset-free control with active constraints by incrementally introducing more complex scenarios. A review of the model requirements for asymptotic stability (condition (2)) in section 1.1 is presented. A necessary and sufficient result for constrained offset-free steady-state target determination is then demonstrated. SISO systems are first considered for simplicity and then MIMO systems are discussed. Although only the model steady-state gain matrix is available in practice, it is necessary to consider the true plant steady-state gain matrix in the sequel in order to demonstrate the conditions under which constrained offset can appear. The effect of using the model, as opposed to the true plant, can only be determined by characterizing the true plant behavior.

4.1 Model requirements for asymptotic stability

The conditions resulting in closed-loop instability of the unconstrained MPC can be determined from the Nyquist stability theorem (Skogestad and Hovd [1994]). Provided that the controller has integral action in all channels and the controller model is strictly proper, if:

$$\det(G_p)/\det(G_m) \begin{cases} < 0 \text{ for } P_m - P_p \text{ even} \\ > 0 \text{ for } P_m - P_p \text{ odd} \end{cases} \qquad (18)$$

then the closed-loop system is unstable (where P is the number of unstable open loop poles). Provided the correct number of unstable open-loop poles have been identified in the model, the requirements on the plant accuracy for closed-loop stability require the determinant of the steady-state gain matrix to have the correct sign. Further limitations on plant/model mismatch to achieve the true optimal steady-state target are presented in this section.

[1] A setpoint could be chosen that is feasible with respect to the model, but not with respect to the plant, which would amount to an equivalent scenario.

[2] Alternatively the MCAS could be included in the SSTO, with effectively the same results.

4.2 SISO systems, input constraints

Theorem 1. For non-integrating SISO systems with linearly independent input constraints, if conditions 1-5 in section 1.1 are fulfilled but constraints are active at steady state, then there is zero constrained offset in the controlled variables if and only if the steady-state model gain [3] has the same sign as that of the plant.

Proof: Theorem 1 can be proved by establishing the SSTO necessary and sufficient optimality conditions.

Necessary condition: Consider the following SSTO; at steady state with active constraints, we have:

$$\begin{array}{l} y^*_{sm} = y_\infty = g_p u_\infty + d_p = g_m u_\infty + d_m \\ u^*_s = u_\infty = u_{max} \text{ or } u_{min} \end{array} \tag{19}$$

The following optimization gives the steady-state Karush-Kuhn-Tucker (KKT) conditions necessary and sufficient for optimality, for either the plant or the model:

$$J_s(u^*_s) = \min_{u_s}(y_s - r)^2 = \min_{u_s}\ (g u_s + d - r)^2 \tag{20}$$

$$\text{s. t.: } \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{F} u_s - \underbrace{\begin{bmatrix} u_{max} \\ -u_{min} \end{bmatrix}}_{f} \leq 0 \tag{21}$$

$$\begin{aligned} \mathscr{L}(u_s, \lambda) &= u_s^2 g^2 + 2u_s g(d - r) + (d - r)^2 \\ &\quad + \lambda^T (F u_s - f) \end{aligned} \tag{22}$$

$$\begin{array}{l} \nabla_{u_s} \mathscr{L}(u^*_s, \lambda) = 2g^2 u^*_s + 2g(d - r) + F^T \lambda = 0 \\ \lambda \geq 0,\ F u^*_s - f \leq 0,\ \text{diag}(\lambda)(F u^*_s - f) = 0 \\ \nabla^2_{u_s} \mathscr{L}(u^*_s, \lambda) = 2g^2 > 0 \end{array} \tag{23}$$

From the dual feasibility condition in (23)

$$g u^*_s + (d - r) + \frac{\lambda_1 - \lambda_2}{2g} = 0,\ r = y^*_s + \frac{\lambda_1 - \lambda_2}{2g} \tag{24}$$

From (19) we know that the true plant and the model have converged to the same point (u_∞, y_∞), and both have the same desired setpoint, r. The SSTO will have chosen the model target y^*_{sm} to be optimal, so the necessary conditions of optimality will be satisfied for the model. However, $y^*_{sm}(= y_\infty) \neq y^*_{sp}$ may be true, where y^*_{sp} is the result of a hypothetical SSTO with true plant steady-state gain data. From (24), for $y^*_{sm} = y^*_{sp}$ [4]:

$$u^*_s = \begin{cases} u_{max} : \left.\begin{array}{l} \lambda_1 > 0 \\ \lambda_2 = 0 \end{array}\right\} \dfrac{\lambda_{1m}}{g_m} = \dfrac{\lambda_{1p}}{g_p} \Rightarrow \dfrac{g_m}{g_p} = \dfrac{\lambda_{1m}}{\lambda_{1p}} \\ u_{min} : \left.\begin{array}{l} \lambda_1 = 0 \\ \lambda_2 > 0 \end{array}\right\} \dfrac{\lambda_{2m}}{g_m} = \dfrac{\lambda_{2p}}{g_p} \Rightarrow \dfrac{g_m}{g_p} = \dfrac{\lambda_{2m}}{\lambda_{2p}} \end{cases} \tag{25}$$

For both instances of (25), $\text{sign}(g_m) = \text{sign}(g_p)$ necessarily.

