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Square-Root Algorithms of Recursive Least-Squares Wiener Estimators in Linear Discrete-Time Stochastic Systems ⋆

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Abstract: This paper addresses the QR decomposition and UD factorization based square-root algorithms of the recursive least-squares (RLS) Wiener fixed-point smoother and filter. In the RLS Wiener estimators, the Riccati-type difference equations for the auto-variance function of the filtering estimate are included. Hence, by the roundoff errors, in the case of the small value of the observation noise variance, under a single precision computation, the auto-variance function becomes asymmetric and the estimators tend to be numerically instable. From this viewpoint, in the proposed square-root RLS Wiener estimators, in each stage updating the estimates, the auto-variance function of the filtering estimate is expressed in a symmetric positive semi-definite matrix and the stability of the RLS Wiener estimators is improved. In addition, in the square-root RLS Wiener estimation algorithms, the variance function of the state prediction error is expressed as a symmetric positive semi-definite matrix in terms of the UD factorization method.

1. INTRODUCTION

In the single precision computation of the Kalman filter, roundoff errors, due to large a priori uncertainty or small value of the variance of the observation noise, cause numerical instability of the filtering estimate Katayama [1983], Grewal and Andrews [1993]. To overcome the numerical difficulties, some computation methods of the Kalman filter have been presented. One method, to reduce or avoid the roundoff errors, is to declare the variables, concerned with the Riccati-type difference equations in the Kalman filter, by the double precision in the computer program for the filtering estimate. In the roundoff computation of the Kalman filter under the single precision, as the number of iteration for updating the filtering estimate increases, the a priori error variance function, i.e. the prediction error variance function, in the Kalman filter tends to be asymmetric. Hence, in each stage updating the filtering estimate, it is required to express the a priori error variance function in a symmetric positive semi-definite matrix. Bucy and Joseh [1968] demonstrated improved numerical stability by rearranging the Kalman filter for observational update. As square-root algorithms, Potter square-root factorization Battin [1964], Bierman UD factorization Bierman [1977], square partitioned UD factorization Morf and Kailath [1975], etc. Bellantoni and Dodge [1967], Andrews [1968], Morf et al. [1978] are developed. In Morf and Kailath [1975], for the Riccati recursion for the variance of the state prediction error, so-called square-root array (or factored) estimation algorithm that propagates square-root factors of the variance function is proposed. Similarly, in Hassibi et al. [2000], by the QR decomposition, the square-root algorithm of the H-infinity filter is proposed.

As an application of the QR decomposition, a square-root algorithm is obtained on a linearized model of two-dimensional shallow water equations for the prediction of tides and storm surges Verlaan and Heemink [1997]. The QR decomposition method is applied also to the recursive least-squares (RLS) adaptive filter Cioffi [1990],Diniz [2002] for possible implementation in systolic arrays.

As an alternative to the Kalman estimators, the RLS Wiener fixed-point smoother and filter Nakamori [1995] are known. This paper presents the QR decomposition and UD factorization based square-root computation algorithms of the RLS Wiener fixed-point smoother and filter. In the RLS Wiener estimators of [**Theorem 1**], the Riccati-type difference equations for the auto-variance function of the filtering estimate are calculated directly. However, by the roundoff errors, in the case of the small value of the observation noise, under the single precision computation, the auto-variance function becomes asymmetric and the estimators tend to be numerically instable. From this viewpoint, in the calculation of the RLS Wiener estimates, in each stage updating the estimates, it is required that the auto-variance function of the filtering estimate is expressed in a symmetric positive semi-definite matrix.

In the QR decomposition, concerned with the Riccati-type difference equations, the prediction error variance function of (17) is required as a priori information. Here, in terms of the UD factorization of a matrix, whose elements includes the information of the variance of the filtering estimate at the previous stage, the system matrix and the variance of the state vector, as shown in (20), the prediction error variance function is expressed as a symmetric positive semi-definite matrix. By the QR decomposition of a matrix $A^T(k-1)$ in (22), the updated auto-variance function $S(k)$ of the filtering estimate of the state vector at time k is

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given by (30). However, in terms of the UD factorization as (31), $S(k)$ is represented by (32) as a symmetric matrix and its positive semi-definiteness is assured. Hence, it is considered that the proposed square-root RLS Wiener estimators of [**Theorem 2**] improve the stability of the RLS Wiener estimators of [**Theorem 1**].

In a numerical simulation example of section 4, the square-root RLS Wiener estimation algorithms of [**Theorem 2**], based on the QR decomposition and the UD factorization, are compared, in estimation accuracy, with the RLS Wiener estimation algorithms of [**Theorem 1**].

