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Stability Analysis of Fuzzy Systems: membership-shape and polynomial approaches

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Abstract: This paper outlines two contributions to stability analysis of fuzzy systems: knowledge of the membership function shape (actually, constraints on the membership function values) and polynomial approaches. Both ideas reduce the conservativeness in the stability analysis of a nonlinear system when expressed as a fuzzy model by using membership shape information and by allowing a more general class of "fuzzy" systems than the widely-used Takagi-Sugeno one with linear consequents. In this way, the gap between fuzzy and nonlinear control gets smaller.

Keywords: fuzzy control, stability analysis, nonlinear control, polynomial systems, linear matrix inequalities, sum-of-squares

1. INTRODUCTION

Takagi-Sugeno (TS) (Takagi and Sugeno (1985)) fuzzy systems may be used as universal approximations of continuous nonlinear dynamic systems, at least in a compact region including the origin; nonlinear (fuzzy) controllers may be designed on them. Hence, fuzzy control is a particular nonlinear control technique. However, there are some sources of conservativeness responsible for a gap between what is achievable with current fuzzy techniques and what could in theory be achieved with a clever use of nonlinear control techniques Khalil (1996); this is the fuzzy-nonlinear gap (Sala (2007)).

Currently, most of the significant results use a linear matrix inequality (LMI) approach, which reached maturity with (Tanaka and Wang (2001)). Basically, the most frequently considered setting is the so called parallel-distributed-compensation (PDC) in which a fuzzy TS controller shares the membership functions with the plant to be controlled. Research has focused in conceiving less conservative LMI conditions, such as the ones in (Kim and Lee (2000); Liu and Zhang (2003)). The conditions in those papers are particular cases of the family of assymptotically necessary and sufficient ones in Sala and Ariño (2007a).

This paper discusses two topics which allow for less conservative stability analysis for nonlinear systems expressed as fuzzy ones:

(a) knowledge of the actual "shape" of the membership functions (actually, knowledge of constraints on the membership function values) and
(b) a polynomial approach allowing more general classes of "fuzzy" models.

Regarding the first issue, it is important to remark that the LMI conditions in the above works do not depend on the "shape" of the membership functions. The values of the membership are needed when implementing a fuzzy controller, but LMI conditions in controller design are usually stated as valid for *any* underlying fuzzy partition. As a result, the conditions may be conservative: a particular nonlinear system (modelled as a fuzzy TS one (Tanaka and Wang (2001))) may be stable, but the LMI conditions may fail to pinpoint the fact. For instance, the system $\dot{x} = \mu_1(z) \cdot x + (1 - \mu_1(z)) \cdot (-x)$ cannot be proved stable for an arbitrary μ_1, $0 \leq \mu_1(z) \leq 1$ (it is unstable for $\mu_1(z) = 1$). However, it *is* stable for, say, $\mu_1 = 0.2 + 0.2sin(x)$ as $\dot{x} = (-1+2\mu_1)x$ is, trivially, an exponentially stable first-order nonlinear system when $\mu_1 \leq b < 0.5$, $b \in \mathbb{R}$.

In summary, there is still some conservativeness to be lifted if knowledge on the shape of the membership functions for a particular TS model is introduced in the LMI framework. Some shape-dependent conditions for PDC regulators appear in (Sala and Ariño (2007b); Ariño and Sala (2007b)), and for non-PDC cases (uncertain memberships) in (Lam and Leung (2005); Sala and Ariño (2008)); however, the conditions in those papers consider inequalities restricting the shape of the memberships which *cannot* consider the relationship between the memberships and the state variables (*i.e.*, nonlinearity!). A preliminary idea on this topic will be presented in this paper.

Regarding the second topic of the discussion in this paper, recent advances allow a generalisation of fuzzy models to polynomial ones via sum-of-squares approaches. This paper discusses perspectives on them and some examples.

The structure of the paper is as follows: next section presents well-known preliminary ideas and notation, section 3 outlines the possibilities of using the knowledge on the membership function shape to relax stability conditions, section 4 discusses how Takagi-Sugeno systems may be extended to a polynomial approach via sum-of-squares tools, and a conclusion section closes the paper. Brief academic examples are provided in sections 3 and 4 to illustrate the presented ideas.