Sufficient condition: If $\text{sign}(g_m) \neq \text{sign}(g_p)$, then satisfaction of (25) for either constraint active would require $\lambda < 0$, violating (23), and so $y^*_{sm} \neq y^*_{sp}$. □

4.3 MIMO systems, input constraints

The same approach can be taken with MIMO systems, comparing the SSTO KKT necessary and sufficient conditions for the true plant and model steady-state gain matrices. The SSTO for a square system with $R_s = 0$, no integrating modes and linearly independent constraints is as follows:

$$J^*_s(y^*_s) = \min_{y_s} \|y_s - r\|^2_{Q_s} = \min_{y_s} J_s(y_s) \tag{26}$$

$$\text{s. t.: } \underbrace{\begin{bmatrix} I \\ -I \end{bmatrix}}_{F} u_s - \begin{bmatrix} u_{max} \\ -u_{min} \end{bmatrix} \leq 0,\ y_s = G u_s + G_d d \tag{27}$$

$$u_s = G^{-1} y_s - G^{-1} G_d d \Rightarrow \tag{28}$$

$$J^*_s(y^*_s) = \min_{y_s} J_s \quad \text{s. t.: } \bar{F} y_s - \bar{f} \leq 0 \tag{29}$$

$$\text{where: } \bar{F} = F G^{-1},\ \bar{f} = f - F G^{-1} G_d d \tag{30}$$

$$\mathscr{L}(y_s, \lambda) = y_s^T Q_s y_s - 2y_s^T Q_s r + r^T Q_s r + \lambda^T \left(\bar{F} y_s - \bar{f}\right) \tag{31}$$

$$\begin{array}{l} \nabla_{y_s} \mathscr{L}(y^*_s, \lambda) = 2Q_s y^*_s - 2Q_s r + \bar{F}^T \lambda = 0 \\ \lambda \geq 0,\ \bar{F} y^*_s - f \leq 0,\ \text{diag}(\lambda)(\bar{F} y^*_s - f) = 0 \\ \nabla^2_{y_s} \mathscr{L}(y^*_s, \lambda) = 2Q_s \end{array} \tag{32}$$

Whatever constraints are active at steady state, as $y_\infty = G_m u_\infty + G_d d_m = G_p u_\infty + G_d d_p$ this must be true for both the plant and the model: there is no uncertainty in u_∞. Due to the necessity and sufficiency [5] conditions of (32), $y_\infty = y^*_{sm} = y^*_{sp}$ iff:

$$\exists \lambda_p : 2Q_s(r - y^*_{sm}) = \bar{F}^T_m \lambda_m = \bar{F}^T_p \lambda_p \tag{33}$$

i.e. the vector $2Q_s(r - y^*_{sm})$ must be a member of the tangent cone of $\bar{F}^T_p \lambda_p$. If the tangent cone of $\bar{F}^T_m \lambda_m$ is a member of the tangent cone of $\bar{F}^T_p \lambda_p$ then (33) is satisfied.

For simplicity, situations will now be considered separately for different numbers of active constraints:

With a single active constraint, i: if there are n_i constraints in F or $\bar{F}$, $\lambda_i > 0, \lambda_j = 0, j \in \{1, \ldots, n_i\}, j \neq i$, and F_i or $\bar{F}_i$ denotes the ith inequality in F_i or $\bar{F}_i$, the same constraint must be active for both the plant and the model (as $u_{sm} = u_{sp}$):

$$\exists \lambda_{pi} : \bar{F}_{mi} \lambda_{mi} = \bar{F}_{pi} \lambda_{pi} = 2Q_s(r - y^*_s) \tag{34}$$

$$\Rightarrow \exists k : \bar{F}_{mi} = k \bar{F}_{pi},\ k \in \mathbb{R},\ k > 0 \tag{35}$$

Therefore, the constraint normal vectors F_{mi} and F_{pi} must be parallel. This is illustrated in Figure 4(a). From (30),

$$\exists k : F_i(G_m^{-1}) = k F_i(G_p^{-1}) = F_i(G_p/k)^{-1} \tag{36}$$

So a sufficient condition for $y^*_{sp} = y^*_{sm}$ is $\exists k : k G_m = G_p,\ k > 0$. In fact, only the i'th column of G_m^{-1} need be linearly dependent of that of G_p^{-1}, as $\lambda_j = 0$. For a 2x2 matrix, this corresponds to the j'th row of G_m being linearly dependent of G_p, but the translation to requirements on G_m (as opposed to G_m^{-1}) is not obvious or simple for higher dimensions.