2. LEAST-SQUARES ESTIMATION PROBLEM

Let an m-dimensional observation equation be given by

$$y(k) = z(k) + v(k),\ z(k) = Hx(k) \tag{1}$$

in linear discrete-time stochastic systems. Here, $z(k)$ is a signal, H is an $m \times n$ observation matrix, and $x(k)$ is a state vector. $v(k)$ is white observation noise. Also, let the state equation for the state vector $x(k)$ be expressed by

$$x(k+1) = \Phi x(k) + \Gamma w(k), \tag{2}$$

where Φ is a system matrix or a state-transition matrix and $w(k)$ is an r-dimensional white noise input. It is assumed that the observation noise and the input noise are mutually independent, the signal and the observation noise are also independent and are zero mean. Let the auto-covariance functions of $v(k)$ and $w(k)$ be given by

$$E[v(k)v^T(j)] = R\delta_K(k-j),\ R > 0, \tag{3}$$

$$E[w(k)w^T(j)] = Q\delta_K(k-j),\ Q > 0. \tag{4}$$

Here, $\delta_K(k-j)$ denotes the Kronecker δ function.

Let $K_x(k,j) = K_x(k-j)$ represent the auto-covariance function of the state vector, and let $K_x(k,j)$ be expressed in the form of

$$K_x(k,j) = \begin{cases} A(k)B^T(j),\ 0 \geq j \geq k, \\ B(k)A^T(j),\ 0 \leq k \leq j, \end{cases} \tag{5}$$

in wide-sense stationary stochastic systems Nakamori [1995]. Here, $A(k) = \Phi^k$, $B^T(j) = \Phi^{-j}K(j,j) = \Phi^{-j}K(0)$, where $K(0)$ represents the variance of $x(k)$.

Let a fixed-point smoothing estimate $\widehat{x}(k,L)$ of $x(k)$ be expressed by

$$\widehat{x}(k,L) = \sum_{i=1}^{L} h(k,i,L)y(i),\ 1 \leq k \leq L, \tag{6}$$

as a linear transformation of the observed value $y(i)$, $1 \leq i \leq L$. In (6), $h(k,i,L)$ is a time-varying impulse response function and k is the fixed point respectively.

Let us consider the estimation problem, which minimizes the mean-square value

$$J = E[||x(k) - \widehat{x}(k,L)||^2] \tag{7}$$

of the fixed-point smoothing error. From an orthogonal projection lemma Sage and Melsa [1971],

$$x(k) - \sum_{i=1}^{L} h(k,i,L)y(i) \perp y(j),\ 0 \leq j,k \leq L, \tag{8}$$

the impulse response function satisfies the Wiener-Hopf equation

$$E[x(k)y^T(j)] = \sum_{i=1}^{L} h(k,i,L)E[y(i)y^T(j)]. \tag{9}$$

Here '$\perp$' denotes the notation of the orthogonality.

Substituting (1) and (3) into (9), we obtain

$$h(k,j,L)R = K(k,j)H^T - \sum_{i=1}^{L} h(k,i,L)HK_x(i,j)H^T. \tag{10}$$

3. SQUARE-ROOT COMPUTATION OF RLS WIENER ESTIMATORS

Under the linear least-squares estimation problem of the signal $z(k)$ in section 2, at first, [**Theorem 1**] shows the RLS Wiener fixed-point smoothing and filtering algorithms Nakamori [1995], which use the covariance information of the signal and observation noise.

[**Theorem 1**] Nakamori [1995]

Let the auto-covariance function $K_x(k,k)$ of $x(k)$ be expressed by (5), and let the variance of white observation noise be R. Then, the RLS Wiener algorithms for the fixed-point smoothing and filtering estimates consist of (11)-(16) in linear discrete-time stochastic systems.

Fixed-point smoothing estimate of the signal $z(k)$ at the fixed point k: $\widehat{z}(k,L) = H\widehat{x}(k,L)$

Fixed-point smoothing estimate of the signal $x(k)$ at the fixed point k: $\widehat{x}(k,L)$

$$\begin{aligned} \widehat{x}(k,L) &= \widehat{x}(k,L-1) + h(k,L,L) \\ &\quad \times (y(L) - H\Phi\widehat{x}(L-1,L-1)) \end{aligned} \tag{11}$$

Smoother gain: $h(k,L,L)$

$$\begin{aligned} h(k,L,L) &= (K_x(k,k)(\Phi^T)^{L-k}H^T \\ &\quad - q(k,L-1)\Phi^T H^T) \\ &\quad \times (R + HK_x(k,k)H^T \\ &\quad - H\Phi S(L-1)\Phi^T H^T)^{-1} \end{aligned} \tag{12}$$

$$\begin{aligned} q(k,L) &= q(k,L-1)\Phi^T + h(k,L,L)H \\ &\quad \times (K_x(L,L) - \Phi S(L-1)\Phi^T), \\ q(k,k) &= S(k) \end{aligned} \tag{13}$$

