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SPI: An Active Queue Management Algorithm for HSTCP ⋆

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Abstract: In this paper, a Scalable Proportional Integral (SPI) active queue management (AQM) algorithm for High Speed TCP (HSTCP) is proposed. Based on the linearization model for HSTCP, we give a stability analysis for the closed loop interconnection systems (HSTCP and SPI AQM). Then, an adaptive parameters setting method for SPI AQM is presented. Using ns2 simulation, we illustrate that SPI shows satisfactory RTT fairness, TCP fairness and convergence performance, also SPI has good robustness in terms of link capacity, the number of connections, round trip time and buffer size.

1. INTRODUCTION

With the growth of link capacity and the emergence of new Internet applications with high-bandwidth demand, TCP's performance is unsatisfactory, especially on high-speed and long distance networks. The main reason for this is the conservative behavior of TCP in adjusting congestion window. This problem leads to the proposal of HighSpeed TCP (HSTCP)[1]. Compared with standard TCP, HSTCP increases the congestion window more aggressively upon receiving ACK packets and decreases the congestion window more gently upon detecting packet losses. For the simplicity of HSTCP and easiness of deploitation in the Internet, HSTCP is advocated by the IETF. However, this mechanism of congestion window will lead to serious RTT unfairness[9], long convergence time and TCP unfairness[8]. In [8], the authors proposed a two-mode congestion avoidance rule to improve TCP fairness. However it can't consider the RTT unfairness and convergence problems. In [7], the convergence rate of HSTCP is improved by decreasing the congestion window more aggressively. But they don't consider the other two problems of HSTCP. In general, it is difficult to improve the performance of HSTCP only by modifying the increasing and decreasing mechanism of the congestion window. In tradition TCP, active queue management (AQM) is introduced to improve the performance of end-to-end congestion control under DropTail queue management, such as RED [2], PI [3], Yellow[6], and so on. Our observation is that the AQM algorithm can improve the HSTCP performance as in tradition TCP. However, there is little work on designing AQM for HSTCP.The basic philosophy of AQM is to drop (or mark, if explicit congestion notification[5] is enabled.) before buffer overflow and to signal the source early to reduce the sending rate. In [10] the authors mainly consider TCP fairness problem. By combining gHSTCP and aARED, they can derive effective utilization of network and good fairness. However they didn't consider the RTT unfairness and the convergence problems of HSTCP. In this paper we will design an AQM which can solve all the limitations of HSTCP under DropTail router. It's known that TCP is ill-suited for high Bandwidth-Delay product (BDP) networks, and HSTCP has better scalable performance in large BDP networks. Those AQM algorithms using control theory based on regular TCP model have low utilization in HSTCP environment for the mismatch model between HSTCP and TCP. In [4], a model of HSTCP based on small time scale linearization is proposed, which has the same structure as the regular TCP model and different parameters.

In this paper, we will design an PI controller for HSTCP based on the proposed model.We call it as Scalable Proportional Integral (SPI) active queue management algorithm. The key idea of SPI is that the control parameters of SPI vary according to the link's bandwidth. We present a stability analysis for the Internet system composed of HSTCP and SPI AQM based on the celebrated Pontryagin Criterion. According to the stability result, we can obtain the stable parameters of SPI.

2. MODEL DESCRIPTION

The congestion avoidance algorithm of HSTCP is specified using three parameters: W_{low} , W_{high} and P_{high}. If the current congestion window is less than W_{low}, HSTCP uses the same response function as standard TCP. Otherwise it uses a new response function. W_{high} and P_{high} are used to specify the upper bound of the HSTCP response function. The default values of W_{low} , W_{high} and P_{high} are 38, 83000 and 10^{-7} respectively.