With two or more active constraints: again it is sufficient that if: $\exists k : k G_m = G_p,\ k > 0 \Rightarrow \bar{F}_m = k \bar{F}_p$, then $\exists \lambda_p = k \lambda_m$, satisfying (33). However, this is not necessary, as there is flexibility in the solution: a number of linear combinations of F_{mi} and F_{pi} will satisfy (33), i.e. the tangent cones overlap. This situation is illustrated in figure 4(b) for two active constraints. This situation is favorable, and should perhaps be a consideration in setpoint selection for zero constrained offset.

[3] Which only exists for systems without integrating modes.

[4] λ_i (where i is the active constraint) > 0 and not ≥ 0 because the constraints are linearly independent, and the set-point is taken to be infeasible, and so there will necessarily be a cost associated with the active constraints for optimality.

[5] $Q_s > 0$, constraints form a convex region.

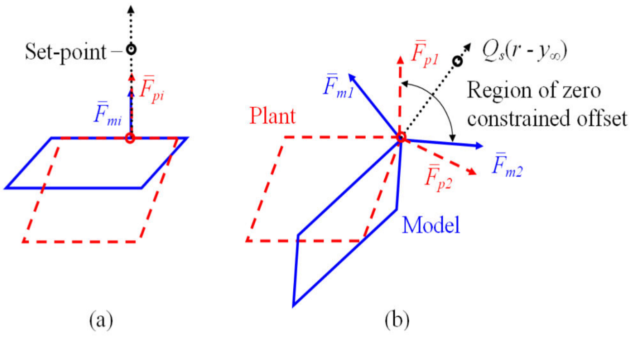

Fig. 4. Scenarios with: (a) zero constrained offset for a single active constraint, $\bar{F}_{mi}$ parallel to $\bar{F}_{pi}$, (b) flexibility in achieving constrained offset-free control with two active constraints.

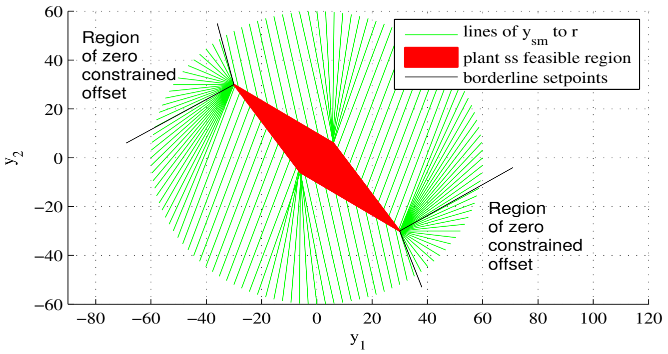

Fig. 5. Regions of zero constrained offset for motivating example.

Finally, to illustrate the different situations, the setpoint in the motivating example was rotated around, outside of the feasible region, and the simulations performed to steady state. Plotted in figure 5 are lines from y_∞ to r for each of these setpoints. It is easy to then identify the regions of zero constrained offset.

5. CONCLUSIONS

This paper has identified a problem that exists with modern MPC methods whereby when a constraint is active at steady state, there can be constrained offset in the controlled variables. The aim of this research activity is to determine the conditions with respect to the plant model accuracy for constrained offset-free control. This paper has begun this research effort by illustrating the problem with a simple motivating example. The example illustrated that with constraints active at steady state, the true optimal operating point is not always achieved through integral action, even though the closed-loop system does settle at a point which is optimal with respect to the model used in the SSTO. It was also noted that if a reference-governor type approach is employed with pseudo-setpoints, the control horizon needs to be sufficient to allow "creeping" to the optimal steady state.

Theoretical investigations using KKT conditions of optimality determined necessary and sufficient condition for SISO plants to attain the true constrained optimum. This result coincides with the necessary condition for asymptotic stability, that the sign of the determinant of both the plant and the model's steady state gain must be the same. However, for MIMO plants, for a single active constraint it is necessary for the rows of G_m^{-1} and G_p^{-1} corresponding to the active constraints to be linearly dependent, meaning that a very accurate model is required. With multiple active constraints, there are set-point regions where constrained zero-offset control is possible with significant modelling inaccuracies. A more general sufficient condition for constrained offset-free control is that G_m differs from G_p by only a scalar gain.

The practical implications of these results are that constrained offset-free control for unreachable set-points is difficult, but is possible if uncertainty in G can be reduced to scalar multiplicative uncertainty. As single active constraints are a common scenario, perhaps an alternative *adaptive* arrangement to the bias update approach might be possible for revising constraint gradients by collecting data around an operating point with active constraints.

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