2. PRELIMINARIES

In many situations, Lyapunov-based conditions for stability or performance of a fuzzy control system may be expressed in the form

$$\Xi(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z(t))\mu_j(z(t))x(t)^T Q_{ij} x(t) > 0 \quad \forall\, x \neq 0 \tag{1}$$

subject to symmetry ($Q_{ij} = Q_{ij}^T$, without loss of generality) and the fuzzy partition condition $\sum_{i=1}^r \mu_i = 1$, which implies:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j = 1 \tag{2}$$

A typical example of the use of condition (1) is proving quadratic stability of the fuzzy system

$$\dot{x} = \sum_{i=1}^{r} \mu_i (A_i x + B_i u) \tag{3}$$

with a fuzzy PDC state-feedback controller

$$u = -\sum_{i=1}^{r} \mu_i K_i x \tag{4}$$

yielding a closed loop

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j (A_i - B_i K_j) x$$

Introducing the notation $G_{ij} = A_i - B_i K_j$, $\Lambda_{ij} = \frac{1}{2}(G_{ij} + G_{ji})$, matrices Q_{ij} in (1) are (Kim and Lee (2000)):

$$Q_{ii} = -G_{ii}^T P - P G_{ii} \tag{5}$$

$$Q_{ij} = -\Lambda_{ij}^T P - P \Lambda_{ij} \tag{6}$$

where $P > 0$ is a symmetric matrix, which defines a Lyapunov function $V(x) = x^T P x$, to be obtained via LMI algorithms (Boyd et al. (1994); Tanaka and Wang (2001)). If K_j is also to be designed, introducing a decay-rate performance requirement $\|x\| \leq \sigma e^{-\alpha t}$,

$$Q_{ij} = -(A_i X + X A_i{}^T - B_i M_j - M_j^T B_i^T + 2\alpha X) \tag{7}$$

where $X > 0$ and $M_i = K_i X$ are LMI decision variables (for details, see (Tanaka and Wang (2001); Kim and Lee (2000))).

Another example of performance-related condition uses [1] (Tuan et al. (2001)) as Q_{ij} the matrix:

$$\begin{pmatrix} PA_i^T + R_j^T B_{2i}^T + A_i P + B_{2i} R_j & B_{1i} & PC_i^T + R_j^T D_{12i}^T \\ B_{1i}^T & -\gamma I & D_{11i}^T \\ C_i P + D_{12i} R_j & D_{11i} & -\gamma I \end{pmatrix} \tag{8}$$

[1] In this case, x in (1) does not represent the state vector; it must be understood as a vector of artificial variables arising from Schur complements (Boyd et al. (1994)).

in order to prove that there exists a stabilising state-feedback controller such that the H_∞ norm (i.e., $\mathcal{L}_2$ to $\mathcal{L}_2$ induced norm) of a TS fuzzy system given by:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z)(A_i x + B_{1i} v + B_{2i} u) \tag{9}$$

$$y = \sum_{i=1}^{r} \mu_i(z)(C_i x + D_{11i} v + D_{12i} u) \tag{10}$$

is lower than γ.

Positiveness conditions. Of course, requiring $Q_{ij} > 0$ is a trivial sufficient condition for positiveness of (1), but much less conservative conditions appear in literature. A widely-used one is Theorem 2 in (Liu and Zhang (2003)), which is mainly based on the scheme in (Kim and Lee (2000)).

Theorem 1. Expression (1) under fuzzy partition condition holds if there exist matrices $X_{ij} = X_{ji}^T$ such that:

$$X_{ii} \leq Q_{ii} \tag{11}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} \;\; i \neq j \tag{12}$$

$$\mathbf{X} = \begin{pmatrix} X_{11} & \ldots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \ldots & X_{rr} \end{pmatrix} > 0 \tag{13}$$

The above conditions have been further improved, at the expense of higher computational cost. Consider a multi-dimensional index variable $\mathbf{i} \in \{1, \ldots, r\}^n$ where r is the number of rules and n is an arbitrary complexity parameter. Denote by $\mathcal{P}(\mathbf{i})$ all the permutations if $\mathbf{i}$. Then, the results in Liu and Zhang (2003); Fang et al. (2006) are particular cases of the ones in (Sala and Ariño (2007a)):