In HSTCP the senders adopt AIMD mechanism to adjusting the congestion window. The HSTCP congestion control can be expressed as following

$$ACK : W \leftarrow W + \frac{a(W)}{W} \qquad (1)$$
$$LOSS : W \leftarrow (1 - b(W)) \times W$$

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where

$$a(W) = \frac{2 \times W^2 \times b(W) \times P(W)}{2 - b(W)}$$
$$b(W) = \frac{log(W) - log(W_{low})}{log(W_{high}) - log(W_{low})}(b_{high} - 0.5) + 0.5$$
$$P(W) = \exp\left[\begin{array}{c}\frac{log(W) - log(W_{low})}{log(W_{high}) - log(W_{low})}\times \\ (log(P_{high}) - log(P_{low})) + log(P_{low})\end{array}\right]$$

When W is larger than W_{low}, the value of $a(W)$ is more than 1 and the value of $b(W)$ is less than 0.5. This means that HSTCP increases W more aggressively and decreases W more gently than standard TCP when $W > W_{low}$. In [4], the author proposed a model of HSTCP, which can be described as:

$$\begin{aligned}\dot{W} &= \frac{a(W)}{R} - (\frac{b(W)W^2}{R} + \frac{a(W)\cdot b(W)}{2R})p(t-R) \\ \dot{q} &= -C + N\frac{W}{R} \\ R &= T_p + \frac{q}{C}\end{aligned} \quad (2)$$

where W is the congestion window size, N is the number of users, R is the round trip time, q is the queue size, p is the per-packet drop rate, $a(W)$ and $b(W)$ are the increasing and decreasing parameters respectively.

The linear model of HSTCP can be derived based on linearization at the operation point (W_0, q_0, p_0).

$$\begin{aligned}\delta\dot{W} &= -K_1 \cdot \delta W - K_2 \cdot \delta p(t-R) \\ \delta\dot{q} &= K_3 \cdot \delta W - K_4 \cdot \delta q\end{aligned} \quad (3)$$

where

$$\begin{cases} K_1 \approx \dfrac{1.25a(W_0)}{W_0 R} \\ K_2 = \dfrac{b(W_0)\cdot W_0 \cdot (2W_0 + a(W_0))}{2W_0} \\ K_3 = \dfrac{N}{R} \\ K_4 = \dfrac{1}{R} \end{cases}$$

The block diagram of the linearized HSTCP/AQM closed-loop control system can be described in Fig.1.

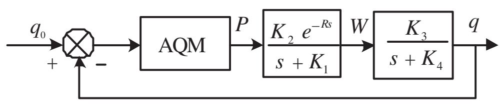

Fig. 1. Block diagram of the linearized HSTCP/AQM closed-loop control system

3. ALGORITHM DESIGN

Based on the above analysis, we can get the open loop transfer function of system as following

$$G_o(s) = \frac{K_2 K_3}{(s + K_1)(s + K_4)} e^{-Rs} \quad (4)$$

The controller is chosen as PI controller denoted by $G_c(s)$.

$$G_c(s) = K_p + \frac{K_I}{s} \quad (5)$$

For $K_1 \ll K_4$ in the real networks, we choose $\frac{K_I}{K_p} \approx K_1$, which can improve the dynamic performance. So the open loop transfer function can be described approximately as

$$G(s) = \frac{K_p K_2 K_3}{s(s + K_4)} e^{-Rs} \quad (6)$$

For convenience, let $K = K_p K_2 K_3, a = K_4, R = \tau$,then the closed loop differential equation can be obtained based on Fig.1 and Eq.(6).

$$\ddot{e}(t) + a\dot{e}(t) + Ke(t - \tau) = 0 \quad (7)$$

Let $t = \tau\xi$, where ξ is a positive integer variable, then $dt = \tau d\xi$. The normalized closed-loop equation is

$$\frac{de^2(\xi)}{\tau^2 d\xi} + a\frac{de(\xi)}{\tau d\xi} + Ke(\xi - 1) = 0 \quad (8)$$

The characteristic equation is

$$\left(s^2 + a\tau s\right)e^s + K\tau^2 = 0 \quad (9)$$

Theorem *Let the root of the equation* $\tan\omega = \frac{a\tau}{\omega}$ *in the interval* $\left[0, \frac{\pi}{2}\right]$ *is* ω_{01}*, the necessary and sufficient condition of stability for the closed loop system (7) is that the parameters satisfy the following restraint*

$$K\tau^2 < \left(\omega_{01}^2 + a^2\tau^2\right)\cos\omega_{01}$$

The detailed proof is given in the Appendix.

