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Adaptive Regulator for Uncertain Linear Minimum Phase Systems with Unknown Undermodeled Exosystems ⋆

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Abstract: The design of regulators is addressed for uncertain minimum phase linear systems with known bounds, known upper bound on system order, known relative degree, known high frequency gain sign and for exosystems with unknown order and unknown frequencies with known upper bound. A new adaptive output error feedback control algorithm is proposed which guarantees exponential convergence of the output error into a region which decreases with the order of the unmodeled exosystem dynamics. Exponential regulation is obtained when the regulator exactly models all of the exosystem excited frequencies, while asymptotic regulation is achieved when the regulator overmodels the actual exosystem.

1. INTRODUCTION

The basic result from linear regulator theory states that disturbances and/or reference signals generated by an exosystem, whose eigenvalues do not coincide with the zeros of a stabilizable and detectable linear system, can be rejected and/or tracked by an output error feedback control which incorporates the exosystem itself (internal model principle): it is remarkable that the system is not required to be minimum phase (see Davison (1976); Francis and Wonham (1976)). On the other hand, a fundamental result from adaptive control theory states that arbitrary smooth output reference signals with known time derivatives can be asymptotically tracked even though the system is uncertain, provided that the system order, the relative degree and the high frequency gain sign are known and the system is minimum phase (see for instance Sastry and Bodson (1989); Marino and Tomei (1995)).

It is of course of interest to explore whether the regulator theory may be extended to classes of models and/or exosystems containing uncertain or unknown parameters. For instance, as far as the regulator theory is concerned, when disturbances contain sinusoidal terms it is not reasonable to assume that their frequencies or their maximum number are known, or, equivalently, that the exosystem and its order are known. As far as the adaptive control is concerned, the knowledge of the output reference signal and its time derivatives is a restrictive assumption when only the tracking error is available for measurements. It would be desirable to bridge the gap between the two theories and to develop a regulator theory for uncertain linear systems leading to the design of regulators which, on the basis of the tracking error, can track reference signals and/or reject disturbances which are generated by a linear exosystem whose parameters and order are unknown. Such a theory is very much needed in applications such as noise cancellation, synchronization, active suspensions, excentricity compensation, learning control with periodic references of uncertain period, pointing systems subject to periodic disturbances. Some results in this direction have been already developed. In Marino and Tomei (2003) it is shown how to incorporate an adaptive internal model in the regulator design using adaptive observers when the exosystem is unknown: the system is required to be known and the exact number of frequencies contained in the reference and in the disturbance is assumed to be known. In Marino and Santosuosso (2004) and Marino and Tomei (2005) this last assumption is relaxed and only an upper bound on the exosystem order is required. An indirect adaptive approach is followed in Marino and Santosuosso (2004) where the regulator design incorporates an adaptive internal model whose order is identified on line. A direct adaptive approach gives a simpler algorithm in Marino and Tomei (2005) which does not require the identification of the exosystem order and of the exosystem frequencies but it is restricted to minimum phase systems. In Marino and Tomei (2006) this approach has been extended to uncertain plants. Related results for nonlinear systems may be found in Ding (2003); Ye and Huang (2003).

The aim of this paper is to present a new adaptive regulator which is restricted to minimum phase systems but allows for uncertain linear systems with known bounds, known relative degree and high frequency gain sign and for exosystems of any unknown finite order with uncertain frequencies with known bounds. The main novelty is that an upper bound on the system order is no longer required. The proposed robust adaptive output error feedback control algorithm relies on an exosystem which may undermodel the actual exosystem generating the disturbances and /or the output reference. It achieves asymptotic regulation with bounded closed loop signals when the adaptive regulator overmodels the actual exosystem which generates all disturbances and references. Exponential regulation is obtained when the regulator exactly models the excited frequencies of the actual exosystem. When the adaptive regulator undermodels the actual exosystem the

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tracking error is exponentially reduced to a residual bound which decreases as the order of the unmodeled exosystem dynamics decreases: robustness is achieved with respect to unmodeled exosystem dynamics. An example is worked out and simulated to illustrate the proposed control algorithm.