Theorem 2. Expression (1) under fuzzy partition condition holds if there exists a multi-dimensional arrangement of matrices (tensors) fulfilling, for all $\mathbf{i}$:

$$\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} Q_{j_1 j_2} > \sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} \frac{1}{2}(X_{\mathbf{j}} + X_{\mathbf{j}}^T) \tag{14}$$

and the inequality (with complexity $n-2$):

$$\sum_{\mathbf{k}\in B_{n-2}} \mu_{\mathbf{k}} \xi^T \begin{pmatrix} X_{(\mathbf{k},1,1)} & \cdots & X_{(\mathbf{k},1,r)} \\ \vdots & \ddots & \vdots \\ X_{(\mathbf{k},r,1)} & \cdots & X_{(\mathbf{k},r,r)} \end{pmatrix} \xi > 0 \quad \text{for } \xi \neq 0 \tag{15}$$

In a suitable recursive framework, it can be proved that the above conditions become *necessary and sufficient* with $n \to \infty$, and establish some "tolerance" parameter for finite n (Sala and Ariño (2007a)).

The discussed conditions are the state-of-the-art in fuzzy systems, but they are membership-independent (valid for *any* shape) and Q_{ij} stem from TS models with *linear* consequents. These restrictions may be partially lifted in stability analysis of fuzzy systems, as discussed in the sequel.

3. KNOWLEDGE OF MEMBERSHIP SHAPE

This section outlines how the knowledge of membership-shape information in the form of "when state x fulfills some

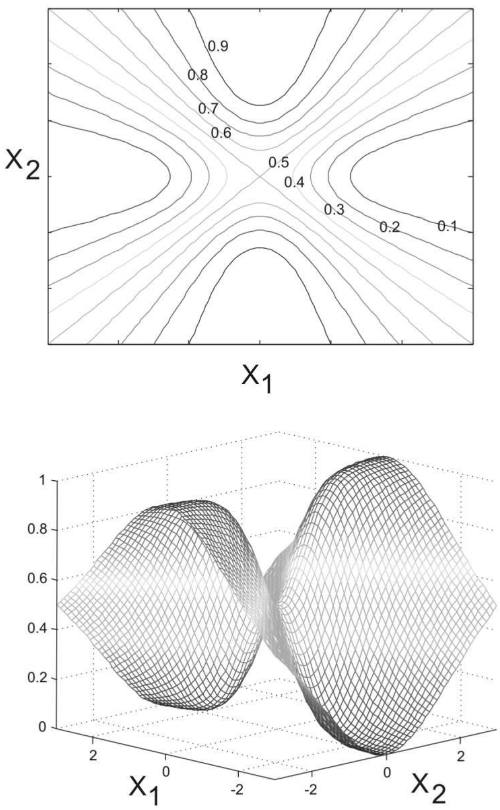

Fig. 1. Membership function μ_1 for the example in the main text.

quadratic inequality, it is known that the membership functions fulfill another inequality" may be used to prove stability of some TS systems for which the standard approach fails.

3.1 Main Result

The following example shows how to incorporate some information about the actual shape of the membership functions.

Example. Consider the system $\dot{x} = \sum_{i=1}^{2} \mu_i(x) A_i x$ with

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0.5 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & -1 \end{pmatrix}$$

with membership functions given by $\mu_1(x) = x_1^2/(x_1^2+x_2^2)*(1-exp(-x_1^2-x_2^2))+0.5*exp(-x_1^2-x_2^2)$, $\mu_2(x) = 1-\mu_1(x)$ (μ_1 is depicted in figure 1).

Note that both A_i are unstable, hence the usual fuzzy control approach of testing feasibility of $A_i^T P + PA_i < 0$ fails.

However, the nonlinear system is stable, and that fact will be proved below.

Indeed, note that the diagonals are the level set for $\mu = 0.5$. In fact, we can set up a cover with the regions:

$$R_1 : x_1^2 - x_2^2 \geq 0 \quad R_2 : -x_1^2 + x_2^2 \geq 0$$

and $\mu_1 \leq \mu_2$ in R_1, $\mu_1 \geq \mu_2$ in R_2.