According to the above description, we can see the designed parameters are related with the network parameters, such as round trip time, bandwidth, the number of connections, and so on. From the Theorem, we can get the stable region for the closed-loop system.

$$\begin{aligned}K_P &< \frac{\left(\omega_{01}^2 + a^2\tau^2\right)\cos\omega_{01}}{K_2 K_3 \tau^2} \\ K_I &= K_1 K_p\end{aligned}$$

Let $\bar{K}_p$ is the upper bound of proportional parameter K_P. Substituting K_2,K_3 to $\bar{K}_p$, $\bar{K}_p$ can be approximated as the follows with the fact $a(\omega_0) \ll \omega_0$.

$$\bar{K}_P \approx \frac{\left(\omega_{01}^2 + a^2\tau^2\right)\cos\omega_{01}}{b(W_0)NW_0^2}$$

Let $\beta = 0.69 + 0.12\log\left(\frac{N}{R}\right)$, and assume that only the bandwidth is changed and other network parameters are unchanged, then the proportional parameter can be determined approximately as following

$$\frac{\bar{K}_{p2}}{\bar{K}_{p1}} \approx \frac{C_1^2 b_{01}}{C_2^2 b_{02}}$$

where $b_{0i} = \beta - 0.12\log C_i$, $(i = 1, 2)$. Let $b = K_p$, $a = Kp + K_I$, $K_I = K_1 K_p$, the marking probability will update as following

$$p(n) = p(n-1) + a(q_0 - q(n)) - b(q_0 - q(n-1)) \quad (10)$$

Let $T_p = 100ms$, $N = 20$, $q_0 = 1500$, according to the theorem, we can derive the relationship of the upper bound of K_p and the link bandwidth. The parameters a and b can be achieved further. To implement SPI, we should use a small data chain table to maintain the parameters a and b. For the SPI router, it can know its output bandwidth. According to the bandwidth it checks out data chain table, and it determines which group of the parameters should be used. The data chain table is shown in Tab.1.

Table1 Parameters of SPI

Bandwidth	$K_p(\times 10^{-5})$	$a(\times 10^{-5})$	$b(\times 10^{-5})$
100Mbps	0.736306621	0.1120	0.1090
200Mbps	0.201100314	0.0520	0.0500
300Mbps	0.094457961	0.0316	0.0300
400Mbps	0.055358088	0.0214	0.0200
500Mbps	0.036616373	0.0200	0.0187
600Mbps	0.026142733	0.0188	0.0176
700Mbps	0.019673896	0.0168	0.0157
800Mbps	0.015386593	0.0128	0.0118
900Mbps	0.012392074	0.0118	0.0109
1000Mbps	0.010213900	0.0108	0.0100

4. SIMULATION RESULTS

In designing SPI AQM, our main goals are to maximize link utilization, minimize queuing delays, reduce transient response time and achieve stable and robust operation. We implement SPI algorithm in ns2 and validate its performance by simulation. The simulation network topology is shown in Fig.2. There is only one bottleneck link lying between node A and node B. Its capacity is 1000 Mbps and the delay is 10ms; other links have 2400 Mbps and X1ms or X2ms delay, the size of all buffers is 3000 packets (the bandwidth-delay product is 12500 packets, 3000 is only 24 percents.). Queue A and Queue B are controlled by SPI and DropTail scheme, respectively.

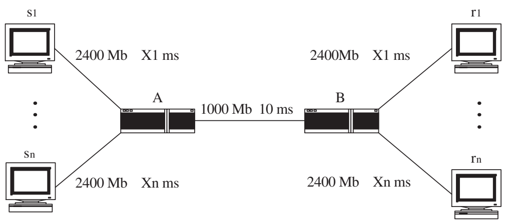

Fig. 2. Network topology

4.1 Experiment 1

In this experiment, we will evaluate the performance of SPI under dynamic traffic scenario using the network topology as shown in Fig.2. Initially at $t = 0s$, 40 FTP is established, 20 FTP are terminated at $t = 100s$, and another 100 FTP are established at $t = 200s$, all of the flows have the propagation round trip time of 100ms. The simulation sustains 300 seconds. The simulation result is shown in Fig.3. We can clearly see that SPI can track the reference queue length under dynamic traffic load, and the shorter response time for SPI is achieved.

