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On instrumental variable-based methods for errors-in-variables model identification

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Abstract: In this paper, the problem of identifying stochastic linear discrete-time systems from noisy input/output data is addressed. The input noise is supposed to be white, while the output noise is assumed to be coloured. Some methods based on instrumental variable techniques are studied and compared to a least squares bias compensation scheme with the help of Monte Carlo simulations.

Keywords: System identification; errors-in-variables; instrumental variable; bias compensation.

1. INTRODUCTION

Identification of errors-in-variables (EIV) models has been an active domain of research in the recent years (see *e.g.* Mahata and Garnier (2006); Mahata (2007); Diversi et al. (2007); Pintelon and Schoukens (2007); Söderström (2008); Thil et al. (2007, 2008)). A survey paper gathering most of the recent developments has been recently published Söderström (2007). Among the 'classical' approaches using second order statistics, the instrumental variable (IV) methods have not received considerable interest. An explanation can be that the asymptotic accuracy of the *basic* IV method can be far from the Cramér-Rao lower bound Söderström (2007). However, there is a large freedom in the choice of the instrument vector, and many ideas stem from the basic IV principle, *e.g.* iterative methods Young (1970, 1984); Gilson and Van den Hof (2005); Young et al. (2008) or the use of higher-order statistics Inouye and Tsuchiya (1991); Inouye and Suga (1994).

The aim of this paper is to present some IV-based methods dedicated to the EIV system identification problem. It is organized as follows. The framework is presented in Section 2. In Section 3, two IV variants are proposed to consistently identify EIV models, while in Section 4 the principle of a bias compensated least squares scheme is recalled. A bias compensated IV scheme is then proposed in Section 5. Finally, before concluding, the performances of the proposed methods are assessed by means of a simulation example in Section 6.

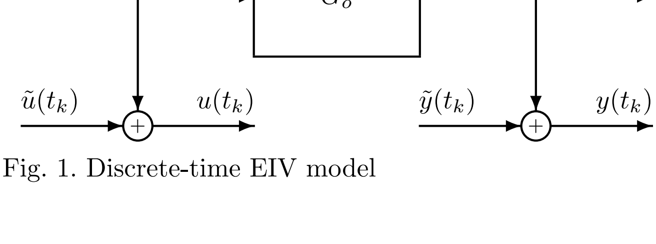


Fig. 1. Discrete-time EIV model

2. ERRORS-IN-VARIABLES FRAMEWORK

Consider the linear time-invariant errors-in-variables (EIV) system represented in Figure 1. The noise-free input and output signals are related by:

$$y_o(t_k) = G_o(q)u_o(t_k) \tag{1}$$

where q is the forward operator and $G_o(q)$ is the transfer operator of the 'true' system. The input/output signals are both contaminated by noise sequences, denoted as $\tilde{u}$ and $\tilde{y}$ respectively. The data-generating system is thus characterized by:

$$\mathcal{S}: \begin{cases} y_o(t_k) = G_o(q)u_o(t_k) \\ u(t_k) = u_o(t_k) + \tilde{u}(t_k) \\ y(t_k) = y_o(t_k) + \tilde{y}(t_k) \end{cases} \tag{2}$$

It is then parameterized as follows:

$$\mathcal{G}(\boldsymbol{\theta}): \begin{cases} y(t_k) = G(q,\boldsymbol{\theta})\left(u(t_k) - \tilde{u}(t_k)\right) + \tilde{y}(t_k) \\ G(q,\boldsymbol{\theta}) = B(q^{-1},\boldsymbol{\theta})/A(q^{-1},\boldsymbol{\theta}) \\ A(q^{-1},\boldsymbol{\theta}) = 1 + a_1 q^{-1} + ... + a_{n_a} q^{-n_a} \\ B(q^{-1},\boldsymbol{\theta}) = b_1 q^{-1} + ... + b_{n_b} q^{-n_b} \end{cases} \tag{3}$$

with $n_a \geqslant n_b$ and $\boldsymbol{\theta}^T = [a_1 \ \dots \ a_{n_a} \ b_1 \ \dots \ b_{n_b}]$. The input noise $\tilde{u}$ is assumed to be white, while the output noise $\tilde{y}$ is assumed to be given as the output of a moving average filter driven by a white noise

$$\mathcal{H}: \begin{cases} \tilde{u}(t_k) = e_{\tilde{u}}(t_k) \\ \tilde{y}(t_k) = C(q^{-1})e_{\tilde{y}}(t_k) \\ C(q^{-1}) = 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c} \end{cases} \tag{4}$$

where $e_{\tilde{u}}$ and $e_{\tilde{y}}$ are white noise sources of variance $\lambda_{e_{\tilde{u}}}$ and $\lambda_{e_{\tilde{y}}}$ respectively. Equation (3) can be rewritten as

$$y(t_k) = \boldsymbol{\varphi}^T(t_k)\boldsymbol{\theta} + v(t_k,\boldsymbol{\theta}) \tag{5}$$

$$v(t_k,\boldsymbol{\theta}) = \tilde{y}(t_k) - \tilde{\boldsymbol{\varphi}}^T(t_k)\boldsymbol{\theta} \tag{6}$$

where the regression vector is given as

$$\boldsymbol{\varphi}^T(t_k) = \big[-y(t_{k-1}) \ \dots \ -y(t_{k-n_a}) \quad u(t_{k-1}) \ \dots \ u(t_{k-n_b})\big] \tag{7}$$

The problem of identifying this errors-in-variables model is concerned with consistently estimating the parameter vector $\boldsymbol{\theta}$ from the noisy data $\{u(t_k), y(t_k)\}_{k=1}^{N}$.

