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Multiple-Level Quantized Innovation Kalman Filter

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Abstract: In this paper, we study a general multiple-level quantized innovation Kalman filter (MLQ-KF) for estimation of linear dynamic stochastic systems. First, given a multi-level quantization of innovation, we derive the corresponding MMSE filter in terms of the given quantization levels under the assumption that the innovation is approximately Gaussian. By optimizing the filter with respect to the quantization levels, we obtain an optimal quantization scheme and the corresponding optimal MLQ-KF. The optimal filter is given in terms of a simple Riccati difference equation as in the standard Kalman filter. For the case of 1-bit transmission, our proposed optimal filter gives a better performance than the sign-of-innovation filter (SOI-KF) Ribeiro et al. [2006]. The convergence of the MLQ-KF to the standard Kalman filter is established.

1. INTRODUCTION

Quantization has been well studied in digital signal processing and control where a signal with continuous values is quantized due to a finite word-length of microprocessor Williamson [1991]. In most of existing works, uniform quantization is adopted and the corresponding quantization error is either neglected or considered as an additive white noise when discussing performance problems in control and filtering.

Recently, the low cost and low power consumption wireless sensor networks(WSNs) have attracted significant interests due to their potentials in military surveillance, environmental monitoring, health care, home and other commercial applications Akyildiz et al. [2002b]. Important constraints in WSNs are their low-quality sensors, limited energy and bandwidth Akyildiz et al. [2002a,b]. Many researchers are currently engaged in developing energy-efficient algorithms for network coverage Cardei and Wu [2004], Krasnopeev et al. [2005], decentralized detection Xiao and Luo [2005] and estimation Wong and Brockett [1997], Reibeiro and Giannakis [2006a,b], Luo [2005] by utilizing the quantized messages from sensors. To minimize the communication cost, only limited information can be transmitted through networks, harsh quantization is usually needed. In this situation, better and more efficient quantization schemes than the uniform quantization are to be sought and the effects of the quantization error are to be evaluated.

In the aforementioned references, quantization is carried out for sensor's observations. A larger quantization noise will be generated if the observed values are large, which results in larger information loss and leads to a lower estimation accuracy. An interesting distributed estimation approach based on the sign of innovation (SOI) has been developed for dynamic stochastic systems in Ribeiro et al. [2006] where only transmission of innovation of a single bit is required. However, a small innovation is quantized to 1 or -1, which may give rise to a large estimation error.

In this paper, we consider a general multiple-level quantized innovation Kalman filter for linear discrete-time stochastic systems. By extending the SOI-KF, we shall develop a very general multi-level quantized innovation filtering scheme. For a given set of quantization levels, we first derive an MMSE filter under the assumption that the innovation is approximately Gaussian. The filter is further optimized with respect to the quantization levels to give an optimal MLQ-KF. The solution to the optimal filter is given in terms of a simple Riccati recursion as in the standard Kalman filter. Our result shows that the quantized innovation filter of 2-bit gives a filtering performance close to the standard Kalman filter. For 1-bit transmission, our proposed filter gives a better performance than the sign-of-innovation filter in Ribeiro et al. [2006]. Convergence of the MLQ-KF to the standard Kalman filter when the number of quantization levels goes to ∞ is established. Simulations demonstrate that the proposed filter gives a better performance than the SOI-KF when the innovation is quantized to 1 bit and a close to the standard Kalman filter performance when the innovation is quantized into 2 bits.

The rest of paper is organized as follows. Problem formulation is delineated in Section II. State estimation based on the MLQ-KF is considered for a scalar measurement model in Section III. In this section, we first consider the case of quantized innovation of 1-bit, then the result is extended to the general multi-level quantization case. Performance analysis on the MLQ-KF is given in Section IV. Simulation studies are carried out in Section V. Some conclusions are drawn in Section VI.

2. PROBLEM FORMULATION

Consider the discrete-time stochastic system:

$$\mathbf{x}(n) = \mathbf{A}(n)\mathbf{x}(n-1) + \mathbf{w}(n) \tag{1}$$

$$y(n) = \mathbf{h}^T(n)\mathbf{x}(n) + v(n) \tag{2}$$

where $\mathbf{x}(n) \in \mathbb{R}^p$ is the state, $y(n) \in \mathbb{R}$ is the observation, and $\mathbf{w}(n) \in \mathbb{R}^p$ and $v(n) \in \mathbb{R}$ are Gaussian white noises with zero means and variances $\mathbf{W}(n)$ and $\sigma_v^2(n)$, respectively. $\mathbf{A}(n) \in \mathbb{R}^{p\times p}$ and $\mathbf{h}(n) \in \mathbb{R}^n$ are respectively bounded time-varying matrix and vector. Note that we shall consider the scalar measurement case first which will then be extended to the case of vector measurement.