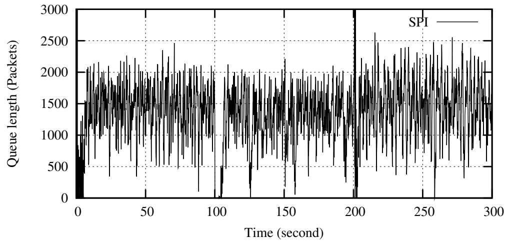

Fig. 3. Performance of SPI under dynamic traffic load

4.2 Experiment 2

In this section, we will show the merits of HSTCP/SPI. As illustrated above, HSTCP/DropTail has three serious limitations: RTT unfairness, slow convergence rate and TCP unfairness. We believe that the SPI AQM router can ameliorate these limitations. First we will test the RTT unfairness of HSTCP/SPI and HSTCP/DropTail using the simulation topology as shown in Fig.2. The propagation round trip time of the flow from $s2$ to $r2$ ($flow2$) is 50ms (X2=7.5ms). The propagation round trip time of the flow from $s1$ to $r1$ ($flow1$) varies from 100ms to 250ms. Other parameters of the topology are same as experiment 1. Simulation results are shown in Tab.2, which show that HSTCP/SPI shows such a little RTT unfairness which can be permitted in the Internet.

Table2 RTT unfairness test of DropTail and SPI

	1:2	1:3	1:4	1:5
DropTail	17.43:1	67.97:1	117.39:1	313.82:1
SPI	2.69:1	3.11:1	3.68:1	4.63:1

Next the convergence performance of HSTCP/SPI is considered. The topology is shown in Fig.2, set X1 and X2 equal to 20ms, thus $flow1$ and $flow2$ have the same propagation round trip time 100ms. Other parameters are the same as in experiment 1. We start $flow1$ at $t = 0ms$, $flow2$ starts at $100ms$, the congestion window size is shown in Fig.4. We can see that after about 100 seconds the two flows converge to fairness, however the convergence time of HSTCP/DropTail is about 550 seconds.

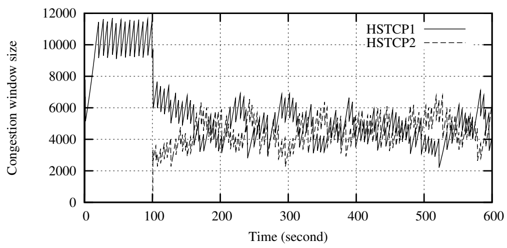

Fig. 4. Convergence test of SPI

TCP fairness is an important criterion to evaluate the performance of HSTCP. It means that the HSTCP should only make better use of the free available bandwidth, instead of stealing bandwidth from other flows. The aggressive behavior of HSTCP flows may severely degrade the performance of regular TCP flows whenever the network path is highly utilized [8]. We believe that the SPI AQM router can improve the TCP fairness of HSTCP. As an illustration, we conducted the following simulations. The network topology is shown in Fig.2. $flow1$ uses the

HSTCP agent, and *flow*2 uses the TCP Sack agent. The congestion window size is shown in Fig.5. We can see that when HSTCP and TCP Sack compete the available bandwidth, HSTCP doesn't steal bandwidth from TCP Sack, thus it has better TCP fairness. We also tested the cases where one and two HSTCP flows are competing with several regular TCP flows, which are illustrated in Tab.3. The simulation results are shown in Fig.6. The results show that HSTCP/SPI has a satisfactory TCP fairness performance.