2.1 Notations

As the input/output noises are additive, linear functions of the measured signals can be broken down into two parts, one made up of the noise-free signals contribution (denoted with a 'o' subscript), the other of the noises contribution (denoted with the '$\tilde{}$' sign). For example, the regression vector $\boldsymbol{\varphi}$ can be decomposed as

$$\boldsymbol{\varphi}(t_k) = \boldsymbol{\varphi}_o(t_k) + \tilde{\boldsymbol{\varphi}}(t_k) \tag{8}$$

To keep the notations compact in the sequel, it is also convenient to define

$$\bar{\boldsymbol{a}}^T = \left[1 \ \boldsymbol{a}^T\right] = [1 \ a_1 \ \dots \ a_{n_a}] \tag{9}$$
$$\boldsymbol{b}^T = [b_1 \ \dots \ b_{n_b}] \tag{10}$$
$$\bar{\boldsymbol{y}}^T(t_k, n) = \left[y(t_k) \ \boldsymbol{y}^T(t_k, n)\right] \tag{11}$$
$$\boldsymbol{y}^T(t_k, n) = [y(t_{k-1}) \ \dots \ y(t_{k-n})] \tag{12}$$
$$\boldsymbol{u}^T(t_k, n) = [u(t_{k-1}) \ \dots \ u(t_{k-n})] \tag{13}$$

Moreover, let $\boldsymbol{w}_1(t_k)$, $\boldsymbol{w}_2(t_k)$ be two vectors (of the same size) and $s(t_k)$ be a scalar. The following notations are used in the sequel for the covariance vectors and matrices

$$\boldsymbol{R}_{w_1 w_2} = \bar{E}\{\boldsymbol{w}_1(t_k)\boldsymbol{w}_2^T(t_k)\} \tag{14}$$
$$\boldsymbol{r}_{w_1 s} = \bar{E}\{\boldsymbol{w}_1(t_k)s(t_k)\} \tag{15}$$

where $\bar{E}\{\cdot\}$ stands for (see Ljung (1999))

$$\bar{E}\{f(t_k)\} = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} E\{f(t_k)\} \tag{16}$$

Lastly, an estimate of the parameter vector $\boldsymbol{\theta}$ from N samples of input/output data and its asymptotic version are written as

$$\hat{\boldsymbol{\theta}} \underset{N\to\infty}{\longrightarrow} \boldsymbol{\theta}^\star \qquad \text{w.p.1} \tag{17}$$

2.2 Assumptions

The following assumptions are needed

A1. The system (1) is asymptotically stable, and all the system modes are observable and controllable;
A2. The signals u_o, $e_{\tilde{u}}$ and $e_{\tilde{y}}$ are stationary, ergodic and zero-mean;
A3. The signals $e_{\tilde{u}}$ and $e_{\tilde{y}}$ are assumed to be uncorrelated with each other and with the input u_o.

3. INSTRUMENTAL VARIABLE METHODS

Let $\boldsymbol{z}(t_k)$ be a vector of instruments of dimension $n_z \geqslant n_a + n_b$. An eXtended IV estimator of $\boldsymbol{\theta}$ is given by Söderström and Stoica (1983)

$$\boldsymbol{\theta}^\star_{\mathsf{xiv}} = \left(\boldsymbol{R}_{z\varphi}^T\boldsymbol{W}\boldsymbol{R}_{z\varphi}\right)^{-1}\boldsymbol{R}_{z\varphi}^T\boldsymbol{W}\boldsymbol{r}_{zy} \tag{18a}$$
$$= \boldsymbol{\theta} + \left(\boldsymbol{R}_{z\varphi}^T\boldsymbol{W}\boldsymbol{R}_{z\varphi}\right)^{-1}\boldsymbol{r}_{zv} \tag{18b}$$

where $\boldsymbol{W}$ is a positive definite weighting matrix. In the sequel no weighting is made, and thus $\boldsymbol{W} = \boldsymbol{I}$. For the estimator (18) to be unbiased, the vector of instruments $\boldsymbol{z}(t_k)$ must be uncorrelated with the composite error $v(t_k, \boldsymbol{\theta})$, while being 'as correlated as possible' with the regression vector $\boldsymbol{\varphi}(t_k)$, so that the matrix to be inverted in (18) is not ill-conditioned, which is summarized as

$$\boldsymbol{r}_{zv} = 0 \tag{19}$$
$$\boldsymbol{R}_{z\varphi} \text{ is nonsingular} \tag{20}$$

There is no unique way to define $\boldsymbol{z}(t_k)$. Two different ways to handle the EIV problem are presented in this paper.

3.1 First approach: xiv1 method

The first approach has been explored in Söderström and Mahata (2002), considering the case of white noises on both input and output. The instrument vector is chosen to contain delayed inputs only

$$\boldsymbol{z}^T(t_k) = \boldsymbol{u}^T(t_{k-d_1}, n_{z_u}) \tag{21}$$
$$= \left[u(t_{k-1-d_1}) \ \dots \ u(t_{k-d_1-n_{z_u}})\right] \tag{22}$$

Since $\tilde{u}$, $\tilde{y}$ and u_o are uncorrelated

$$\boldsymbol{r}_{zv} = \bar{E}\{\boldsymbol{z}(t_k)v(t_k,\boldsymbol{\theta})\} \tag{23}$$
$$= \bar{E}\Big\{\boldsymbol{z}(t_k)\left(\tilde{\bar{\boldsymbol{y}}}^T(t_k, n_a)\bar{\boldsymbol{a}} - \tilde{\boldsymbol{u}}^T(t_k, n_b)\boldsymbol{b}\right)\Big\} \tag{24}$$
$$= -\bar{E}\Big\{\tilde{\boldsymbol{u}}(t_{k-d_1}, n_{z_u})\tilde{\boldsymbol{u}}^T(t_k, n_b)\Big\}\boldsymbol{b} \tag{25}$$

Thus, if the delay is such that $d_1 \geqslant n_b$, the vector of instruments $\boldsymbol{z}(t_k)$ satisfies (19). Besides, since $\tilde{u}$ and $\tilde{y}$ are uncorrelated, it is also true when the output noise is coloured.