Filtering estimate of the signal $z(k)$: $\widehat{z}(k,k) = H\widehat{x}(k,k)$

Filtering estimate of $x(k)$: $\widehat{x}(k,k)$

$$\begin{gathered} \widehat{x}(k,k) = \Phi\widehat{x}(k-1,k-1) + K_g(k) \\ \times(y(k) - H\Phi\widehat{x}(k-1,k-1)),\ \widehat{x}(0,0) = 0 \end{gathered} \tag{14}$$

Auto-variance function of the filtering estimate $\widehat{x}(k,k)$: $S(k)$

$$\begin{gathered} S(k) = \Phi S(k-1)\Phi^T \\ +K_g(k)H(K_x(k,k) - \Phi S(k-1)\Phi^T), \\ S(0) = 0 \end{gathered} \tag{15}$$

Filter gain $K_g(k)$

$$K_g(k) = (K_x(k,k)H^T - \Phi S(k-1)\Phi^T H^T) \times (R + HK_x(k,k)H^T - H\Phi S(k-1)\Phi^T H^T)^{-1} \quad (16)$$

In the calculation of (12), for the matrix inversion $(R + HK_x(k,k)H^T - H\Phi S(k-1)\Phi^T H^T)^{-1}$ to exist, since $R > 0$ from (3), the positive semi-definiteness $HK_x(k,k)H^T - H\Phi S(k-1)\Phi^T H^T \geq 0$ must be guaranteed. Similarly, in the calculation of the Riccati-type difference equations (15) for the variance function $S(k)$ of the filtering estimate $\widehat{x}(k,k)$, $HK_x(k,k)H^T - H\Phi S(k-1)\Phi^T H^T \geq 0$ must be satisfied for the nonsingular matrix condition $R + HK_x(k,k)H^T - H\Phi S(k-1)\Phi^T H^T > 0$ to be valid.

In the following, taking into account also of this point, let us consider on the QR decomposition and UD factorization based square-root algorithms of the RLS Wiener estimators of [**Theorem 1**].

Let the prediction error variance function $K_x(k,k) - H\Phi S(k-1)\Phi^T$ be represented by

$$K_{\widetilde{x}}(k-1,k-1) = K_x(k,k) - \Phi S(k-1)\Phi^T. \quad (17)$$

From (16), (15) is written as

$$S(k) = \Phi S(k-1)\Phi^T + K_{\widetilde{x}}(k-1,k-1)H^T \times (R + HK_{\widetilde{x}}(k-1,k-1)H^T)^{-1} HK_{\widetilde{x}}(k-1,k-1). \quad (18)$$

In [**Theorem 1**], from the function $S(k-1)$ etc., $S(k)$ is updated. In terms of a factorization of

$$\begin{bmatrix} K_x(k-1,k-1) & \Phi S^{0.5}(k-1) \\ S^{0.5}(k-1)\Phi^T & I \end{bmatrix} = \begin{bmatrix} U_1(k-1) & \Phi S^{0.5}(k-1) \\ 0 & I \end{bmatrix} \begin{bmatrix} D_1(k-1) & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} U_1^T(k-1) & 0 \\ S^{0.5}(k-1)\Phi^T & I \end{bmatrix}$$
$$= \begin{bmatrix} U_1(k-1)D_1(k-1)U_1^T(k-1) + \Phi S(k-1)\Phi^T & \Phi S^{0.5}(k-1) \\ S^{0.5}(k-1)\Phi^T & I \end{bmatrix}, \quad (19)$$

it is seen that $K_{\widetilde{x}}(k-1,k-1)$ is given by

$$K_{\widetilde{x}}(k-1,k-1) = U_1(k-1)D_1(k-1)U_1^T(k-1) \quad (20)$$

as a symmetric positive semi-definite matrix.

Let a square matrix $A(k-1)$ be given by

$$A(k-1) = \begin{bmatrix} R^{0.5} & HK_{\widetilde{x}}^{0.5}(k-1,k-1) \\ 0 & K_{\widetilde{x}}^{0.5}(k-1,k-1) \end{bmatrix}, \quad (21)$$

and let its transpose be expressed in terms of a decomposition of

$$A^T(k-1) = Q(k-1)\Re(k-1), \quad (22)$$

where $Q(k-1)$ is an orthogonal matrix with $Q(k-1)Q^T(k-1) = Q^T(k-1)Q(k-1) = I$, which is an $m+n$ identity matrix. Here, let $\Re^T(k-1)$ be an $m+n$ square lower triangular matrix expressed by

$$\Re^T(k-1) = \begin{bmatrix} X(k-1) & 0 \\ Y(k-1) & Z(k-1) \end{bmatrix}. \quad (23)$$