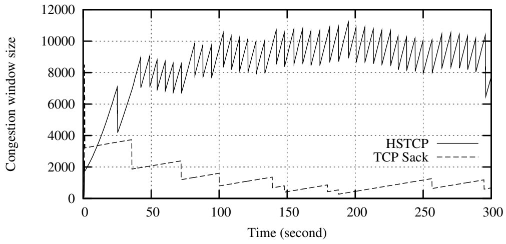

Fig. 5. TCP fairness test of SPI

Table 3 TCP fairness test of DropTail and SPI

Categories of simulations	A	B	C	D	E	F
Number of HSTCP	1	1	1	2	2	2
Number of TCP	4	6	8	4	6	8

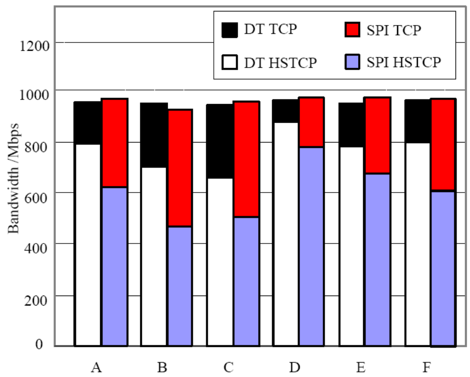

Fig. 6. TCP fairness test of HSTCP/DropTail and HSTCP/SPI

4.3 Experiment 3

The performance and robustness of the SPI scheme are explored with respect to different number of HSTCP connections, round trip time, link capacity and buffersize. We only consider two performance indexes, link utilization and packet loss ratio. For the robustness test of number of connections we conduct the simulation for 300s with the same setting as in experiment 1, except that the number of connections varies from 5 to 50. For the robustness text of other indexes, the propagation round trip time varies from 20ms to 400ms , the link capacity varies from 100Mbps to 1000Mbps and the buffersize varies from 1000 packets to 5000 packets. The throughput and packet loss ratio are plotted in Fig.7 and Fig.8 respectively.

From Fig.7, we can see that SPI can guarantee high link utilization. In the worst case, RTT is 400ms, the link utilization is higher than 85%. Fig.7(c) shows that the larger the round trip time is, the lower the link utilization will be. This is because we don't mainly consider the effect of the delay, and we think that other control methods,such as interior model control, can be used to alleviate the effect. Fig.8 shows that the packet loss ratio is low (in the worst case it is not higher than 0.14%). Thus, from the link utilization and packet loss ratio, we can say that SPI is robust to the number of connections, buffersize, round trip time and link capacities.

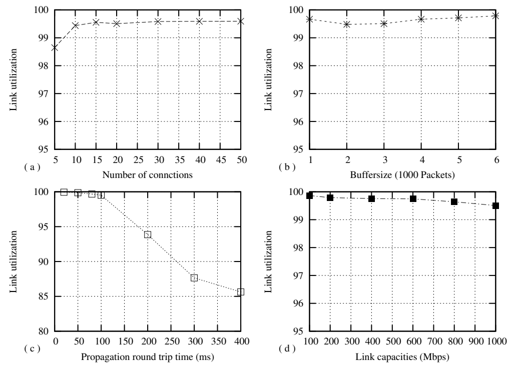

Fig. 7. Throughput performance test of SPI

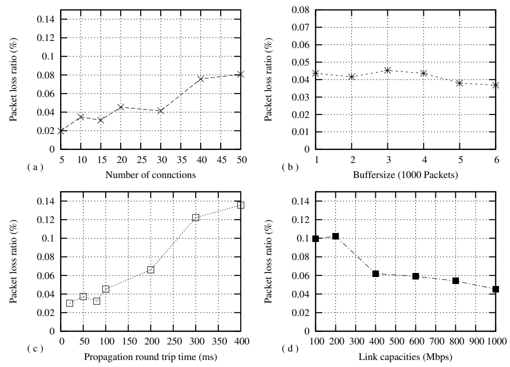

Fig. 8. Packet loss ratio test of SPI

5. CONCLUSION

A scalable PI (SPI) AQM for HSTCP is developed in this paper. Based on the transfer function of HSTCP, we have designed a PI controller, and the stability of the closed-loop system is proved using *Pontryagin* criterion. We also figure out the relation between the parameters of SPI and the stability of the system, and develop an adaptive design of the parameter. SPI only needs to maintain a small data chain table in the AQM router, thus SPI nearly doesn't heighten the burden of the router and will be easy to implement in the network. Through simulations we find that SPI can overcome the limitations of HSTCP/DropTail. SPI also has a good robustness performance. Of course, our work still have some limitations such as RTT is assumed to be known and constant to compute the parameters. How to estimate the dominated delay and number of users is a very interesting future work.