3.2 Second approach: xiv2 method

The xiv1 method satisfies the conditions (19)-(20) and therefore leads to unbiased estimates. However, better results may be obtained by using an instrument vector more closely related to the regression vector. Indeed, it can be seen as a way to avoid – as much as possible – cases where the matrix $\boldsymbol{R}_{z\varphi}$ has a bad conditioning. To this end, let us define an instrument vector containing delayed inputs and delayed outputs as

$$\boldsymbol{z}^T(t_k) = \left[-\boldsymbol{y}^T(t_{k-d_2}, n_{z_y}) \ \boldsymbol{u}^T(t_{k-d_3}, n_{z_u})\right] \tag{26}$$
$$= \big[-y(t_{k-1-d_2}) \ \dots \ -y(t_{k-d_2-n_{z_y}}) \quad u(t_{k-1-d_3}) \ \dots \ u(t_{k-d_3-n_{z_u}})\big] \tag{27}$$

Since $\tilde{u}$, $\tilde{y}$ and u_o are uncorrelated

$$\boldsymbol{r}_{zv} = \bar{E}\{\boldsymbol{z}(t_k)v(t_k,\boldsymbol{\theta})\} \tag{28}$$
$$= \bar{E}\left\{\begin{matrix} -\boldsymbol{y}(t_{k-d_2}, n_{z_y})\left(\tilde{\bar{\boldsymbol{y}}}^T(t_k, n_a)\bar{\boldsymbol{a}} - \tilde{\boldsymbol{u}}^T(t_k, n_b)\boldsymbol{b}\right) \\ \boldsymbol{u}(t_{k-d_3}, n_{z_u})\left(\tilde{\bar{\boldsymbol{y}}}^T(t_k, n_a)\bar{\boldsymbol{a}} - \tilde{\boldsymbol{u}}^T(t_k, n_b)\boldsymbol{b}\right)\end{matrix}\right\}$$
$$= \bar{E}\left\{\begin{matrix} -\tilde{\boldsymbol{y}}(t_{k-d_2}, n_{z_y})\tilde{\bar{\boldsymbol{y}}}^T(t_k, n_a)\bar{\boldsymbol{a}} \\ -\tilde{\boldsymbol{u}}(t_{k-d_3}, n_{z_u})\tilde{\boldsymbol{u}}^T(t_k, n_b)\boldsymbol{b}\end{matrix}\right\} \tag{29}$$

Hence, if the delays satisfy $d_2 \geqslant n_a + n_c$ and $d_3 \geqslant n_b$, the vector of instruments $\boldsymbol{z}(t_k)$ satisfies (19). Besides, since the instrument vector (26) is closely linked to the regression vector, the matrix $\boldsymbol{R}_{z\varphi}$ is expected to have a better conditioning than when using the instrument vector (21). Therefore, the xiv2 method is expected to yield more accurate estimates of the parameter vector than the xiv1 method.

4. BIAS COMPENSATED LS

In the simulation example presented in Section 6, the proposed methods are compared with the bels[1] method introduced in Zheng (2002), one of the few existing methods that can handle the coloured output noise case. Here is given a quick overview of that method, as well as a way to improve its performances.

The goal of the bels method is to compensate the bias of the least squares (LS) estimate, which, in the case of white noise on input and coloured noise on output, is given by

$$\boldsymbol{\theta}^\star_{\mathsf{ls}} = \boldsymbol{\theta} - \boldsymbol{R}_{\varphi\varphi}^{-1}\begin{bmatrix}\boldsymbol{r}_{\tilde{y}\tilde{y}} + \boldsymbol{R}_{\tilde{y}\tilde{y}}\boldsymbol{a} \\ \lambda_{e_{\tilde{u}}}\boldsymbol{b}\end{bmatrix} \tag{30}$$

$$= \boldsymbol{\theta} - \boldsymbol{R}_{\varphi\varphi}^{-1}\boldsymbol{M}(\boldsymbol{\theta})\boldsymbol{\nu}_{\tilde{u},\tilde{y}} \tag{31}$$

where

$$\boldsymbol{\nu}^T_{\tilde{u},\tilde{y}} = \begin{bmatrix}\lambda_{\tilde{y}} & \boldsymbol{r}^T_{\tilde{y}\tilde{y}} & \lambda_{e_{\tilde{u}}}\end{bmatrix} \tag{32}$$

$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix}\boldsymbol{M_1} + \boldsymbol{M_2}(\boldsymbol{a}) \\ \boldsymbol{M_3}(\boldsymbol{b})\end{bmatrix} \tag{33}$$

and

- $\boldsymbol{M_1} = [0 \ \boldsymbol{I}_{n_a} \ 0] \ \in \mathbb{R}^{n_a \times (n_a+2)}$;
- $\boldsymbol{M_2}(\boldsymbol{a}) = \sum_{k=1}^{n_a} \boldsymbol{\mathcal{I}}_{n_a}(k)\boldsymbol{a}\boldsymbol{\tau}_k^T \ \in \mathbb{R}^{n_a \times (n_a+2)}$;
- $\boldsymbol{\tau}_k \in \mathbb{R}^{(n_a+2)\times 1}$ is the vector whose kth component is 1 and the others 0's;
- $\boldsymbol{\mathcal{I}}_n(k)$ is a square matrix of dimension n whose kth upper diagonal and kth lower diagonal are made of 1's, and 0's everywhere else (by convention $\boldsymbol{\mathcal{I}}_n(0) = \boldsymbol{I}_n$);
- $\boldsymbol{M_3}(\boldsymbol{b}) = \boldsymbol{b}\,\boldsymbol{\tau}^T_{n_a+2} \ \in \mathbb{R}^{n_b \times (n_a+2)}$.