We consider the sensor network configuration where the estimation centre has sufficient power to broadcast its predicted output and the corresponding prediction error covariance to its sensors. The sensors have limited power and hence their transmission of information should be limited. Here, we assume that the energy cost for receiving a message is much lower than that of transmitting the message. Also, in the case of large sensor number, due to the limited bandwidth, only limited number of bits can be received by the estimation centre. We note that it is more efficient to transmit innovation than the raw sensor measurement. That is, for a given estimation accuracy, a much lower number of bits is required to be transmitted for the innovation than the measurement. However, given the bit number (number of quantization levels), what is the optimal quantization that will give rise to the optimal estimation and how to compute the optimal estimate? In the following, we shall propose an optimal quantization scheme and the corresponding quantized innovation based optimal filtering.

A sensor makes an observation $y(n)$ and computes the innovation $\epsilon(n) := y(n) - \hat{y}(n|n-1)$, where $\hat{y}(n|n-1) = \mathbf{h}^T(n)\hat{\mathbf{x}}(n|n-1)$ together with the variance of the innovation $\sigma_\epsilon^2(n) = \mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)$ are received by the sensor from the estimator centre with $\hat{\mathbf{x}}(n|n-1)$, the one-step predictor of the state. In Ribeiro et al. [2006], the so-called sign-of-innovation Kalman filter (SOI-KF) has been studied where the innovation is quantized to 1 if it is positive and -1 if it is negative. The scheme is simple but obviously not optimal. For the case of small innovation, there is simply no need to transmit the innovation since the prediction is reasonably accurate in this case and quantizing the innovation to 1 or -1 will introduce a greater error to the estimation. In this paper, we shall study a general multi-bit (multi-level) quantization filtering. When specialized to the 1-bit case, it will provide a better quantization and filtering than the SOI-KF. Denote the normalized innovation as $\bar{\epsilon}(n)$, i.e. $\bar{\epsilon}(n) = \epsilon(n)/\sigma_\epsilon(n)$. We shall quantize the normalized innovation into (2N+1) levels, 0 and $\pm z_j$, $j = 1, 2, \cdots, N$. Let $\bar{z}_j = \sigma_\epsilon z_j > 0$. We consider a symmetric quantizer for ϵ given by

$$b(n) := \begin{cases} z_N, & \bar{z}_N < \epsilon(n) \\ z_{N-1}, & \bar{z}_{N-1} < \epsilon(n) \le \bar{z}_N \\ \vdots & \vdots \\ z_1, & \bar{z}_1 < \epsilon(n) \le \bar{z}_2 \\ 0, & -\bar{z}_1 < \epsilon(n) \le \bar{z}_1 \\ -z_1, & -\bar{z}_2 < \epsilon(n) \le -\bar{z}_1 \\ \vdots & \vdots \\ -z_N, & \epsilon(n) \le -\bar{z}_N \end{cases} \tag{3}$$

When $b(n) = 0$, it will not be transmitted to the estimation center. Hence, for the same 1-bit transmission, we can have 3 levels unlike that in Ribeiro et al. [2006] where the innovation is quantized to 1 as long as it is positive and -1 if it is negative. Intuitively, our approach should perform better since one more quantization level is added. The question is how to design the quantization levels z_j that will give rise to the optimal estimation. Note that since quantized innovation of zero will not be transmitted, the number of bits required for the $(2N+1)$-level quantization is $n_b \ge [log_2(2N)]$, where $[\cdot]$ is the integer round-up function. It should be noted that due to the presence of quantizer, the innovation may not remain Gaussian. However, as in Ribeiro et al. [2006], it is assumed that the innovation is approximately Gaussian. Note that with more quantization levels it can be argued that the innovation resulted from our quantization should be closer to a Guassian distribution than that of Ribeiro et al. [2006]. We further assume that there exists no error in transmitting the quantized message $b(n)$. Our goal in this paper is to find and analyze the MMSE (minimum mean square estimator) with the innovation quantized into $2N+1$ levels. Denote $\mathbf{b}_{0:n} = \{b(0), b(1), \cdots, b(n)\}$. Since the MMSE is obtained by the conditional expectation, therefore, if $\hat{\mathbf{x}}(n|n)$ denotes the MMSE of $\mathbf{x}(n)$ given $\mathbf{b}_{0:n}$, we have

$$\hat{\mathbf{x}}(n|n) := E[\mathbf{x}(n)|\mathbf{b}_{0:n}] = \int_{\mathbb{R}^p} \mathbf{x}(n) p[\mathbf{x}(n)|\mathbf{b}_{0:n}] d\mathbf{x}(n) \tag{4}$$