6. ACKNOWLEDGMENT

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Appendix A. PROOF OF THE THEOREM

Pontryagin Criterion: Consider the polynomial $H(s) = \sum_{l=0}^{L}\sum_{m=0}^{M} b_{lm} s^l (e^s)^m = 0$, with the principal term $b_{LM} s^L (e^s)^M$. Let $G(\omega), F(\omega)$ are the real and imaginary part of $H(j\omega)$, respectively. In order for all the roots of equation $H(s) = 0$ to have negative real parts, it is sufficient and necessary that 1) $F(\omega) = 0$ (or $G(\omega) = 0$) has $4kM + L$ roots in the interval $[-2k\pi + \varepsilon, 2k\pi + \varepsilon]$, where k is a positive integer and ε is an appropriate constant. 2) Each zero of $F(\omega) = 0$ (or $G(\omega) = 0$), to be denoted by ω_i,must satisfy $G(\omega_i)\dot{F}(\omega_i) > 0$(or $-\dot{G}(\omega_i)F(\omega_i) > 0$).

From the characteristic equation (9), we can get $L = 2, M = 1$. Let $F(\omega) = 0$, i.e.

$$\omega(\omega \sin\omega - a\tau\cos\omega) = 0 \qquad \text{(A.1)}$$

The roots of Eq.(11) include two parts, $\omega = 0$ and the roots of Eq.(12)

$$\omega \sin\omega - a\tau\cos\omega = 0 \qquad \text{(A.2)}$$

The intersection points of curves $\tan\omega$ and $\frac{a\tau}{\omega}$ are the roots of Eq.(12). It is obviously that the roots of Eq.(12) distribute periodically, for $\tan\omega$ is a period function. The distribution of these roots is shown in Fig.9. We choose $\varepsilon = \frac{\pi}{2}$.

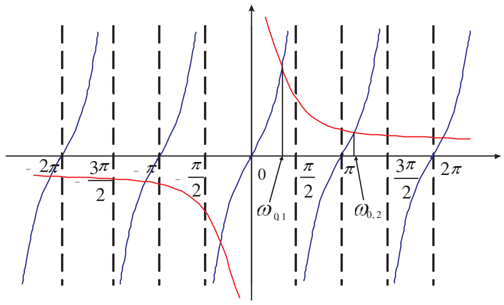

Fig. A.1. The location of roots of Eq.(12)

it can be known both $\tan\omega$ and $\frac{a\tau}{\omega}$ are odd functions, and the points of intersection of two curves are symmetrical with the origin. When $k = 1$, $\omega \in \left[-2\pi + \frac{\pi}{2}, 2\pi + \frac{\pi}{2}\right]$, and there are 5 intersection points. This is to say, Eq.(12) has 5 solutions and they are located in the interval $\left(-\frac{3\pi}{2}, -\pi\right), \left(-\frac{\pi}{2}, 0\right), \left(0, \frac{\pi}{2}\right), \left(\pi, \frac{3\pi}{2}\right), \left(2\pi, \frac{5\pi}{2}\right)$, respectively. Considering the zero solution , there are 6 roots. So $N(1) = 4kM + L|_{k=1} = 6$, and the condition(1) of the lemma is satisfied with $k = 1$. From the Fig.9, we can see that with the increase of k every 1, the number of the intersection points of two curves increases 4. That is there are $4kM + L$ roots in the interval $\left[-2\pi + \frac{\pi}{2}, 2\pi + \frac{\pi}{2}\right]$ for all k.