If an estimate of $\boldsymbol{\nu}_{\tilde{u},\tilde{y}}$ is available, then the bias of the LS method can be subtracted to obtain a consistent estimate of $\boldsymbol{\theta}$

$$\boldsymbol{\theta}^\star_{\mathsf{bels}} = \boldsymbol{\theta}^\star_{\mathsf{ls}} + \boldsymbol{R}_{\varphi\varphi}^{-1}\boldsymbol{M}\left(\boldsymbol{\theta}\right)\boldsymbol{\nu}_{\tilde{u},\tilde{y}} \tag{34}$$

4.1 Estimation of $\boldsymbol{\nu}_{\tilde{u},\tilde{y}}$

The estimate of the vector $\boldsymbol{\nu}_{\tilde{u},\tilde{y}}$ is obtained by solving a system of linear equations.

A first equation is obtained from the quadratic error of the LS method, given as Zheng (2002)

$$J(\boldsymbol{\theta}^\star_{\mathsf{ls}}) = \bar{E}\left\{e(t_k, \boldsymbol{\theta}^\star_{\mathsf{ls}})^2\right\} \tag{35}$$

$$= \underbrace{\left\{\boldsymbol{\tau}_1^T + \boldsymbol{\theta}^{\star T}_{\mathsf{ls}}\boldsymbol{M}(\boldsymbol{\theta}) + \boldsymbol{a}^T\boldsymbol{M_3}\right\}}_{=\boldsymbol{q}(\boldsymbol{\theta}^\star_{\mathsf{ls}},\boldsymbol{\theta})}\boldsymbol{\nu}_{\tilde{u},\tilde{y}} \tag{36}$$

To obtain the other n_a+1 equations, the degree of the numerator of $G(q,\boldsymbol{\theta})$ is increased by n_a+1, the new parameters introduced this way having a true value equal to 0. It can then be shown that the vector $\boldsymbol{\nu}_{\tilde{u},\tilde{y}}$ satisfies Zheng (2002)

$$\boldsymbol{R}^T_{\varphi\mu}\boldsymbol{R}^{-1}_{\varphi\varphi}\boldsymbol{M}(\boldsymbol{\theta})\boldsymbol{\nu}_{\tilde{u},\tilde{y}} = \boldsymbol{r}_{\mu y} - \boldsymbol{R}^T_{\varphi\mu}\boldsymbol{\theta}^\star_{\mathsf{ls}} \tag{37}$$

where

$$\boldsymbol{\mu}(t_k) = [u(t_{k-n_b-1}) \ \ldots \ u(t_{k-n_b-n_a-1})]^T \tag{38}$$

4.2 The bels algorithm

1. Initialization: $\hat{\boldsymbol{\theta}}^0_{\mathsf{bels}} = \hat{\boldsymbol{\theta}}_{\mathsf{ls}}$;
2. Iteration until convergence:

   2.1. Computation of $\hat{\boldsymbol{\nu}}^{i+1}_{\tilde{u},\tilde{y}}$ by solving:

$$\begin{bmatrix}\hat{\boldsymbol{R}}^T_{\varphi\mu}\hat{\boldsymbol{R}}^{-1}_{\varphi\varphi}\boldsymbol{M}(\hat{\boldsymbol{\theta}}^i_{\mathsf{bels}}) \\ \boldsymbol{q}(\hat{\boldsymbol{\theta}}_{\mathsf{ls}},\hat{\boldsymbol{\theta}}^i_{\mathsf{bels}})\end{bmatrix}\hat{\boldsymbol{\nu}}^{i+1}_{\tilde{u},\tilde{y}} = \begin{bmatrix}\hat{\boldsymbol{r}}_{\mu y} - \hat{\boldsymbol{R}}^T_{\varphi\mu}\hat{\boldsymbol{\theta}}_{\mathsf{ls}} \\ J(\hat{\boldsymbol{\theta}}_{\mathsf{ls}})\end{bmatrix} \tag{39}$$

   2.2. Computation of $\hat{\boldsymbol{\theta}}^{i+1}_{\mathsf{bels}}$:

$$\hat{\boldsymbol{\theta}}^{i+1}_{\mathsf{bels}} = \hat{\boldsymbol{\theta}}_{\mathsf{ls}} + \hat{\boldsymbol{R}}^{-1}_{\varphi\varphi}\boldsymbol{M}(\hat{\boldsymbol{\theta}}^i_{\mathsf{bels}})\hat{\boldsymbol{\nu}}^{i+1}_{\tilde{u},\tilde{y}} \tag{40}$$

4.3 Extension to an over-determined system

In the method proposed in Zheng (2002), the estimate of the statistical properties of the noises, contained in the vector $\boldsymbol{\nu}_{\tilde{u},\tilde{y}}$, is obtained by solving the system of linear equations (39), composed of n_a+2 equations in the n_a+2 unknowns. However, it appears that the method gives rather crude estimates of $\boldsymbol{\nu}_{\tilde{u},\tilde{y}}$, especially when the signal-to-noise ratio is low, or when there are many parameters to estimate (in the numerical example of Zheng (2002) a first-order model is considered; when the order of the model increases, the performances of the algorithm deteriorate). This crude estimation of $\boldsymbol{\nu}_{\tilde{u},\tilde{y}}$ may lead to convergence problems, and crude estimates of the parameter vector as well.

An idea to overcome this problem is to use more equations to estimate the noise statistical properties. Indeed, increasing the numerator's degree by a number n_a+1+n_{bels} where $n_{\mathsf{bels}} \geqslant 1$ yields an over-determined system, which can be solved in a least squares sense to obtain a better estimate of $\boldsymbol{\nu}_{\tilde{u},\tilde{y}}$. The following vector

$$\boldsymbol{\mu}(t_k) = [u(t_{k-n_b-1}) \ \ldots \ u(t_{k-n_b-n_a-1-n_{\mathsf{bels}}})]^T \tag{41}$$

is therefore used instead of (38). The resulting parameter estimate is expected to be more accurate, which in turn implies a quicker convergence of the algorithm. The associated method is denoted abels[2].