Note that given the optimal estimates of the state and output at $k-1$, the one-step ahead predictors of the state and observation are given by Ribeiro et al. [2006]

$$\hat{\mathbf{x}}(n|n-1) := E[\mathbf{x}(n)|\mathbf{b}_{0:n-1}] = \mathbf{A}(n)\hat{\mathbf{x}}(n-1|n-1) \tag{5}$$

$$\hat{y}(n|n-1) := E[y(n)|\mathbf{b}_{0:n-1}] = \mathbf{h}^T(n)\hat{\mathbf{x}}(n|n-1) \tag{6}$$

We define the error covariance matrices of the state filter and predictor as:

$$\mathbf{P}(n|n) := E[(\hat{\mathbf{x}}(n|n) - \mathbf{x}(n))(\hat{\mathbf{x}}(n|n) - \mathbf{x}(n))^T] \tag{7}$$

$$\mathbf{P}(n|n-1) := E[(\hat{\mathbf{x}}(n|n-1) - \mathbf{x}(n))(\hat{\mathbf{x}}(n|n-1) - \mathbf{x}(n))^T] \tag{8}$$

It is well known that

$$\mathbf{P}(n|n-1) = \mathbf{A}(n)\mathbf{P}(n-1|n-1)\mathbf{A}^T(n) + \mathbf{W}(n) \tag{9}$$

3. KALMAN FILTERING WITH MULTIPLE-LEVEL QUANTIZED INNOVATION

In this section, we shall first consider the case when $N = 1$. We derive the filtering error covariance matrix in terms of the quantization level z_1 and by minimizing the covariance with respect to z_1 we obtain the optimal 1-bit quantization scheme and the corresponding optimal Kalman filter (1-LQ-KF). We then extend this result to the general cases of $N > 1$.

3.1 Kalman filter with quantized innovation of 1-bit

For $N = 1$, the quantized innovation is reduced to

$$b(n) := \begin{cases} z_1, & \bar{z}_1 < \epsilon(n) \\ 0, & -\bar{z}_1 < \epsilon(n) \le \bar{z}_1 \\ -z_1, & \epsilon(k) \le -\bar{z}_1 \end{cases} \tag{10}$$

When $z_1 \to 0^+$, it reduces to the case of SOI-KF Ribeiro et al. [2006].

Theorem 1. For the stochastic dynamic system described by (1) and (2), if $p[\mathbf{x}(n)|\mathbf{b}_{0:n-1}] = \mathcal{N}[\mathbf{x}(n); \hat{\mathbf{x}}(n|n-1), \mathbf{P}(n|n-1)]$, the Kalman filter with quantized innovation of single bit can be given by

$$\begin{aligned}\hat{\mathbf{x}}(n|n) = &\hat{\mathbf{x}}(n|n-1) + f_1(n) \\ &\times \frac{\mathbf{P}(n|n-1)\mathbf{h}(n)}{\sqrt{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}}\end{aligned} \tag{11}$$

and the corresponding filtering error covariance is

$$\begin{aligned}\mathbf{P}(n|n) = &\mathbf{P}(n|n-1) - \frac{2\phi^2(n)}{\alpha_{z_1}} \\ &\times \frac{\mathbf{P}(n|n-1)\mathbf{h}(n)\mathbf{h}^T(n)\mathbf{P}(n|n-1)}{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}\end{aligned} \tag{12}$$

where $\hat{\mathbf{x}}(n|n-1)$ and $\mathbf{P}(n|n-1)$ are respectively given by (5) and (9), $\phi(x) := \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})$, $\alpha_{z_1} := Q(z_1) = \int_{z_1}^{\infty} \phi(x)dx$, and $f_1(n) := \frac{\phi(z_1)}{\alpha_{z_1}} Sgn(b(n))$ with

$$Sgn(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Obviously, when $z_1 \to 0^+$, then (11) and (12) are identical to the recursive formula in Ribeiro et al. [2006].

Proof: First, due to the symmetry of the problem, it is sufficient to prove the case $b(n) = z_1$. To establish (11), recall that the conditional mean can be obtained by averaging $\mathbf{x}(n)$ over the posterior pdf, $p[\mathbf{x}(n)|\mathbf{b}_{0:n}]$:

$$\hat{\mathbf{x}}(n|n) := E[\mathbf{x}(n)|\mathbf{b}_{0:n}] = \int_{\mathbb{R}^n} \mathbf{x}(n) p[\mathbf{x}(n)|\mathbf{b}_{0:n}] d\mathbf{x}(n) \tag{13}$$