From (22), since $A(k-1) = \Re^T(k-1)Q^T(k-1)$, the relationship $A(k-1)Q(k-1) = \Re^T(k-1)$ is valid. Namely, the orthogonal matrix $Q(k-1)$ transforms $A(k-1)$ to the lower triangular matrix $\Re^T(k-1)$ as

$$\begin{bmatrix} R^{0.5} & HK_{\widetilde{x}}^{0.5}(k-1,k-1) \\ 0 & K_{\widetilde{x}}^{0.5}(k-1,k-1) \end{bmatrix} Q(k-1) = \begin{bmatrix} X(k-1) & 0 \\ Y(k-1) & Z(k-1) \end{bmatrix}. \quad (24)$$

From the calculation

$$\begin{bmatrix} R^{0.5} & HK_{\widetilde{x}}^{0.5}(k-1,k-1) \\ 0 & K_{\widetilde{x}}^{0.5}(k-1,k-1) \end{bmatrix} Q(k-1)Q^T(k-1) \times \begin{bmatrix} R^{0.5} & HK_{\widetilde{x}}^{0.5}(k-1,k-1) \\ 0 & K_{\widetilde{x}}^{0.5}(k-1,k-1) \end{bmatrix}^T = \begin{bmatrix} X(k-1) & 0 \\ Y(k-1) & Z(k-1) \end{bmatrix} \begin{bmatrix} X(k-1) & 0 \\ Y(k-1) & Z(k-1) \end{bmatrix}^T,$$

it follows that

$$X(k-1)X^T(k-1) = R + HK_{\widetilde{x}}(k-1,k-1)H^T, \quad (25)$$

$$X(k-1)Y^T(k-1) = HK_{\widetilde{x}}(k-1,k-1), \quad (26)$$

$$Z(k-1)Z^T(k-1) + Y(k-1)Y^T(k-1) = K_{\widetilde{x}}(k-1,k-1). \quad (27)$$

From (25)-(27), it is found that

$$\begin{aligned} Z(k-1)Z^T(k-1) &= K_{\widetilde{x}}(k-1,k-1) - Y(k-1)Y^T(k-1) \\ &= K_{\widetilde{x}}(k-1,k-1) \\ &\quad - Y(k-1)X^T(k-1)(X(k-1)X^T(k-1))^{-1} \\ &\quad \times X(k-1)Y^T(k-1) \\ &= K_{\widetilde{x}}(k-1,k-1) - K_{\widetilde{x}}(k-1,k-1)H^T \\ &\quad \times (R + HK_{\widetilde{x}}(k-1,k-1)H^T)^{-1} \\ &\quad \times HK_{\widetilde{x}}(k-1,k-1). \end{aligned} \quad (28)$$

From (17), taking account of (15) and (16), we obtain

$$\begin{aligned} Z(k-1)Z^T(k-1) &= K_{\widetilde{x}}(k-1,k-1) - \Phi S(k-1)\Phi^T \\ &\quad - (K_x(k-1,k-1) - \Phi S(k-1)\Phi^T)H^T \\ &\quad \times (R + H(K_x(k-1,k-1) - \Phi S(k-1)\Phi^T)H^T)^{-1} \\ &\quad \times H(K_x(k-1,k-1) - \Phi S(k-1)\Phi^T) \\ &= K_x(k-1,k-1) - S(k). \end{aligned} \quad (29)$$

Hence, the updated value $S(k)$, at time k, is calculated by

$$S(k) = K_x(k-1,k-1) - Z(k-1)Z^T(k-1). \quad (30)$$

Also, in terms of a UD factorization of

$$\begin{bmatrix} K_x(k-1,k-1) & Z(k-1) \\ Z^T(k-1) & I \end{bmatrix} = \begin{bmatrix} \widetilde{U}(k-1) & Z(k-1) \\ 0 & I \end{bmatrix} \times \begin{bmatrix} \widetilde{D}(k-1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{U}^T(k-1) & 0 \\ Z^T(k-1) & I \end{bmatrix}$$
$$= \begin{bmatrix} \widetilde{U}(k-1)\widetilde{D}(k-1)\widetilde{U}^T(k-1) + Z(k-1)Z^T(k-1) & Z(k-1) \\ Z^T(k-1) & I \end{bmatrix}, \quad (31)$$

$S(k)$ is given by

$$S(k) = K_x(k-1,k-1) - Z(k-1)Z^T(k-1) = \widetilde{U}(k-1)\widetilde{D}(k-1)\widetilde{U}^T(k-1) \quad (32)$$

as a symmetric positive semi-definite matrix. Now, let us summarize the above result in [**Theorem 2**].