5. BIAS COMPENSATED IV

The bias compensated least squares schemes need to estimate a vector $\boldsymbol{\nu}$ containing some values of the noises' autocovariance functions. However, this is a difficult task, especially when the size of this vector increases. Indeed, when the noises are both white, only their variance has to be estimated and the bias compensated least squares schemes achieve satisfactory results. When the output noise is coloured, however, the size of this vector is directly linked to the order of the estimated model, and the performances deteriorate.

On another hand, the IV methods can be applied in fairly general noise conditions, but defining an instrument vector uncorrelated with the noises (see (19)) is often in contradiction with the second requirement (20). Indeed, when the value of the delays appearing in the instrument vector increases, it becomes 'less correlated' with the regression vector.

Here is proposed a method which avoids both these disadvantages. A bias compensated instrumental variable is used to obtain an estimate of the parameter vector $\boldsymbol{\theta}$: in the first step the coloured output noise is handled by an instrumental variable, while in a second step a bias compensation scheme is applied to remove the bias induced

[1] for "Bias Eliminated Least Squares".

[2] for "Augmented Bias Eliminated Least Squares".

by the input noise. By allowing the IV estimator to give biased estimates, the instrument vector can be chosen 'more correlated' with the regression vector, while only the input noise variance is needed to be estimated for the bias compensation scheme. Note that the idea of compensating the IV bias was used in Yang et al. (1993).
Define a vector of instruments containing inputs and delayed outputs as

$$\boldsymbol{z}^T(t_k) = \left[-\boldsymbol{y}^T(t_{k-d_4}, n_{z_y}) \;\; \boldsymbol{u}^T(t_k, n_{z_u})\right] \tag{42}$$

An IV estimator of $\boldsymbol{\theta}$ is then given by

$$\boldsymbol{\theta}^\star_{\mathsf{iv3}} = \boldsymbol{R}_{z\varphi}^{-1}\boldsymbol{r}_{zy} = \boldsymbol{\theta} + \boldsymbol{R}_{z\varphi}^{-1}\boldsymbol{r}_{zv} \tag{43}$$

Assume that $d_4 \geqslant n_a + n_c$. Then

$$\boldsymbol{r}_{zv} = \bar{E}\left\{ \begin{matrix} -\tilde{\boldsymbol{y}}(t_{k-d_4}, n_{z_y})\tilde{\bar{\boldsymbol{y}}}^T(t_k, n_a)\bar{\boldsymbol{a}} \\ -\tilde{\boldsymbol{u}}(t_k, n_{z_u})\tilde{\boldsymbol{u}}^T(t_k, n_b)\boldsymbol{b} \end{matrix} \right\} \tag{44}$$

$$= -\lambda_{e_{\tilde{u}}} \begin{bmatrix} \boldsymbol{0}_{n_{z_y}\times n_a} & \boldsymbol{0}_{n_{z_y}\times n_b} \\ \boldsymbol{0}_{n_{z_u}\times n_a} & \boldsymbol{I}_{n_{z_u}\times n_b} \end{bmatrix} \boldsymbol{\theta} \tag{45}$$

$$= -\lambda_{e_{\tilde{u}}} \boldsymbol{M_4}\boldsymbol{\theta} \tag{46}$$

Thus finally

$$\boldsymbol{\theta}^\star_{\mathsf{iv3}} = \boldsymbol{R}_{z\varphi}^{-1}\boldsymbol{r}_{zy} = \boldsymbol{\theta} - \lambda_{e_{\tilde{u}}}\boldsymbol{R}_{z\varphi}^{-1}\boldsymbol{M_4}\boldsymbol{\theta} \tag{47}$$

If an estimate of the input noise variance $\lambda_{e_{\tilde{u}}}$ is available, then the bias of the estimator (43) can be subtracted to obtain a consistent estimate of $\boldsymbol{\theta}$

$$\boldsymbol{\theta}^\star_{\mathsf{bciv}} = \boldsymbol{\theta}^\star_{\mathsf{iv3}} + \lambda_{e_{\tilde{u}}}\boldsymbol{R}_{z\varphi}^{-1}\boldsymbol{M_4}\boldsymbol{\theta} \tag{48}$$

5.1 Estimation of $\lambda_{e_{\tilde{u}}}$

To obtain an estimate of $\lambda_{e_{\tilde{u}}}$ the idea is to use the correlation between the composite noise $v(t_k, \boldsymbol{\theta})$ and the 'residual' r_1, defined as

$$r_1(t_k, d, \boldsymbol{\theta}) = y(t_{k-d}, n_a) - \boldsymbol{z}^T(t_k)\boldsymbol{\theta} \tag{49}$$

$$= \bar{\boldsymbol{a}}^T\bar{\boldsymbol{y}}(t_{k-d}, n_a) - \boldsymbol{b}^T\boldsymbol{u}(t_k, n_b) \tag{50}$$

Then, if $d > n_a + n_c$

$$J_1(d, \boldsymbol{\theta}) = \bar{E}\{r_1(t_k, d, \boldsymbol{\theta})v(t_k, \boldsymbol{\theta})\} \tag{51}$$

$$= \bar{\boldsymbol{a}}^T\bar{E}\left\{\tilde{\bar{\boldsymbol{y}}}(t_{k-d}, n_a)\tilde{\bar{\boldsymbol{y}}}^T(t_k, n_a)\right\}\bar{\boldsymbol{a}} + \boldsymbol{b}^T\bar{E}\left\{\tilde{\boldsymbol{u}}(t_k, n_b)\tilde{\boldsymbol{u}}^T(t_k, n_b)\right\}\boldsymbol{b} \tag{52}$$

$$= \lambda_{e_{\tilde{u}}}\boldsymbol{b}^T\boldsymbol{b} \tag{53}$$

If an estimate of $\boldsymbol{b}$ is available, then an estimate of the input noise variance is obtained as

$$\hat{\lambda}_{e_{\tilde{u}}} = \frac{J_1(d, \hat{\boldsymbol{\theta}})}{\hat{\boldsymbol{b}}^T\hat{\boldsymbol{b}}} \qquad \text{for } d > n_a + n_c \tag{54}$$