Define the set of roots of Eq.(12) as

$$S = \{\omega | \omega \sin\omega = a\tau\cos\omega\}$$

which is consisted of three parts denoted as S_0, S^-, S^+, i.e. $S = \{S_0, S^-, S^+\}$, where

$$S_0 = \{0\}$$
$$S^- = \left\{\omega | \omega \in S, \omega \in \left(-k\pi - \frac{\pi}{2}, k\pi\right)\right\}, k = 0, 1, 2, ...$$
$$S^+ = \left\{\omega | \omega \in S, \omega \in \left(k\pi, k\pi + \frac{\pi}{2}\right)\right\}, k = 0, 1, 2, ...$$

For it is hold $G(\omega)\dot{F}(\omega) = [-G(\omega)][-\dot{F}(\omega)]$, we let

$$G(\omega) = \omega^2\cos\omega + a\tau\omega\sin\omega - K\tau^2$$
$$F(\omega) = \omega^2\sin\omega - a\tau\omega\cos\omega$$

That does not affect the results of the following discussion. We have

$$\dot{F}(\omega) = (a\tau + 2)\omega\sin\omega + (\omega^2 + a\tau)\cos\omega$$

It is hold that $G(\omega)\dot{F}(\omega)|_{\omega=0} = ak\tau^3 > 0$ for $\omega \in S_0$. And the solution of zero satisfies the condition 2 of the criterion. For every element of set S can made Eq.(12) hold, we can get

$$\dot{F}(\omega) = (a^2\tau^2 + a\tau + \omega^2)\cos\omega \qquad (\omega \in S)$$

It is obviously that $G(\omega), \dot{F}(\omega)$ are even functions, and they are symmetrical with the imaginary axis. When we study the sign of $G(\omega)\cdot\dot{F}(\omega)$, we can consider the positive axis only, i.e. $\omega \in S^+$. We let $S^+ = \{S_o^+, S_e^+\}$, where

$$S_e^+ = \left\{ \omega | \omega \in S^+, \omega \in \left(k\pi, k\pi + \frac{\pi}{2} \right) \right\}, \; k = 0, 2, 4, ...$$
$$S_o^+ = \left\{ \omega | \omega \in S^+, \omega \in \left(k\pi, k\pi + \frac{\pi}{2} \right) \right\}, \; k = 1, 3, 5, ...$$

From Fig.9, on the positive axis, function $\frac{a\tau}{\omega}$ is a decreasing function for all $\omega \in S^+$; however, function $\tan \omega$ is an increasing function in every period. There exist two points of intersection in the interval $[2k\pi, 2(k+1)\pi], k = 0, 1, 2, ...$, denoted as $\omega_{k,1} \in S_e^+$, $\omega_{k,2} \in S_0^+$, respectively. And the following relations are held.

$$\begin{aligned} \cos \omega_{k,1} &> 0, \forall k \\ \cos \omega_{k,2} &< 0, \forall k \\ \cos \omega_{k+1,1} &> \cos \omega_{k,1}, \forall k \\ \omega_{k+1,1} &> \omega_{k,1} > 0, \forall k \\ \omega_{k+1,2} &> \omega_{k,2} > 0, \forall k \end{aligned}$$

Note that $G(\omega)\dot{F}(\omega) = ((\omega^2 + a^2\tau^2)\cos\omega - K\tau^2)(a^2\tau^2 + a\tau + \omega^2)\cos\omega \;\; \forall \omega \in S$. For all $\omega_{k,1} \in S_e^+$, $G(\omega)$ is an increasing function for the subscript k. Furthermore, according to the theorem, we know $K\tau^2 < (\omega_{0,1}^2 + a^2\tau^2)\cos\omega_{0,1}$, we can draw the conclusion that $G(\omega) > 0$ for all $\omega_{k,1} \in S_e^+$, and $G(\omega)\dot{F}(\omega) > 0$. For $\omega_{k,2} \in S_0^+$, it is easy to know that $G(\omega) < 0, \dot{F}(\omega) < 0$, we can get the same conclusion that $G(\omega)\dot{F}(\omega) > 0$ for $\omega_{k,2} \in S_0^+$.

Based on the above analysis, it holds that $G(\omega)\dot{F}(\omega) > 0$ for all $\omega \in S^+$. For the symmetry, we can draw the final conclusion that $G(\omega)\dot{F}(\omega) > 0$ holds for all $\omega \in S$. According to the criterion, we can know the considered system is stable.