[**Theorem 2**]

Let the auto-covariance function $K_x(k,s)$ of $x(k)$ be expressed by (5), and let the variance of white observation noise be R. Then, the QR decomposition and UD factorization based square-root RLS Wiener estimation algorithms for the fixed-point smoothing and filtering estimates consist of (33)-(41) in linear discrete-time stochastic systems.

Fixed-point smoothing estimate of the signal $z(k)$ at the fixed point k: $\widehat{z}(k,L) = H\widehat{x}(k,L)$

Fixed-point smoothing estimate of the signal $x(k)$ at the fixed point k: $\widehat{x}(k,L)$

$$\widehat{x}(k,L) = \widehat{x}(k,L-1) + h(k,L,L) \times (y(L) - H\Phi\widehat{x}(L-1,L-1)) \quad (33)$$

Smoother gain: $h(k,L,L)$

$$\begin{aligned} h(k,L,L) &= (K_x(k,k)(\Phi^T)^{L-k}H^T - q(k,L-1)\Phi^T H^T) \\ &\quad \times (R + HU_1(L-1)D_1(L-1)U_1^T(L-1)H^T)^{-1} \\ &= (K_x(k,k)(\Phi^T)^{L-k}H^T - q(k,L-1)\Phi^T H^T)(X(L-1)X^T(L-1))^{-1} \end{aligned} \quad (34)$$

Here, $X(L-1)$ is obtained similarly with the calculation of $X(k-1)$ in (39) and $X(L-1)X^T(L-1)$ is a positive definite matrix from (20) and (25).

$$\begin{aligned} q(k,L) &= q(k,L-1)\Phi^T + h(k,L,L)HU_1(L-1)D_1(L-1)U_1^T(L-1), \\ q(L,L) &= S(L) \end{aligned} \quad (35)$$

Filtering estimate of $z(k)$: $\widehat{z}(k,k) = H\widehat{x}(k,k)$

Filtering estimate of $x(k)$: $\widehat{x}(k,k)$

$$\begin{aligned} \widehat{x}(k,k) &= \Phi\widehat{x}(k-1,k-1) + K_g(k) \\ &\times (y(k) - H\Phi\widehat{x}(k-1,k-1)),\ \widehat{x}(0,0) = 0 \end{aligned} \quad (36)$$

Auto-variance function of the filtering estimate $\widehat{x}(k,k)$: $S(k)$

$$S(k) = K_x(k-1,k-1) - Z(k-1)Z^T(k-1) = \widetilde{U}(k-1)\widetilde{D}(k-1)\widetilde{U}^T(k-1), \quad (37)$$

Here, $\widetilde{U}(k-1$ and $\widetilde{D}(k-1)$ are calculated by the UD factorization of

$$\begin{bmatrix} K_x(k-1,k-1) & Z(k-1) \\ Z^T(k-1) & I \end{bmatrix} = \begin{bmatrix} \widetilde{U}(k-1) & Z(k-1) \\ 0 & I \end{bmatrix} \times \begin{bmatrix} \widetilde{D}(k-1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{U}^T(k-1) & 0 \\ Z^T(k-1) & I \end{bmatrix}. \quad (38)$$

$Z(k-1)$ on the left hand side of (38) is obtained, in terms of the orthogonal matrix $Q(k-1)$, with $Q(k-1)Q^T(k-1) = Q^T(k-1)Q(k-1) = I$, by the lower triangularization as

$$\begin{aligned} &\begin{bmatrix} R^{0.5} & HK_{\widetilde{x}}^{0.5}(k-1,k-1) \\ 0 & K_{\widetilde{x}}^{0.5}(k-1,k-1) \end{bmatrix} Q(k-1) = \begin{bmatrix} X(k-1) & 0 \\ Y(k-1) & Z(k-1) \end{bmatrix}, \\ &K_{\widetilde{x}}(k-1,k-1) = K_x(k,k) - \Phi S(k-1)\Phi^T. \end{aligned} \quad (39)$$

Filter gain $K_g(k)$

$$\begin{aligned} K_g(k) &= U_1(k-1)D_1(k-1)U_1^T(k-1)H^T \\ &\quad \times (R + HU_1(k-1)D_1(k-1)U_1^T(k-1)H^T)^{-1} \\ &= U_1(k-1)D_1(k-1)U_1^T(k-1)H^T \\ &\quad \times (X(k-1)X^T(k-1))^{-1} \end{aligned} \quad (40)$$

Herein, $X(k-1)$ is calculated by (39).

$K_{\widetilde{x}}(k-1,k-1)$ in (39) is UD factorized as $K_{\widetilde{x}}(k-1,k-1) = U_1(k-1)D_1(k-1)U_1^T(k-1)$.