2. MAIN RESULT

Consider the regulator problem for an observable minimum phase system of constant known relative degree ρ

$$\begin{aligned}\dot{x} &= A_c x + ay + \frac{1}{\beta} bu + Pw,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}\\ y &= C_c x,\ y \in \mathbb{R}\\ \dot{w} &= Rw,\ w \in \mathbb{R}^{2m+1}\\ y_r &= qw\end{aligned} \tag{1}$$

in which n is the known upper bound on the unknown system order $\bar{n}$, the pair (A_c, C_c) is in observer canonical form

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ 0 & 0 & \cdots & 0\end{bmatrix}, \qquad C_c = [\,1\ 0\ \cdots\ 0\,]$$

$a = [a_1, \dots, a_n]^T$, $b = [0, \dots, 0, 1, b_{\rho+1}, \dots, b_n]^T$ are unknown vectors, $\beta > 0$ is an unknown positive real and P, R, q are unknown matrices; the polynomial $s^{n-\rho} + b_{\rho+1}s^{n-\rho-1} + \cdots + b_n$ has all roots with negative real part. The unknown parameters of the system, grouped in the vector $\alpha = [a_1, \dots, a_n, \frac{1}{\beta}, b_{\rho+1}, \dots, b_n]^T$, belong to the known region Ω_1, $\alpha \in \Omega_1 \subset \mathbb{R}^{2n-\rho+1}$. The disturbances Pw and reference signals to be tracked qw are both generated by a linear exosystem $\dot{w} = Rw$, $w \in \mathbb{R}^{2m+1}$. We assume that the spectrum of the matrix R of the exosystem is $\{0, \pm j\omega_i, 1 \le i \le m\}$ with ω_i unknown distinct positive parameters and m an unknown integer so that the characteristic polynomial of R is given by

$$s\prod_{i=1}^{m}(s^2 + \omega_i^2) \triangleq s(s^{2m} + \theta_1 s^{2m-2} + \cdots + \theta_m)\ .$$

Let $\bar{m}$ be an estimate of the unknown number m which will be used by the controller. We assume that the unknown parameters of the estimated exosystem, grouped in the vector $\theta = [\theta_1, \dots, \theta_{\bar{m}}]^T$, belong to a known closed ball Ω_2 centered at the origin of radius r_{Ω_2}, $\theta \in \Omega_2 \subset \mathbb{R}^{\bar{m}}$. The regulator problem is stated in the following definition.

Definition 2.1. We say that the regulator problen is globally solvable for system (1) if there exists a dynamic (adaptive) output error feedback control $\dot{\nu} = f_\nu(\nu, y - y_r)$, $\nu(0) = \nu_0$, $\nu \in \mathbb{R}^{s+\bar{m}}$, $u = f_u(\nu, y - y_r)$, such that all closed loop signals are bounded and, for any initial condition $(x(0), w(0)$: (i) if $\bar{m} > m$, then $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$; (ii) if $\bar{m} = m$, then $|y(t) - y_r(t)| \le c_a e^{-c_b t}$, $\forall t \ge 0$; (iii) if $\bar{m} < m$, then $|y(t) - y_r(t)| \le c_a e^{-c_b t} + \varphi(m - \bar{m})$, $\forall t \ge 0$, in which $\varphi(\cdot)$ is a class-k function and c_a, c_b are positive reals depending on the initial conditions.

Theorem 2.1. Consider system (1). There exists an adaptive dynamic output error feedback control which globally solves the regulator problem.