Improvement by filtering The estimation of the variance $\lambda_{e_{\tilde{u}}}$ relies on the estimation of $\boldsymbol{b}$. It is thus preferable to obtain the best estimation possible of this vector. To this end, a filtered version of (54) is used. This filtering should not be viewed as a way to emphasize some spectrum part, but rather as a way to introduce *a priori* knowledge to the problem (*i.e.* the filter coefficients).
Define a moving average filter $F(q^{-1})$ as

$$F(q^{-1}) = 1 + \sum_{k=1}^{n_f} f_k q^{-k} \tag{55}$$

A second residual is then introduced

$$r_2(t_k, d, \boldsymbol{\theta}) = \bar{\boldsymbol{a}}^T\bar{\boldsymbol{y}}(t_{k-d}, n_a) - \boldsymbol{b}^T\boldsymbol{u}_f(t_k, n_b) \tag{56}$$

where $\boldsymbol{u}_f(t_k, n_b) = F(q^{-1})\boldsymbol{u}(t_k, n_b)$. Then, if $d > n_a + n_c$

$$J_2(d, \boldsymbol{\theta}) = \bar{E}\{r_2(t_k, d, \boldsymbol{\theta})v(t_k, \boldsymbol{\theta})\} \tag{57}$$

$$= \bar{\boldsymbol{a}}^T\bar{E}\left\{\tilde{\bar{\boldsymbol{y}}}(t_{k-d}, n_a)\tilde{\bar{\boldsymbol{y}}}^T(t_k, n_a)\right\}\bar{\boldsymbol{a}} + \boldsymbol{b}^T\bar{E}\left\{\tilde{\boldsymbol{u}}_f(t_k, n_b)\tilde{\boldsymbol{u}}^T(t_k, n_b)\right\}\boldsymbol{b} \tag{58}$$

$$= \lambda_{e_{\tilde{u}}}\boldsymbol{b}^T\boldsymbol{M}_F\,\boldsymbol{b} \tag{59}$$

where $\boldsymbol{M}_F$ is the following Tœplitz matrix

$$\boldsymbol{M}_F = \begin{bmatrix} f_0 & 0 & \dots & 0 \\ f_1 & f_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ f_{n_b-1} & f_{n_b-2} & \dots & f_0 \end{bmatrix} \tag{60}$$

Hence, if an estimate of $\boldsymbol{b}$ is available, then an estimate of the input noise variance is obtained as

$$\hat{\lambda}_{e_{\tilde{u}}} = \frac{J_2(d, \hat{\boldsymbol{\theta}})}{\hat{\boldsymbol{b}}^T\boldsymbol{M}_F\hat{\boldsymbol{b}}} \qquad \text{for } d > n_a + n_c \tag{61}$$

Remark 1. The bciv method requires the knowledge of $\boldsymbol{\theta}$ (in the expression of the bias and in the estimate of the input noise variance). Hence, it has to be iterative. Furthermore, since (54) is true for any $d > n_a + n_c$, the variance estimation can be improved using n_{bciv} values of d instead of one.

5.2 The bciv *algorithm*

1. Initialization: $\hat{\boldsymbol{\theta}}^0_{\mathsf{bciv}} = \hat{\boldsymbol{\theta}}_{\mathsf{iv3}}$
2. Iteration until convergence:
   2.1. Estimation of $\lambda_{e_{\tilde{u}}}$ using (54) or (61)
   2.2. Computation of $\hat{\boldsymbol{\theta}}^{i+1}_{\mathsf{bciv}}$

$$\hat{\boldsymbol{\theta}}^{i+1}_{\mathsf{bciv}} = \hat{\boldsymbol{\theta}}_{\mathsf{iv3}} + \hat{\lambda}^i_{e_{\tilde{u}}}\hat{\boldsymbol{R}}_{z\varphi}^{-1}\boldsymbol{M_4}\hat{\boldsymbol{\theta}}^i_{\mathsf{bciv}} \tag{62}$$

In the numerical example Section, the algorithm using (54) to obtain the input variance estimate is referred to as bciv, while the algorithm using (61) is denoted fbciv [3] (the number of values of $J_1(d, \boldsymbol{\theta})$ used to estimated the input noise variance is noted n_{bciv}).

6. NUMERICAL SIMULATIONS

The following system is considered

$$G_o(q) = \frac{q^{-1} + 0.5q^{-2}}{1 - 1.5q^{-1} + 0.7q^{-2}} \tag{63}$$

The noise-free input is defined as Söderström and Mahata (2002)

$$u_o(t_k) = \frac{1}{1 - 0.2q^{-1} + 0.5q^{-2}} e_{u_o}(t_k) \tag{64}$$

where e_{u_o} is a white noise source. The coloured noise model is defined in (4) with

$$C(q^{-1}) = 1 - 0.2q^{-1} \tag{65}$$

The variances of the white noises are then set to $\lambda_{e_{\tilde{u}}} = 0.14$ and $\lambda_{e_{\tilde{y}}} = 1.45$ to obtain a signal-to-noise ratio (SNR) equal to about 10 dB on both input and output, with

$$\text{SNR} = 10\log_{10}\left(\frac{P_{x_o}}{P_{\tilde{x}}}\right) \tag{66}$$

[3] for "Filtered Bias Compensated Instrumental Variable".