By applying the Bayes' rule, we have

$$p[\mathbf{x}(n)|\mathbf{b}_{0:n}] = \frac{p[b(n)|\mathbf{x}(n), \mathbf{b}_{0:n-1}] p[\mathbf{x}(n)|\mathbf{b}_{0:n-1}]}{p[b(n)|\mathbf{b}_{0:n-1}]} \tag{14}$$

where

$$p[b(n) = 1|\mathbf{b}_{0:n-1}] = Pr\{\epsilon(n) > \bar{z}_1|\mathbf{b}_{0:n-1}\} = \alpha_{z_1} \tag{15}$$

and

$$\begin{aligned}&p[b(n) = 1|\mathbf{x}(n), \mathbf{b}_{0:n-1}] = Pr\{\epsilon(n) > \bar{z}_1|\mathbf{x}(n)\} \\ &= Pr\{v(n) > \mathbf{h}^T(n)(\hat{\mathbf{x}}(n|n-1) - \mathbf{x}(n)) + \bar{z}_1|\mathbf{x}(n)\} \\ &= Q\left[\frac{\mathbf{h}^T(n)(\hat{\mathbf{x}}(n|n-1) - \mathbf{x}(n)) + \bar{z}_1}{\sigma_v(n)}\right]\end{aligned} \tag{16}$$

Then, letting $\tilde{\mathbf{x}}(n) = \mathbf{x}(n) - \hat{\mathbf{x}}(n|n-1)$ and substituting it to (13) with the consideration of (14), (15) and (16) lead to

$$\begin{aligned}\hat{\mathbf{x}}(n|n) = &\hat{\mathbf{x}}(n|n-1) + \frac{1}{\alpha_{z_1}} \int_{\mathbb{R}^p} \tilde{\mathbf{x}}(n) Q\left[-\frac{\mathbf{h}^T(n)\tilde{\mathbf{x}}(n) - \bar{z}_1}{\sigma_v(n)}\right] \\ &\times \frac{exp[-\frac{1}{2}\tilde{\mathbf{x}}^T(n)\mathbf{P}^{-1}(n|n-1)\tilde{\mathbf{x}}(n)]}{(2\pi)^{n/2} det^{1/2}[\mathbf{P}(n|n-1)]} d\tilde{\mathbf{x}}(n)\end{aligned} \tag{17}$$

Similar to Ribeiro et al. [2006], we have proved that the integral part of (17) can be reduced to

$$\mathbf{l}(\bar{z}_1, n) = \phi(z_1) \frac{\mathbf{P}(n|n-1)\mathbf{h}(n)}{\sqrt{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}} \tag{18}$$

which yields

$$\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1) + \frac{\mathbf{l}(\bar{z}_1, n)}{\alpha_{z_1}} Sgn(b(n)) \tag{19}$$

for $b(n) = z_1$ as assumed. That is, (11) follows. Substituting $f_1(n)$ into (19) and subtracting $\mathbf{x}(n)$ on the both sides of (19), then the filtering error variance can be computed as

$$\begin{aligned}\mathbf{P}(n|n) = &\mathbf{P}(n|n-1) + \mathbf{k}(n)\mathbf{k}^T(n)E[f_1^2(n)] \\ &-2\mathbf{k}(n)E[f_1(n)\mathbf{x}^T(n)]\end{aligned} \tag{20}$$

where

$$\mathbf{k}(n) = \frac{\mathbf{P}(n|n-1)\mathbf{h}(n)}{\sqrt{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}} \tag{21}$$

$$\begin{aligned}E[f_1(n)\mathbf{x}^T(n)] &= E\{E[f_1(n)\mathbf{x}^T(n)|f_1(n)]\} \\ &= \alpha_{z_1}\frac{\phi(z_1)}{\alpha_{z_1}} E\left[\mathbf{x}^T(n)|b(n) = z_1\right] \\ &\quad -\alpha_{z_1}\frac{\phi(z_1)}{\alpha_{z_1}} E\left[\mathbf{x}^T(n)|b(n) = -z_1\right] \\ &= 2\frac{\phi^2(z_1)}{\alpha_{z_1}}\mathbf{k}^T(n)\end{aligned} \tag{22}$$

$$E[f_1^2(n)] = 2\alpha_{z_1}\frac{\phi^2(z_1)}{\alpha_{z_1}^2} = 2\frac{\phi^2(z_1)}{\alpha_{z_1}} \tag{23}$$

Substituting the above into (20), we have (12).

3.2 Kalman filter with multiple-level quantized innovation

When transmission of multi-bit quantized innovation is allowed in the sensor network, it will result in a finer quantization of innovation and hence a better state estimation. In this subsection, we shall extend our study of the single-bit case to the general multi-bit case. We consider the quantized innovation in (3).