Let us define $Q_i = A_i^T P + PA_i$ (the well-known quadratic stability condition) and check the condition

$$x^T(\mu_1 Q_1 + \mu_2 Q_2)x > 0 \qquad (16)$$

on R_1.

In that case, we can add $(\mu_1-\mu_2)x^T T_1 x$ (which is negative) for any $T_1 > 0$.

The result: if there exist $T_1 > 0$ so that for all $x \in R_1$ the expression $x^T(\mu_1(Q_1+T_1)+\mu_2(Q_2-T_1))x > 0$ holds, then (16) is positive on R_1.

To further reduce conservativeness, now we apply the S-procedure, for the region R_1, given by $x^T B_1 x \geq 0$ with

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The result is the assertion below:

if there exists $T_1 \geq 0$ and $\tau_1 \geq 0$ so that $\mu_1(Q_1 + T_1) + \mu_2(Q_2 - T_1) - \tau_1 B_1 > 0$ then (16) holds on R_1.

Similarly, with R_2 (with an associated shape matrix $B_2 = -B_1$) we get the assertion:

if there exists $T_2 \geq 0$ and $\tau_2 \geq 0$ so that $\mu_1(Q_1 - T_2) + \mu_2(Q_2 + T_2) + \tau_2 B_1 > 0$ then (16) holds on R_2.

So we get the LMIs in decision variables P (appearing in Q_1 and Q_2), T_1, T_2, τ_1, τ_2:

$$Q_1 + T_1 - \tau_1 B_1 > 0 \qquad (17)$$

$$Q_2 - T_1 - \tau_1 B_1 > 0 \qquad (18)$$

$$Q_1 - T_2 + \tau_2 B_1 > 0 \qquad (19)$$

$$Q_2 + T_2 + \tau_2 B_1 > 0 \qquad (20)$$

$$T_1 > 0,\ T_2 > 0,\ \tau_1 > 0,\ \tau_2 > 0 \qquad (21)$$

which yield feasible, proving the existence of a quadratic Lyapunov function for this particular nonlinear system even if there is no common Lyapunov function for the family of fuzzy systems with vertex models A_1 and A_2.

In summary, the example shows that quadratic stability of nonlinear systems may be tackled via a fuzzy approach if the state-space is divided in *sectors* for which some bounds on the membership can be computed. In a general case, there might be more than 2 sectors.

Based on the ideas in the example, the following result can be proved.

Theorem 3. Consider a subset $\Omega \in \mathbb{R}^n$ so that there exist known matrices C and B such that

$$x^T C x \geq 0,\ \mu^T(x) B \mu(x) \geq 0 \quad \forall x \in \Omega$$

Then, the conditions:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j x^T Q'_{ij} x \geq 0 \quad \forall x \in \Omega \qquad (22)$$

$$Q'_{ij} = Q_{ij} - \tau_{ij} C - \beta_{ij} R \qquad (23)$$

$$R - \nu C \geq 0 \qquad (24)$$

$$\tau_{ij} \geq 0,\ \nu \geq 0 \qquad (25)$$

are a sufficient condition for:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j x^T Q_{ij} x \geq 0 \quad \forall x \in \Omega \tag{26}$$

Proof: Indeed, consider any $x \in \Omega$, then

$$x^T R x \geq \nu x^T C x \geq 0$$

Hence,

$$x^T R x \mu^T B \mu = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\beta_{ij} x^T R x \geq 0$$

So, for any $x \in \Omega$:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j x^T Q_{ij} x \geq \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j x^T (Q_{ij} - \beta_{ij} R) x$$

and, furthermore, as $x^T C x \geq 0$,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j x^T Q_{ij} x \geq \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j x^T (Q_{ij} - \beta_{ij} R - \tau C) x$$

hence if (23) holds, (26) does as well.

The above result can be applied if a cover $\{\Omega_1, \ldots, \Omega_p\}$ of the state space (or a subset of it including the origin in which Lyapunov functions are searched) is known so that the conditions in the above theorem hold for some known C_i, B_i.

Note also that linear restrictions can be made quadratic by multiplying by $\sum_i \mu_i$ in order to apply the above theorem.

Remark: The division of the state space in zones reminds of the ideas of piecewise Lyapunov functions (Johansson (1999)); the Lyapunov function is quadratic in the approach in this paper. An interesting idea could be combining both approaches.