Proof. Consider first the case $\bar{m} < m$ with $\rho = 1$. Since system (1) is observable and minimum phase (and therefore stabilizable) and the eigenvalues of the exosystem are on the imaginary axis, there exists a solution (π, c) to the regulator matrix equations (see Davison (1976); Francis and Wonham (1976); Wonham (1979))

$$\begin{aligned}\pi R &= A_c\pi + aq + \frac{1}{\beta}bc + P\\ q &= C_c\pi\end{aligned} \tag{2}$$

so that defining $x_r = \pi w$, $u_r = cw$, we can write

$$\begin{aligned}\dot{x}_r &= A_c x_r + ay_r + \frac{1}{\beta}bu_r + Pw\\ y_r &= C_c x_r\ .\end{aligned}$$

The regulator error equations ($\tilde{x} = x - x_r$, $e = y - y_r$) become

$$\begin{aligned}\dot{\tilde{x}} &= A_c\tilde{x} + ae + \frac{1}{\beta}b(u - cw)\\ e &= C_c\tilde{x}\\ \dot{w} &= Rw, \quad w \in \mathbb{R}^{2m+1}\ .\end{aligned} \tag{3}$$

Without loss of generality, we can assume that the pair (R, c) is observable [if this is not true, we may consider only the observable part of (R, c) obtained by Kalman decomposition] so that to generate u_r we may equivalently consider its observer canonical form

$$\begin{aligned}\dot{w}_c &= R_c w_c, \qquad w_c \in \mathbb{R}^{2m+1}\\ u_r &= \gamma_c w_c\end{aligned} \tag{4}$$

with

$$R_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ -\theta_1 & 0 & 1 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ -\theta_m & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}, \qquad \gamma_c = [\,1\ 0\ \cdots\ 0\,]\ .$$

Since the order of the exosystem is not known, we assume that its value is $2\bar{m}+1$ ($\bar{m} < m$, for the first part of the proof) so that from (3), (4) and using $\bar{m}$ in place of m, we obtain

$$\begin{aligned}\dot{\tilde{x}} &= A_c\tilde{x} + ae + \frac{1}{\beta}b[u - \bar{w}_{c1} - r(t)]\\ e &= C_c\tilde{x}\\ \dot{\bar{w}}_c &= \bar{R}_c\bar{w}_c, \quad \bar{w}_c \in \mathbb{R}^{2\bar{m}+1}\end{aligned} \tag{5}$$

in which

$$\bar{R}_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ -\theta_1 & 0 & 1 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ -\theta_{\bar{m}} & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}$$

and

$$w_{c1} = \bar{w}_{c1} + r(t) \tag{6}$$

with $r(t)$ representing the unmodeled $m - \bar{m}$ sinusoids. Defining the change of coordinates

$$\eta_i = \tilde{x}_{i+1} - b_{i+1}\tilde{x}_1, \qquad 1 \le i \le n-1 \tag{7}$$

we have ($\eta = [\eta_1, \ldots, \eta_{n-1}]^T$),

$$\begin{aligned}
\dot{e} &= \eta_1 + b_2 e + a_1 e + \frac{1}{\beta}(u - \bar{w}_{c1} - r) \\
\dot{\eta} &= \begin{bmatrix} -b_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -b_{n-1} & 0 & \cdots & 1 \\ -b_n & 0 & \cdots & 0 \end{bmatrix} \eta \\
&\quad + e \begin{bmatrix} b_3 - b_2^2 + a_2 - b_2 a_1 \\ \vdots \\ b_n - b_2 b_{n-1} + a_{n-1} - b_{n-1} a_1 \\ -b_n b_2 + a_n - b_n a_1 \end{bmatrix} \\
&\triangleq \Gamma\eta + de \\
\dot{\bar{w}}_c &= \bar{R}_c \bar{w}_c \; .
\end{aligned} \tag{8}$$