Theorem 2. Consider the dynamical system (1)-(2) and the multi-level quantization of innovation in (3), then the recursive formulae for computing the state estimator can be given by

$$\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1) + \frac{f_N(n)\mathbf{P}(n|n-1)\mathbf{h}(n)}{\sqrt{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}} \tag{24}$$

$$\begin{aligned}\mathbf{P}(n|n) = &\mathbf{P}(n|n-1) - 2\sum_{k=1}^{N} \frac{[\phi(z_k) - \phi(z_{k+1})]^2}{\alpha_{z_k} - \alpha_{z_{k+1}}} \\ &\times \frac{\mathbf{P}(n|n-1)\mathbf{h}(n)\mathbf{h}^T(n)\mathbf{P}(n|n-1)}{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}\end{aligned} \tag{25}$$

where

$$f_N(n) = \sum_{k=0}^{N} Sgn(b(n)) I_{\{k\}}(b(n)) \frac{\phi(z_k) - \phi(z_{k+1})}{\alpha_{z_k} - \alpha_{z_{k+1}}} \tag{26}$$

with $I_A(\cdot)$ the indicator function given by

$$I_{\{k\}}(b(n)) = \begin{cases} 1, & |b(n)| = z_k \\ 0, & \text{otherwise} \end{cases}$$

Clearly, when $N = 1$, the above filter is simplified to that in Theorem 1.

Proof: For the same reason as in Theorem 1, we only need to prove the case when $b(n)$ is positive. Since

$$\begin{aligned} p[b(n) = z_k|\mathbf{b}_{0:n-1}] &= Pr\{\bar{z}_k < \epsilon(n) \le \bar{z}_{k+1}|\mathbf{b}_{0:n-1}\} \\ &= \alpha_{z_k} - \alpha_{z_{k+1}} \end{aligned} \quad (27)$$

and

$$\begin{aligned} &p[b(n) = z_k|\mathbf{x}(n), \mathbf{b}_{0:n-1}] = Pr\{\bar{z}_k < \epsilon(n) \le \bar{z}_{k+1}|\mathbf{x}(n)\} \\ &= Q\left[-\frac{\mathbf{h}^T(n)\tilde{\mathbf{x}}(n) - \bar{z}_k}{\sigma_v(n)}\right] - Q\left[-\frac{\mathbf{h}^T(n)\tilde{\mathbf{x}}(n) - \bar{z}_{k+1}}{\sigma_v(n)}\right] \end{aligned} \quad (28)$$

Substituting (27) and (28) into (13) yields

$$\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1) + \frac{\mathbf{l}(\bar{z}_k, n) - \mathbf{l}(\bar{z}_{k+1}, n)}{\alpha_{z_k} - \alpha_{z_{k+1}}} \quad (29)$$

By taking into consideration (26) and (29), (24) follows. Next, apply the theorem of total probability as in (21) to obtain

$$\begin{aligned} E[f_N(n)\mathbf{x}^T(n)] &= E\left\{E[f_N(n)\mathbf{x}^T(n)|f_N(n)]\right\} \\ &= 2\sum_{k=1}^{N} \frac{[\phi(z_k) - \phi(z_{k+1})]^2}{\alpha_{z_k} - \alpha_{z_{k+1}}}\mathbf{k}^T(n) \end{aligned} \quad (30)$$

and

$$E[f_N^2(n)] = 2\sum_{k=1}^{N} \frac{[\phi(z_k) - \phi(z_{k+1})]^2}{\alpha_{z_k} - \alpha_{z_{k+1}}} \quad (31)$$

Finally, in view of (20), (30), and (31), we obtain (25).

Remark 1. Theorem 2 gives the MMSE filter based on the given quantized innovation of (3). It is clear that the filter performance relies on the choice of the quantization levels $z_j,\ j = 1, 2, \cdots, N$. The optimal quantizer and filter can be obtained by optimization, which will be discussed in the next section.

4. OPTIMAL FILTERING AND PERFORMANCE ANALYSIS

The mean-squared errors (MSE) of $\hat{\mathbf{x}}(n|n)$ and $\hat{\mathbf{x}}(n|n-1)$ are given by minimizing the $tr[\mathbf{P}(n|n)]$ and $tr[\mathbf{P}(n|n-1)]$ among all possible estimators $\hat{\mathbf{x}}(n|n)$and $\hat{\mathbf{x}}(n|n-1)$.

Define

$$F(z_1, \cdots, z_N) = 2\sum_{k=1}^{N} \frac{[\phi(z_k) - \phi(z_{k+1})]^2}{\alpha_{z_k} - \alpha_{z_{k+1}}}$$

then the optimal filter can be obtained by maximizing $F(z_1, \cdots, z_N)$ under the constraint $0 < z_1 < \cdots < z_N$. Calling the function,

$$[x, y] = fmincon(fun, x0, A, b, Aeq, beq, lb, ub)$$

in the Optimization Toolbox of Matlab, we can approximately obtain the optimal quantization levels for a given N. For $N = 1$, if we set $z_1 \to 0^+$, $\phi(z_1) \to \frac{1}{\sqrt{2\pi}}$ and $\alpha_{z_1} \to \frac{1}{2}$. In this situation, the filter of Theorem 1 is reduced to that of Ribeiro et al. [2006]. However, by