where P_x represents the average power of the signal x. In all the iterative algorithms (bels, abels, bciv and fbciv) the same stop criterion is used, that is

$$\frac{\|\hat{\boldsymbol{\theta}}^{i+1} - \hat{\boldsymbol{\theta}}^{i}\|}{\|\hat{\boldsymbol{\theta}}^{i}\|} < 10^{-3} \quad \text{or} \quad i \geqslant 50 \tag{67}$$

where $\|\cdot\|$ is the Euclidian norm and $\hat{\boldsymbol{\theta}}^{i}$ the estimate obtained at the ith iteration. Hence, if an algorithm has not 'converged' before 50 iterations, it is automatically stopped. The system parameters are estimated on the basis of a data set of length $N = 2000$, and a Monte Carlo simulation of $n_{mc} = 100$ runs is performed. The filter in the fbciv method is chosen as $F(q^{-1}) = 1 + q^{-1} + q^{-2}$, and the delays d_i, $1 \leqslant i \leqslant 4$ are set to their minimum value. The estimates of the parameter vector and the input noise variance are given in Table 1, as well as the normalized root mean square error, defined as

$$\text{NRMSE} = \sqrt{\frac{1}{n_{mc}} \sum_{i=1}^{n_{mc}} \frac{\|\hat{\boldsymbol{\theta}}^{i} - \boldsymbol{\theta}_o\|^2}{\|\boldsymbol{\theta}_o\|^2}} \tag{68}$$

Besides, since some methods do not always converge before 50 iterations, the Table 1 also contains 'C^{ce}', the percentage of simulations for which the algorithms have converged. The Bode diagrams are plotted on Figure 2.

6.1 Discussion

In this example, the bels algorithm fails to give good results because of a too high number of parameters to be estimated. Note that it does not lead to a reliable estimation of the input noise variance. However, its performance is greatly improved by using of the suggested over-determined system to estimate the noise statistical properties. Although still not very good, the estimation of input noise variance is improved, leading to better parameter estimates. The two IV methods provide unbiased results with reasonable standard deviation. Note that, as expected, the use of past outputs in the xiv2 method allows to get overall better estimates than the xiv1 method. Finally, the bias-compensated instrumental variable methods give, at first, poor results, but are greatly improved by using of an over-determined system to estimate the input noise variance. However, thanks to the prefiltering operation, the proposed fbciv algorithm provides accurate results, in particular the estimates of the input noise variance obtained with this algorithm are (very) close to the true value.

7. CONCLUSION

This paper was concerned with identification of parametric errors-in-variables models, assuming white noise on input and coloured noise on output. Two extended instrumental variable methods, as well as a bias compensated instrumental variable have been proposed. The performances of the proposed methods have been compared to a bias compensated least squares algorithm through a numerical simulation. Future work will include the analysis of the influence of the various user parameters (the number of equations added in abels, the filter in fbciv, etc).

REFERENCES

R. Diversi, R. Guidorzi, and U. Soverini. Maximum likelihood identification of noisy input-output models. *Automatica*, 43(2):464–472, 2007.

M. Gilson and P. Van den Hof. Instrumental variable methods for closed-loop system identification. *Automatica*, 41(2):241–249, February 2005.

Y. Inouye and Y. Suga. Identification of linear systems with noisy input using input-output cumulants. *International Journal of Control*, 59(5):1231–1253, May 1994.

Y. Inouye and H. Tsuchiya. Identification of linear systems using input-output cumulants. *International Journal of Control*, 53(6):1431–1448, June 1991.

L. Ljung. *System Identification: Theory for the User.* Prentice Hall, Englewood Cliffs, 2nd edition, 1999.

K. Mahata. An improved bias-compensation approach for errors-in-variables model identification. *Automatica*, 43 (8):1339–1354, August 2007.

K. Mahata and H. Garnier. Identification of continuous-time errors-in-variables models. *Automatica*, 49(9): 1470–1490, September 2006.

R. Pintelon and J. Schoukens. Frequency domain maximum likelihood estimation of linear dynamic errors-in-variables models. *Automatica*, 43(4):621–630, 2007.

T. Söderström. Errors-in-variables methods in system identification. *Automatica*, 43(6):939–958, June 2007.

T. Söderström. Extending the Frisch scheme for errors-in-variables identification to correlated output noise. *International Journal of Adaptive Control and Signal Processing*, 22(1):55–73, February 2008.

T. Söderström and K. Mahata. On instrumental variable and total least squares approaches for identification of noisy systems. *International Journal of Control*, 75(6): 381–389, April 2002.

T. Söderström and P. Stoica. *Instrumental Variable Methods for System Identification.* Springer, Berlin, 1983.

S. Thil, H. Garnier, M. Gilson, and K. Mahata. Continuous-time model identification from noisy input/output measurements using fourth-order cumulants. In *46th IEEE Conference on Decision and Control (CDC'2007)*, New Orleans, LA, USA, December 2007.

S. Thil, H. Garnier, and M. Gilson. Third–order cumulants based methods for continuous–time errors–in–variables model identification. *Automatica*, 44(3), March 2008.