The further change of coordinates

$$e = e, \quad \eta = \eta, \quad \bar{w} = -\bar{w}_c + \beta e \begin{bmatrix} 0 \\ \theta_1 \\ 0 \\ \theta_2 \\ \vdots \\ 0 \\ \theta_{\bar{m}} \\ 0 \end{bmatrix} \tag{9}$$

transforms (8) into

$$\begin{aligned}
\begin{bmatrix} \beta\dot{e} \\ \dot{\bar{w}} \end{bmatrix} &= A_c \begin{bmatrix} \beta e \\ \bar{w} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \theta_1 \\ 0 \\ \theta_2 \\ \vdots \\ 0 \\ \theta_{\bar{m}} \\ 0 \end{bmatrix} [\beta\eta_1 + \beta(b_2 + a_1)e + u - r] \\
\dot{\eta} &= \Gamma\eta + de
\end{aligned} \tag{10}$$

Define the filtered transformation ($\xi_i \in \mathbb{R}^{2\bar{m}+1}$)

$$\begin{aligned}
\dot{\xi}_i &= D\xi_i + [\,0 \;\; I_{2\bar{m}+1}\,] E_{2i+1} u \\
\mu_i &= [\,1 \; 0 \; \cdots \; 0\,]\,\xi_i, \qquad 1 \le i \le \bar{m}
\end{aligned} \tag{11}$$

$$z = \begin{bmatrix} \beta e \\ \bar{w} \end{bmatrix} - \begin{bmatrix} 0 \\ \sum_{i=1}^{\bar{m}} \xi_i \theta_i \end{bmatrix}, \quad z \in \mathbb{R}^{2\bar{m}+2} \tag{12}$$

with

$$D = \begin{bmatrix} -d_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_{2\bar{m}+1} & 0 & \cdots & 1 \\ -d_{2\bar{m}+2} & 0 & \cdots & 0 \end{bmatrix}$$

a Hurwitz matrix, $I_{2\bar{m}+1}$ the $(2\bar{m}+1) \times (2\bar{m}+1)$ identity matrix and E_j a vector of suitable dimension with all zero entries except for the jth element which is equal to 1. From (11) and (12), we obtain

$$\begin{aligned}
\dot{z} &= A_c z + \sum_{i=1}^{\bar{m}} \theta_i E_{2i+1}[\beta\eta_1 + \beta(b_2 + a_1)e] \\
&\quad + d\mu^T\theta + E_1[u + \beta\eta_1 + \beta(b_2 + a_1)e - r] \\
\beta e &= C_c z
\end{aligned} \tag{13}$$

with $\mu = [\mu_1, \ldots, \mu_{\bar{m}}]^T$ and $d = [1, d_2, \ldots, d_{2\bar{m}+2}]^T$. Define the linear change of coordinates

$$\chi_i = z_{i+1} - d_{i+1} z_1, \qquad 1 \le i \le 2\bar{m}+1 \; . \tag{14}$$

From (13) and (14), we have

$$\begin{aligned}
\beta\dot{e} &= \chi_1 + u + \beta[\eta_1 + (b_2 + a_1)e] + d_2\beta e + \mu^T\theta - r \\
&\triangleq \chi_1 + u - r + c_1 e + \beta\eta_1 + \mu^T\theta \\
\dot{\chi} &= D\chi + \beta e \begin{bmatrix} d_3 - d_2^2 \\ \vdots \\ d_{2\bar{m}+2} - d_2 d_{2\bar{m}+1} \\ -d_{2\bar{m}+2} d_2 \end{bmatrix} - \begin{bmatrix} d_2 \\ \vdots \\ d_{2\bar{m}+2} \end{bmatrix} [u + \beta(\eta_1 \\
&\quad + (b_2 + a_1)e) - r] + \sum_{i=1}^{\bar{m}} \theta_i E_{2i}[\beta\eta_1 + \beta(b_2 + a_1)e] \\
&\triangleq D\chi + f_1 e + f_2 \eta_1 - \bar{d}u + \bar{d}r
\end{aligned} \tag{15}$$

with f_1 and f_2 unknown vectors, c_1 unknown scalar and $\bar{d}$ a known vector. Introduce the observer

$$\dot{\hat{\chi}} = D\hat{\chi} - \bar{d}u \tag{16}$$

with error dynamics ($\tilde{\chi} = \chi - \hat{\chi}$)

$$\dot{\tilde{\chi}} = D\tilde{\chi} + f_1 e + f_2 \eta_1 + \bar{d}r \; . \tag{17}$$