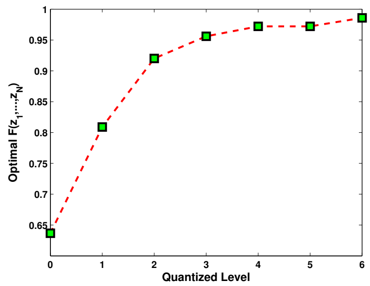

Fig. 1. Optimal $F(z_1, z_2, \cdots, z_N)$ versus N,SOI-KF Ribeiro et al. [2006] is treated as N=0 for the special case of MLQ-KF

maximizing $F(z_1)$ the optimal level is $z_1^* = 0.612$, which leads to $F(z_1) = 0.8098$. Apparently, it gives a much improved filter than the SOI-KF in Ribeiro et al. [2006]. The optimal filter associated with the optimal 1-bit quantized innovation is given in the theorem below.

Theorem 3. Consider the system (1)-(2) and the 1-bit innovation quantizer (10). The optimal filter is given by

$$\begin{aligned} \hat{\mathbf{x}}(n|n) &= \hat{\mathbf{x}}(n|n-1) + \\ &\quad \frac{1.2240\mathbf{P}(n|n-1)\mathbf{h}(n)b(n)}{\sqrt{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}} \end{aligned} \quad (32)$$

and the corresponding filtering error covariance is

$$\begin{aligned} \mathbf{P}(n|n) &= \mathbf{P}(n|n-1) - \\ &\quad \frac{0.8098\mathbf{P}(n|n-1)\mathbf{h}(n)\mathbf{h}^T(n)\mathbf{P}(n|n-1)}{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}. \end{aligned} \quad (33)$$

For $N = 2$, the optimization of $F(z_1, z_2)$ with respect to z_1 and z_2 leads to $z_1^* = 0.3823$ and $z_2^* = 1.2437$, which gives $F(z_1^*, z_2^*) = 0.9201$. Surprisingly, we can closely approximate the standard Kalman Filter by only quantizing the innovation into 2 bits. Fig 1 shows the relationship between N and $F(z_1^*, \cdots, z_N^*)$.

In the following, we show that the limiting case of the filter associated with the multi-level quantization of innovation converges the clairvoyant Kalman filter.

Theorem 4. Let $\triangle = sup_{k\in\mathbb{N}}\triangle_k$, where $\triangle_k = |z_k - z_{k+1}|$, and $z_k,\ k \in \mathbb{N}$ satisfy that

(1) $z_1 \to 0^+$
(2) $\triangle_k \le \triangle \to 0$
(3) $S(N) = \sum_{k=1}^{N} \triangle_k \to \infty$ as $N \to \infty$.

Then, for $\bar{\epsilon}(n)$ belonging to the interval $(z_k, z_{k+1}]$

$$f_N(n) = \frac{\phi(z_k) - \phi(z_{k+1})}{\alpha_{z_k} - \alpha_{z_{k+1}}} \to z_{k+1} \quad (34)$$

and

$$\sum_{k=1}^{\infty} \frac{[\phi(z_k) - \phi(z_{k+1})]^2}{\alpha_{z_k} - \alpha_{z_{k+1}}} \to \frac{1}{2} \quad (35)$$

Proof: For $\triangle \to 0$, $\triangle z_k \to 0$ as well. Hence

$$\begin{aligned}
&\lim_{z_k\to z_{k+1}} \frac{\phi(z_k)-\phi(z_{k+1})}{\alpha_{z_k}-\alpha_{z_{k+1}}}\\
&= \lim_{\triangle z_k\to 0}\frac{\phi(z_{k+1}-\triangle z_k)-\phi(z_{k+1})}{\int_{z_{k+1}-\triangle z_k}^{z_{k+1}}\phi(t)dt}\\
&= \lim_{\triangle z_k\to 0}\frac{\phi'(z_{k+1}-\triangle z_k)}{\phi(z_{k+1}-\triangle z_k)}\\
&= z_{k+1}
\end{aligned} \tag{36}$$

$$\begin{aligned}
S :&= \sum_{k=1}^{\infty}\frac{[\phi(z_k)-\phi(z_{k+1})]^2}{\alpha_{z_k}-\alpha_{z_{k+1}}}\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\frac{[e^{(-\frac{z_k^2}{2})}-e^{(-\frac{z_{k+1}^2}{2})}]^2}{\int_{z_k}^{z_{k+1}}e^{(-\frac{t^2}{2})}dt}\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\frac{[e^{(-\frac{z_k^2}{2})}-e^{[-\frac{(z_k+\triangle z_k)^2}{2}]}]^2}{e^{-\frac{(z_k+\theta_k\triangle z_k)^2}{2}}\triangle z_k}\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}e^{-\frac{z_k^2}{2}}\frac{[1-e^{-(\frac{(\triangle z_k)^2}{2}-z_k\triangle z_k)}]^2}{\triangle z_k e^{-\frac{(\theta_k\triangle z_k)^2}{2}-\theta_k z_k\triangle z_k}}
\end{aligned} \tag{37}$$