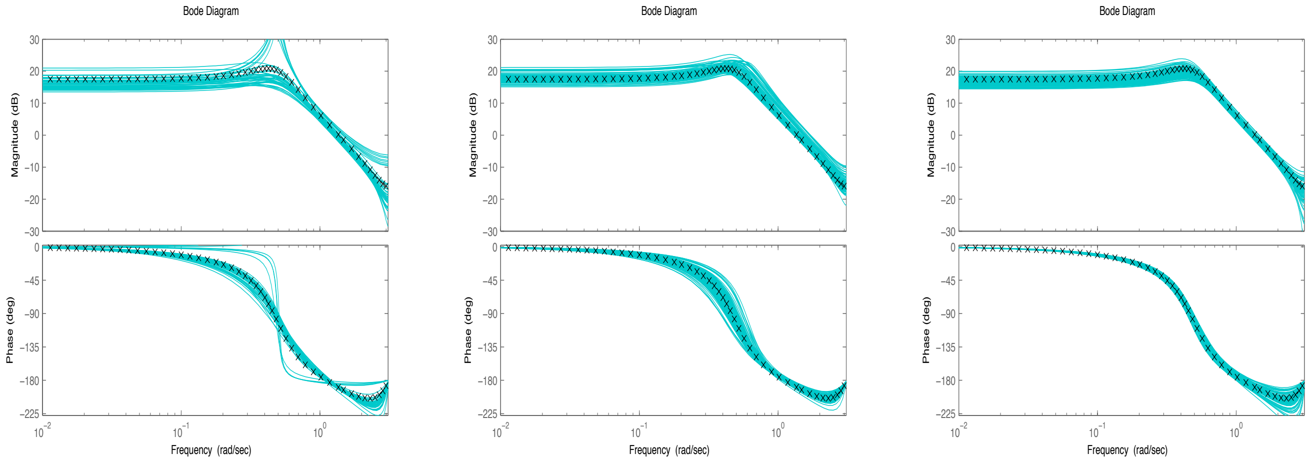
Z.-J. Yang, S. Sagara, and K. Wada. Identification of continuous-time systems from sampled input-output data using bias eliminating techniques. *Control Theory and Advanced Technology*, 9(1):53–75, March 1993.

P.C. Young. An instrumental variable method for real-time identification of a noisy process. *Automatica*, 6(2): 271–287, March 1970.

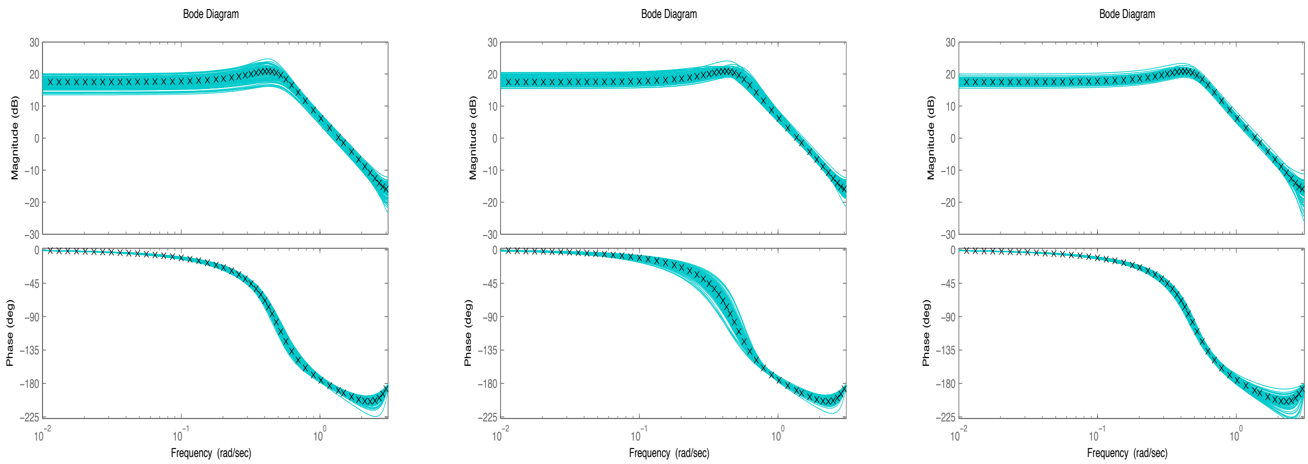
P.C. Young. *Recursive Estimation and Time Series Analysis.* Springer-Verlag, Berlin, 1984.

P.C. Young, H. Garnier, and M. Gilson. Refined instrumental variable identification of continuous-time hybrid Box-Jenkins models. In H. Garnier and L. Wang, editors, *Identification of Continuous-time Models from Sampled Data.* Springer-Verlag, London, 2008.

W.X. Zheng. A bias correction method for identification of linear dynamic errors-in-variables models. *IEEE Transactions on Automatic Control*, 47(7):1142–1147, July 2002.



(a) bels (b) bciv ($n_{\mathsf{bciv}} = 2$) (c) xiv1



(d) abels (e) fbciv (f) xiv2 ($n_{\mathsf{bels}} = 4$)

Fig. 2. Bode diagrams of the true ('x') and estimated models.

	true	bels	abels		xiv1	xiv2	bciv		fbciv
			$n_{\mathsf{bels}} = 3$	$n_{\mathsf{bels}} = 4$			$n_{\mathsf{bciv}} = 1$	$n_{\mathsf{bciv}} = 2$	
a_1	-1.5	-0.696 ± 0.556	-1.488 ± 0.002	-1.488 ± 0.001	-1.476 ± 0.001	-1.499 ± 0.001	-1.462 ± 0.005	-1.490 ± 0.001	-1.498 ± 0.001
a_2	0.7	0.325 ± 0.130	0.691 ± 0.001	0.691 ± 0.001	0.681 ± 0.001	0.699 ± 0.001	0.689 ± 0.001	0.695 ± 0.001	0.697 ± 0.001
b_1	1	0.450 ± 0.279	0.973 ± 0.022	0.972 ± 0.014	0.942 ± 0.012	0.996 ± 0.008	1.305 ± 0.251	1.083 ± 0.044	1.022 ± 0.009
b_2	0.5	0.207 ± 0.061	0.491 ± 0.003	0.494 ± 0.003	0.495 ± 0.006	0.504 ± 0.007	0.629 ± 0.053	0.533 ± 0.009	0.506 ± 0.003
$\lambda_{e_{\tilde{u}}}$	0.14	0.626 ± 12.56	0.069 ± 0.049	0.080 ± 0.028	–	–	0.361 ± 0.148	0.209 ± 0.054	0.155 ± 0.013
C^{ce}	–	47%	100%	100%	–	–	52%	88%	100%
NRMSE	–	66.4%	7.7%	6.3%	6.9%	5.5%	29.0%	11.2%	5.1%

Table 1. Monte Carlo simulation results for $N = 2000$, SNR $\simeq$ 10dB.