Here, $U_1(k-1)$ and $D_1(k-1)$ are calculated by the UD factorization of

$$\begin{bmatrix} K_x(k-1,k-1) & \Phi S^{0.5}(k-1) \\ S^{0.5}(k-1)\Phi^T & I \end{bmatrix} = \begin{bmatrix} U_1(k-1) & \Phi S^{0.5}(k-1) \\ 0 & I \end{bmatrix} \begin{bmatrix} D_1(k-1) & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} U_1^T(k-1) & 0 \\ S^{0.5}(k-1)\Phi^T & I \end{bmatrix}. \quad (41)$$

An alternative square-root estimation algorithms is obtained by replacing $S^{0.5}(k-1)$ with $\widetilde{U}(k-21)\widetilde{D}^{0.5}(k-2)$ in [**Theorem 2**].

4. A NUMERICAL SIMULATION EXAMPLE

Let a scalar observation equation be given by

$$y(k) = z(k) + v(k). \quad (42)$$

Let the observation noise $v(k)$ be zero-mean white Gaussian process with the variance R, $N(0,R)$. Let the auto-covariance function $K(\cdot)$ of the signal $z(k)$ be given by

$$\begin{aligned} &K(0) = \sigma^2, \\ &K(m) = \sigma^2\{\alpha_1(\alpha_2^2-1)\alpha_1^m/[(\alpha_2-\alpha_1)(\alpha_1\alpha_2+1)] \\ &-\alpha_2(\alpha_1^2-1)\alpha_2^m/[(\alpha_2-\alpha_1)(\alpha_1\alpha_2+1)]\},\ m > 0, \end{aligned} \quad (43)$$

$$\begin{aligned} &\alpha_1, \alpha_2 = (-a_1 \pm \sqrt{a_1^2 - 4a_2})/2, \\ &a_1 = -0.1, a_2 = -0.8,\ \sigma = 0.5. \end{aligned} \quad (44)$$

By referring to Nakamori [1997], the observation vector H, the variance $K_x(k,k)$ of the state vector $x(k)$ and the system matrix Φ in the state equation for the state vector are expressed as follows:

$$\begin{aligned} &H = [\,1\ \ 0\,],\ K_x(k,k) = \begin{bmatrix} K(0) & K(1) \\ K(1) & K(0) \end{bmatrix}, \\ &\Phi = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix},\ K(0) = 0.25,\ K(1) = 0.125. \end{aligned} \quad (45)$$

If we substitute (44) and (45) into the RLS Wiener estimation algorithms in [**Theorem 1**] and the square-root estimation algorithms obtained by replacing $S^{0.5}(k-1)$ with $\widetilde{U}(k-21)\widetilde{D}^{0.5}(k-2)$ in [**Theorem 2**], we can calculate the fixed-point smoothing estimate, at the fixed point, and the filtering estimate of the signal recursively.

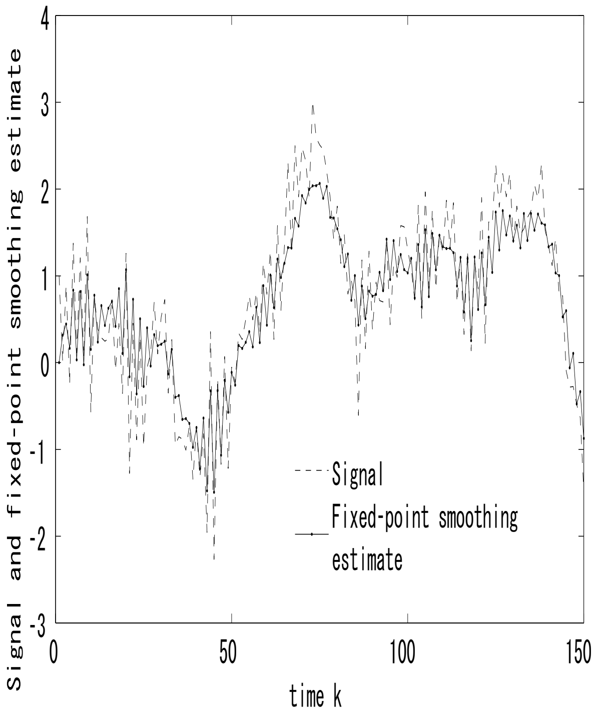

Fig. 1. Signal $z(k)$ and the fixed-point smoothing estimate $\hat{z}(k, k+5)$ in single precision by the square-root RLS Wiener estimation algorithms in [**Theorem 2**] vs. k for the white Gaussian observation noise $N(0, 0.5^2)$.

Table 1 Mean-square values of filtering errors by the square-root RLS Wiener filtering algorithm of [**Theorem 2**] in single precision and by the RLS Wiener filtering algorithm of [**Thorem 1**] in double precision.