4. POLYNOMIAL FUZZY SYSTEMS: PRELIMINARY IDEAS

The second contribution of this paper towards extending the domain of applicability of fuzzy ideas refers to polynomial systems. The ideas presented here are, indeed, preliminary work-in-progress.

This section presents how fuzzy systems can be embedded into the polynomial system class, for which sum-of-squares sufficient stability conditions can be stated, and solved with recent tools, such as the freely available SOSTOOLS package (Prajna et al. (2004)).

4.1 Polynomial-in-membership Fuzzy Systems

A polynomial-in-membership fuzzy system is a natural extension of the TS model to the form:

$$\dot{x} = p_1(\mu)(A_1 x + B_1 u) + p_2(\mu)(A_2 x + B_2 u) + \ldots \tag{27}$$

where p_i are polynomials in the membership functions (μ is a vector $(\mu_1, \mu_2, \ldots, \mu_p)$). Multiplying by $\sum \mu_i$, p_i may be assumed to be homogeneous polynomials of any desired degree.

As an example, consider the nonlinear system

$$\dot{x_1} = (\sin z) x_1 + (\sin z)^2 x_2 + u \tag{28}$$

$$\dot{x_2} = x_1 - x_2 \tag{29}$$

with $z \in [-\pi/2, \pi/2]$.

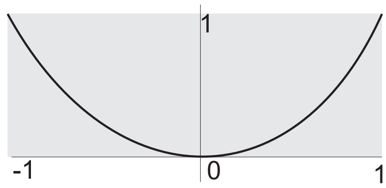

Fig. 2. Range of first-row elements of the TS state matrix: [solid line] polynomial-in-membership representation; [shaded area] independent nonlinearity, standard sector-nonlinearity approach.

If we fuzzify $\sin z = \mu_1 * 1 + \mu_2 * (-1)$, then $(\sin z)^2 = \mu_1^2 - 2\mu_1\mu_2 + \mu_2^2$. Hence, by replacing the expressions in the system's equations, we obtain a polynomial-in-membership TS fuzzy system:

$$\dot{x_1} = (\mu_1 + \mu_2) * (\mu_1 * 1 + \mu_2 * (-1)) x_1 + \tag{30}$$

$$+\mu_1^2 - 2\mu_1\mu_2 + \mu_2^2 x_2 + u \tag{31}$$

$$\dot{x_2} = x_1 - x_2 \tag{32}$$

which results:

$$\dot{x_1} = (\mu_1^2 - \mu_2^2) x_1 + (\mu_1^2 - 2\mu_1\mu_2 + \mu_2^2) x_2 + u \tag{33}$$

$$\dot{x_2} = x_1 - x_2 \tag{34}$$

so that the system can be expressed as:

$$\dot{x} = \sum_{i=1}^{2}\sum_{j=1}^{2}\mu_i\mu_j (A_{ij} x + B u) \tag{35}$$

with $B = (1\ 0)^T$ and

$$A_{11} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; A_{12} = A_{21} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \; A_{22} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and so, stability analysis (and control design) can be carried out less conservatively than considering the nonlinearities $\sin z$ and $(\sin z)^2$ as unrelated ones. Indeed, in the latter case, following standard sector-nonlinearity methodologies (Tanaka et al. (2001)), the system would be expressed as a convex combination of:

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Figure 2 pictorially represents the difference between the standard approach and the polynomial-in-membership approach.

Of course, polynomial-in-membership controllers may be easily designed for polynomial-in-membership plants, using the results in (Sala and Ariño (2007a)). The polynomial order of the controller may not be the same of that of the plant (for instance, (Ariño and Sala (2007a)) proposes higher-order polynomial controllers for a standard Takagi-Sugeno plant). Shape-dependent conditions on this class of systems are considered in Sala and Ariño (2008).

4.2 Polynomial-in-state fuzzy systems

A different generalisation of TS systems involves expressions such as:

$$\dot{x}_j = \sum_{i=1}^{p}\mu_i p_{ij}(x, u) \quad j = 1, \ldots, n \tag{36}$$

where p_{ij} are polynomials in the state and input variables. These are TS models with nonlinear consequent (polynomial, in particular).