where $0 \le \theta_k \le 1$. Using the Taylor expansion for the exponential function in (37), we have $S := S_0 + S_1$

$$\begin{aligned}
S_1 &= \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}z_k^2 e^{-\frac{z_k^2}{2}}\triangle z_k\\
&\xrightarrow{(\triangle\to 0)}\frac{1}{\sqrt{2\pi}}\int_0^{\infty}t^2e^{-t^2/2}dt = 1/2
\end{aligned} \tag{38}$$

and there exist nonnegative integers $0 \le i,j,u,v \le n < \infty$ and $|c_{i,j,v,u}| < \infty$ such that

$$\begin{aligned}
S_0 =&\frac{1}{\sqrt{2\pi}}\sum_{i,j,u,v}\sum_{k=1}^{\infty}[c_{i,j,v,u}e^{(-\frac{z_k^2}{2})}(\triangle z_k)^{2+i}z_k^j\theta_k^u\times\\
&o((\triangle z_k)^v)] < \triangle\sum_{i,j,u,v}c_{i,j,v,u}\frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}z_k^je^{(-\frac{z_k^2}{2})}\triangle z_k\\
&:= C\triangle \to 0 \quad (\text{as } \triangle\to 0)
\end{aligned} \tag{39}$$

where C is a finite constant because for $\triangle \to 0$,

$$\frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}z_k^je^{-\frac{z_k^2}{2}}\triangle z_k \to \frac{1}{\sqrt{2\pi}}\int_0^{\infty}t^je^{-t^2/2}dt < E[X^j] < \infty$$

for standard Gaussian random variable x and $\forall j \in \mathbb{N}$. This completes the proof.

Remark 2. Conditions (2) and (3) of Theorem 4 can be simultaneously satisfied. For example, the power series $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges to ∞. Consequently, for sufficiently large $N \in \mathbb{N}$, $n_k := \frac{1}{N+k} \le \frac{1}{N+1} \to 0$ and $\sum_{k=1}^{\infty}n_k \to \infty$. As $z_k \to z_{k+1}$, then $\bar{\epsilon}(n) = \frac{\epsilon(n)}{\sigma_\epsilon(n)} \to z_{k+1}$. It follows from (34), (35), (24), and (25) that the filter becomes

$$\hat{\mathbf{x}}(n|n) \to \hat{\mathbf{x}}(n|n-1) + \frac{\epsilon(n)\mathbf{P}(n|n-1)\mathbf{h}(n)}{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n)+\sigma_v^2(n)} \tag{40}$$

$$\mathbf{P}(n|n) \to \mathbf{P}(n|n-1) - \frac{\mathbf{P}(n|n-1)\mathbf{h}(n)\mathbf{h}^T(n)\mathbf{P}(n|n-1)}{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n)+\sigma_v^2(n)} \tag{41}$$

which is the standard Kalman Filter.

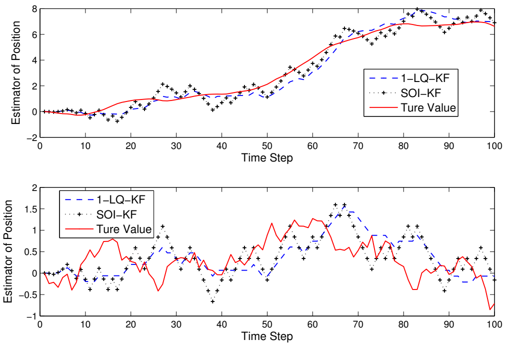

Fig. 2. Comparison of position and velocity tracking performance between the 1-LQ-KF and SOI-KF.

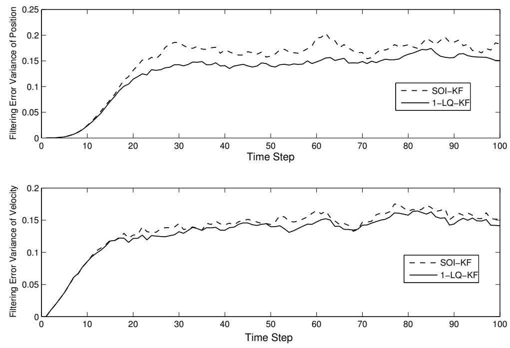

Fig. 3. Comparison between filtering error variances of SOI-KF and 1-LQ-KF obtained by Monte Carlo simulations based on 500 samples.