Observation noise	MSV of filtyering errors by [Theorem 1]	MSV of filtyering errors by [Theorem 2]
$N(0, 0.1^2)$	0.0989	0.0989
$N(0, 0.3^2)$	0.2103	0.2151
$N(0, 0.5^2)$	0.3150	0.32265
$N(0, 1)$	0.4494	0.4575

Table 2 Mean-square values of filtering errors, in the case of the computation to the efficient 4 decimal places, by the square-root RLS Wiener filtering algorithm in [**Theorem 2**] and by the RLS Wiener filtering algorithm in [**Theorem 1**].

Observation noise	MSV of filtyering errors by [Theorem 1]	MSV of filtyering errors by [Theorem 2]
$N(0, 0.1^2)$	0.0937	0.1005
$N(0, 0.3^2)$	0.2602	0.2151
$N(0, 0.5^2)$	0.5101	0.3225
$N(0, 1)$	1.0401	0.4574

Fig.1 illustrates the signal $z(k)$ and the fixed-point smoothing estimate $\widehat{z}(k, k+5)$ in single precision by the square-root RLS Wiener estimation algorithms in [**Theorem 2**] vs. k for the white Gaussian observation noise $N(0, 0.5^2)$. **Fig.2** illustrates the mean-square values (MSVs) of the fixed-point smoothing errors $z(k) -$

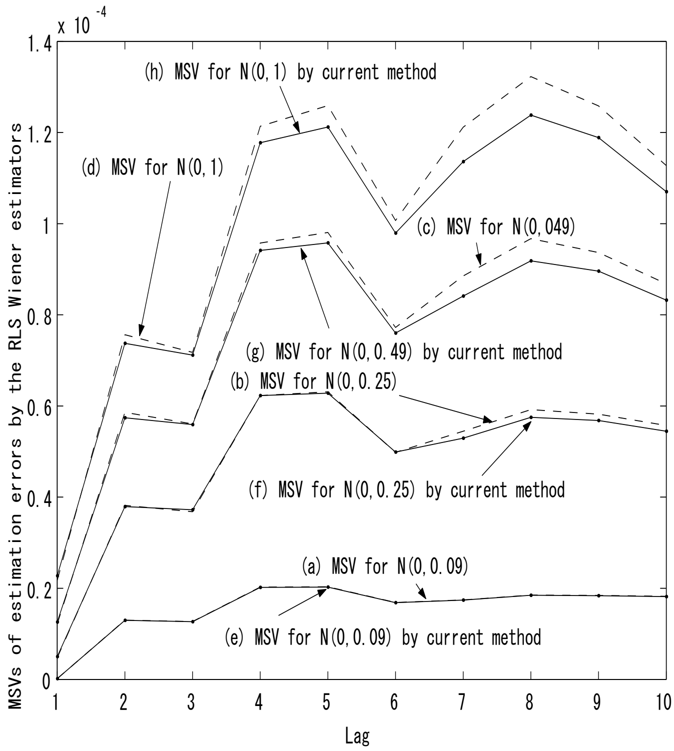

Fig. 2. Mean-square values of the fixed-point smoothing errors $z(k) - \hat{z}(k, k + Lag)$ vs. Lag, $1 \leq Lag \leq 10$, by the RLS Wiener estimation algorithms, in single precision, of [**Theorem 2**] and, in double precision, of [**Theorem 1**] for the observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$, $N(0, 0.7^2)$ and $N(0, 1)$.

$\widehat{z}(k, k + Lag)$ vs. lag, $1 \leq Lag \leq 10$, by the RLS Wiener estimators of [**Theorem 1**] and the square-root RLS Wiener estimators for the observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$, $N(0, 0.7^2)$ and $N(0, 1)$. In **Fig.1** and graphs (e), (f), (g) and (h), in **Fig.2**, the square-root estimates are computed in single precision by MATLAB 7.4. Also, the QR decomposition and the UD factorization Oguni and Dongarra [1988] are computed based on the single precision programs. In graphs (a), (b), (c) and (d) in **Fig.2**, the fixed-point smoothing estimate is calculated in double precision. As shown in **Fig.2**, the MSVs of the fixed-point smoothing errors by the square-root RLS Wiener estimators are less than those by the RLS Wiener estimators of [**Theorem 1**] in the case of the observation noises $N(0, 0.5^2)$ for $7 \leq Lag \leq 10$, $N(0, 0.7^2)$ for $2 \leq Lag \leq 10$, and $N(0, 1)$ for $2 \leq Lag \leq 10$. **Table 1** shows the MSVs of the filtering errors by the square-root RLS Wiener filtering algorithm in single precision and by the RLS Wiener filtering algorithm of [**Theorem 1**] in double precision. From **Table 1,** it is seen that the both filtering algorithms show the almost same estimation accuracies. **Fig.3,** in the case of the computation to the efficient 4 decimal places, illustrates the MSVs of the fixed-point smoothing errors by the estimators of [**Theorem 1**] and the square-root estimators. Graphs (a), (b), (c) and (d) are computed by the estimators in [**Theorem 1**] and graphs (e), (f), (g) and (h) are computed by the square-root estimators. For $lag = 1$, the MSV by the square-root RLS Wiener fixed-point smoother are smaller than that by the RLS Wiener fixed-point smoother for respective