For instance, a nonlinear system

$$\dot{x} = \begin{pmatrix} -x_1^3 - (0.05 + 0.95(\sin x_1)^2)x_1 x_3^2 \\ -(1 + (\sin x_1)^2) * x_2 - x_1 x_2 \\ -x_3 + 3 * x_1^2 * x_3 - 3 * x_3 \end{pmatrix} \quad (37)$$

may be modelled as:

$$\dot{x} = \begin{pmatrix} -x_1^3 - \mu_1 x_1 x_3^2 - 0.05\mu_2 x_1 * x_3^2 \\ -\mu_1 * x_2 - x_1 x_2 + \mu_2 * (-2x_2) \\ -x_3 + 3 * x_1^2 * x_3 - 3 * x_3 \end{pmatrix} \quad (38)$$

where the sector-nonlinearity methodology is used to model only the non-polynomial functions of the state (in this case, $(\sin x_1)^2$ expressed as an interpolation between 0 and 1).

If the sector-nonlinearity approach is used, then perhaps the membership values have been obtained assuming that $x \in \Omega$ where Ω is a compact set containing the origin. Then, if a superset of Ω may be expressed as:

$$g(x) > 0$$

where g is a vector-valued polynomial constraint, such constraints can be also used to relax the requirements of Lyapunov decrescence only in Ω.

4.3 Full-polynomial fuzzy systems

The most general expression would be an expression which is polynomial in the memberships and polynomial in the states and inputs.

$$\dot{x}_j = p_j(\mu, x, u) \quad j = 1, \ldots, n \forall x \in \Omega \quad (39)$$

For instance, a nonlinear system

$$\dot{x} = \begin{pmatrix} -x_1^3 - (0.05 + 0.95(\sin x_1)^2)x_1 x_3^2 \\ -(1 + (\sin x_1)^2) * x_2 - x_1 x_2 \\ -x_3 + 3 * x_1^2 * x_3 - 3(\sin x_1) * x_3 \end{pmatrix} \quad (40)$$

could be modelled as a full-polynomial fuzzy system combining both of the previous approaches (there appear polynomial expressions of the state and of a non-polynomial nonlinearity, the latter modelled as a fuzzy expression interpolating between -1 and 1). Details are straightforward, omitted for brevity.

4.4 summary: a polynomial view of fuzzy systems

The previous ideas show that fuzzy systems may be considered a particular case of polynomial ones, in the sense that fuzzy systems are polynomial systems, i.e., derivatives of state variables are expressed as polynomials in two sets of variables:

- State variables
- Membership functions, which may be themselves a nonlinear (even non-polynomial) function of state variables and other scheduling inputs.

Additional information on the membership functions is that they are positive and all of them add one, as usual, hence there is no loss of generality in assuming that the polynomial fuzzy systems are homogeneous in the membership functions.

In many cases, fuzzy systems are obtained from nonlinear systems from which a sector condition holds, at least locally. In that case, such validity domain should be expressed by polynomial constraints in the state variables in order to overcome some conservativeness (local approach, see below).

4.5 Available tools

Polynomial-in-membership systems: may be approached easily via the Polya-related constructs in (Sala and Ariño (2007a)). Indeed, its application is evident once they are expressed as homogeneous sums:

$$\dot{x} = \sum_{\mathbf{i} \in \mathcal{I}_n} \mu_{\mathbf{i}} (A'_{\mathbf{i}} x + B'_{\mathbf{i}} u) \quad (41)$$

Most, if not all, of the fuzzy literature in analysis and control design can be applied to the above systems with minimal modifications.

Poynomial-in-state and full-polynomial systems: in this case, the tools for polynomial systems (sum of squares) may be applied (Prajna et al. (2004)). Indeed, polynomial fuzzy systems are a particular class of polynomial systems: they are those homogeneous in the scheduling variables μ. However, the ideas in SOSTOOLS apply with very small modifications, as shown in the example below.

Remark on control design: Transforming the resulting expressions to LMI in the second case above is not possible in a general case if both Lyapunov functions and feedback gains are considered decision variables. Indeed, the derivative of the Lyapunov function is a polynomial in $(x, \dot{x})$, and if $\dot{x}$ includes decision variables, products of them appear so the problem is no longer a LMI one.