5. SIMULATION

Consider a simple tracking system

$$\mathbf{x}(n+1) = \begin{bmatrix}1 & \tau\\ 0 & 1\end{bmatrix}\mathbf{x}(n) + \begin{bmatrix}\tau^2/2\\ \tau\end{bmatrix}w(n) \tag{42}$$

$$y(n) = [1,0]\mathbf{x}(n) + v(n) \tag{43}$$

where τ is the sampling period. The state $\mathbf{x}(n) = [s(n), \dot{s}(n)]^T$, where $s(n)$ and $\dot{s}(n)$ are the position and velocity of the target at time τn, respectively. $y(n)$ is the measurement signal, $v(n)$ is the measurement noise with zero and variance σ_v^2 and is independent of the Gaussian noise $w(n)$ which is of zero mean and variance σ_w^2. Our aim is to solve the optimal multi-level quantization Kalman filter (MLQ-KF) and compare it with the standard Kalman filter (OKF) Anderson and Moore [1979], and SOI-KF of Ribeiro et al. [2006].

In the simulation, we set the sampling period $\tau = 0.1s$ and $\sigma_w^2 = 1, \sigma_v^2 = 0.81$ and take the initial value $\mathbf{x}(0) = [0 \;\; 0]^T$ and $P_0 = 0.01I_2$. The superiority of our proposed 1-bit quantized innovation Kalman filter (1-LQ-KF) over the SOI-KF Ribeiro et al. [2006] is showed in Fig 2 and Fig 3 while Fig 4 and Fig 5 illustrate that 2-bit MLQ-KF (2LQ-KF) is sufficient to approximate the standard Kalman filter. Fig 6 demonstrates that the estimated filtering error variances obtained by the Monte Carlo method converge to the theoretic variances computed by the Riccati recursions.

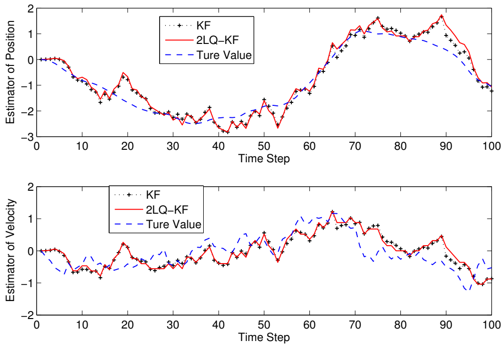

Fig. 4. Comparison of position and velocity tracking performance between the 2LQ-KF and standard Kalman filter.

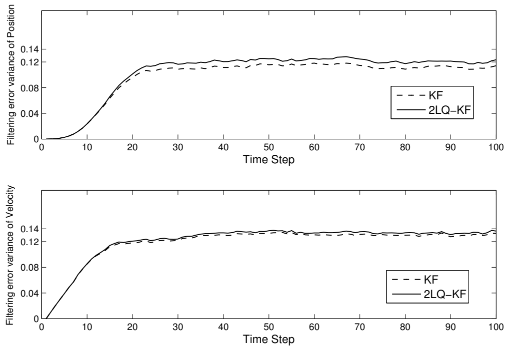

Fig. 5. Comparison of the filtering error variances of the 2LQ-KF and the standard Kalman filter obtained by Monte Carlo simulations based on 500 samples.

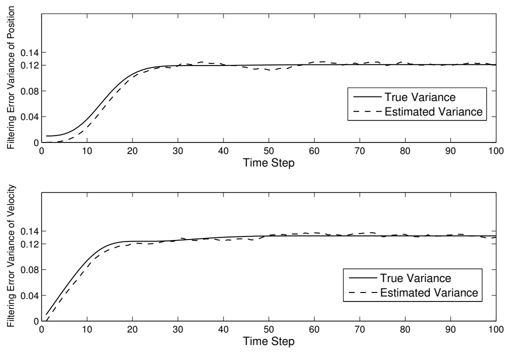

Fig. 6. Computed filtering error variance by the Riccati difference equation and the filtering error variance obtained by Monte Carlo simulations based on 500 samples.

6. CONCLUSIONS

Extending previously proposed sign of innovation Kalman filter, we have developed a very general multi-level quantized innovation filter. By assuming that the innovation is approximately Gaussian, we have derived an optimal multi-level quantization scheme and the corresponding optimal filter. The solution to the optimal filter is given in terms of a simple Reccati recursion as in the standard Kalman filter. Our result shows that the quantized innovation filter of 2-bit gives a filtering performance close to the standard Kalman filter. For 1-bit transmission, Our proposed filter gives a better performance than the sign-of-innovation filter in Ribeiro et al. [2006]. Performance analysis of the multi-level quantized innovation filter has been carried out. It has been shown that the quantized innovation filter converges to the standard Kalman filter when the level of quantization goes to ∞.

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