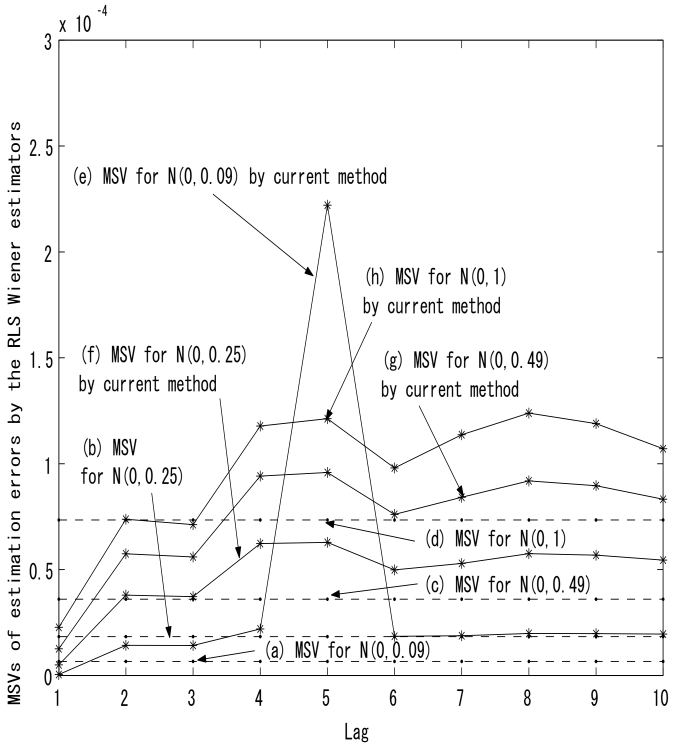

Fig. 3. Mean-square values of the fixed-point smoothing errors $z(k) - \hat{z}(k, k + Lag)$ vs. Lag, $1 \leq Lag \leq 10$, by the RLS Wiener estimation algorithms, in the case of the computation to the efficient 4 decimal places, of [**Theorem 2**] and, in the double precision, of [**Theorem 1**] for the observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$, $N(0, 0.7^2)$ and $N(0, 1)$.

observation noise. **Table 2,** in the case of the computation to the efficient 4 decimal places, shows the MSVs of the filtering errors by the square-root RLS Wiener filtering algorithm of [**Theorem 2**] and by the RLS Wiener filtering algorithm in [**Theorem 1**]. As the variance of the observation noise becomes large, in comparison with the MSV calculated under the double precision, the MSV of the filtering errors by the filter in [**Theorem 1**] has a tendency to increase. However, the proposed square-root filter, in the case of the computation to the efficient 4 decimal places, has almost the same estimation accuracy with the filter of [**Thorem 1**] under double precision. Here, the MSVs of the fixed-point smoothing and filtering errors are calculated by $\sum_{k=1}^{2000}(z(k)-\widehat{z}(k, k+Lag))^2/2000$ and $\sum_{k=1}^{2000}(z(k) - \widehat{z}(k, k))^2/2000$ respectively.

For references, the AR model, which generates the signal process, is given by

$$z(k + 1) = -a_1 z(k) - a_2 z(k - 1) + w(k + 1), \qquad (46)$$

$$E[w(k)w(s)] = Q\delta_K(k - s),\ Q = 0.25. \qquad (47)$$

5. CONCLUSION

This paper, to reduce or avoid the roundoff errors, proposed the QR decomposition and UD factorization based square-root algorithms of the RLS Wiener fixed-point smoother and filter. The numerical simulation results by MATLAB 7.4, for the single precision computation, have shown that the proposed square-root RLS Wiener estimation algorithms have a tendency to improve estimation accuracy for relatively large variance of the observation noise in comparison with the RLS Wiener estimation algorithms under double precision cercumstances. Hence, the proposed square-root estimators have feasible estimation characteristics in comparison with the RLS Wiener estimation algorithms in [**Theorem 1**], which calculate the Riccati-type difference equations directly.

Double-precision variables use 64 bits (8 bytes) of memory storage and the values are accurately represented to 15 decimal places. The proposed square-root filtering algorithm is superior in estimation accuracy with less memory in the case of the computation to the efficient 4 decimal places, to the RLS Wiener filtering algorithm.

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