Iterative LMI approaches might be the only option with current tools, unless specific structures appear in the system model or conservative steps are taken. Only the particular case of polynomial-in-membership systems with linear consequents may be easily handled. These ideas are a matter of current research.

4.6 Example of the polynomial approach

Consider the system (38), and write it as:

$$\dot{x} = \begin{pmatrix} -(\psi_1^2 + \psi_2^2)x_1^3 - \psi_1^2 x_1 x_3^2 - 0.05\psi_2^2 x_1 * x_3^2 \\ -\psi_1^2 * x_2 - (\psi_1^2 + \psi_2^2)x_1 x_2 + \psi_2^2 * (-2x_2) \\ (\psi_1^2 + \psi_2^2) * (-x_3 + 3 * x_1^2 * x_3 - 3 * x_3) \end{pmatrix} \quad (42)$$

where $\psi_i^2 = \mu_i$ and the add-1 condition has been used considering $\psi_1^2 + \psi_2^2 = 1$ in order to convert the system equations into an homogeneous polynomial in the memberships.

Consider now a Lyapunov function $V(x)$ in the form of a degree 4 polynomial in the state variables x_1, x_2, x_3.

Then, in a sum of squares programming package the positiveness of $V(x)$ is set as requiring $V(x)$ to be SOS (SOS is a sufficient conditions for positiveness). The negativeness of $\frac{dV}{dt}$ is set as requiring -$\frac{dV}{dt}$ to be SOS.

Note that the derivative of the Lyapunov function will then be non-positive for any value of ψ_i, i.e., for any non-negative μ_i. As the fuzzy system is an homogeneous polynomial in μ_i, being non-positive in the hyperplane $\sum_i \mu_i = 1$ is equivalent to being non-positive in the first quadrant (indeed an homogeneous polynomial of degree n verifies, for any $\lambda > 0$, $f(\lambda\mu) = \lambda^n f(\mu)$ hence the signs do not change in the first quadrant).

Using SOSTOOLS and SeDuMi, the solver finds, in less than half a second, a 4th-degree Lyapunov function given by (terms lower than 10^{-7} set to zero):

$$V(x) = 8.3931 * x_1^2 + 6.1584 * x_2^2 + 2.0284 * x_3^2 + 12.539 * x_1^4 + \\ + 0.24121e - 2 * x_3^4 + 12.074 * x_1^2 * x_2^2 + 1.5372 * x_1^2 * x_3^2 + \\ + 0.15023 * x_2^2 * x_3^2$$

which is, evidently, SOS and whose time derivative with changed sign is also SOS.

Luckily, the proposed system is stable in all state space. If the search for a Lyapunov function had not been successful, there would have been two options:

- Searching for a higher-degree global Lyapunov function (this approach may quickly exhaust computational resources)
- Pursuing a *local approach* in a region around the origin[2] described by $g(x) \geq 0$. The idea would stem from Karush-Kuhn-Tucker (*Positivstellensatz*) (Prajna et al. (2004)) argumentations, searching for vectors of SOS polynomials τ_1, τ_2 such that $V(x) - \tau_1 g(x)$ and $-dV/dx - \tau_2 g(x)$ are SOS. Indeed, $\{V(x) - \tau_1 g(x) > 0,\ \tau_1 \geq 0\}$ entails $\{V(x) > 0\ \forall\ x, g(x) \geq 0\}$ and similar arguments apply to the derivative equation. τ_1 and τ_2 have the role of (generalized) Lagrange multipliers.

As previously considered, the locality idea may be quite an interesting one for fuzzy systems in which, usually, sector-nonlinearity modelling techniques are locally applied.

5. CONCLUSIONS

This paper has outlined a couple of ideas which contribute to bridge the gap between fuzzy systems and nonlinear systems, at least regarding stability analysis: on one hand, knowledge of the actual membership function shape (which is available in nonlinear control) may reduce conservativeness; on the other hand, the "local" models need not be linear and they may be expanded to polynomial ones, even resorting to polynomial-in-membership fuzzy systems (when polynomial expressions of non-polynomial nonlinearities appear). In this way, fuzzy systems may be considered a particular case of polynomial ones.

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[2] Indeed, if the linearised system is stable, a local quadratic Lyapunov function exists.