Let u be defined as

$$\begin{aligned}
u &= -\hat{\chi}_1 - \mu^T\hat{\theta} - ke \\
\dot{\hat{\theta}} &= g\,\mathrm{Proj}(\mu e, \hat{\theta})
\end{aligned} \tag{18}$$

where $g > 0$ and $k > 0$ are the adaptation and the control gains, and $\mathrm{Proj}(\cdot,\cdot)$ is the smooth projection operator defined as (see Pomet and Praly (1992))

$$\begin{aligned}
&\mathrm{Proj}(\varphi, \hat{\theta}) = \varphi, \text{ if } p_r(\hat{\theta}) \le 0 \\
&\mathrm{Proj}(\varphi, \hat{\theta}) = \varphi, \text{ if } p_r(\hat{\theta}) \ge 0 \text{ and } \langle \mathrm{grad}\; p_r(\hat{\theta}), \varphi \rangle \le 0 \\
&\mathrm{Proj}(\varphi, \hat{\theta}) = \left[I - \frac{p_r(\hat{\theta}) \mathrm{grad}\; p_r(\hat{\theta}) \mathrm{grad}\; p_r(\hat{\theta})^T}{\|\mathrm{grad}\; p_r(\hat{\theta})\|^2} \right] \varphi, \\
&\qquad \text{if } p_r(\hat{\theta}) > 0 \text{ and } \langle \mathrm{grad}\; p_r(\hat{\theta}), \varphi \rangle > 0
\end{aligned}$$

with $p_r(\hat{\theta}) = \frac{\|\hat{\theta}\|^2 - r_{\Omega_2}^2}{\epsilon_r^2 + 2\epsilon_r r_{\Omega_2}}$, and ϵ_r an arbitrary positive real. If $\|\hat{\theta}(0)\| \le r_{\Omega_2}$ then, $\forall t \ge 0$: (a1) $\|\hat{\theta}(t)\| \le r_{\Omega_2} + \epsilon_r$; (a2) $\mathrm{Proj}(\varphi, \hat{\theta})$ is Lipschitz continuous; (a3) $\|\mathrm{Proj}(\varphi, \hat{\theta})\| \le \|\varphi\|$; (a4) $\tilde{\theta}^T \mathrm{Proj}(\varphi, \hat{\theta}) \ge \tilde{\theta}^T \varphi$. Consider the Lyapunov function ($\epsilon > 0$)

$$V = \frac{1}{2}\beta e^2 + \eta^T P_1 \eta + \epsilon \tilde{\chi}^T P_2 \tilde{\chi} + \frac{1}{2g}\tilde{\theta}^T\tilde{\theta} \tag{19}$$

with P_1 and P_2 the symmetric positive definite solutions of $\Gamma^T P_1 + P_1\Gamma = -I$, $D^T P_2 + P_2 D = -I$. From (10), (15), (17) and (18), we have

$$\begin{aligned}\dot V = & e(\tilde\chi_1 + c_1 e + \beta\eta_1 + \mu^T\tilde\theta - ke - r) - \eta^T\eta \\ & +2\eta^T P_1 de - \epsilon\tilde\chi^T\tilde\chi + 2\epsilon\tilde\chi^T P_2(f_1 e + f_2\eta_1 + \bar d r) \\ & -\tilde\theta \mathrm{Proj}(\mu e, \hat\theta)\ . \end{aligned} \quad (20)$$

Since

$$\begin{aligned} e\beta\eta_1 + 2\eta^T P_1 De &\le \frac{1}{4}\|\eta\|^2 + h_1 e^2 \\ e\tilde\chi_1 + 2\epsilon\tilde\chi^T P_2 f_1 e &\le \frac{\epsilon}{4}\|\tilde\chi\|^2 + h_2 e^2 \\ 2\epsilon\tilde\chi^T P_2 f_2\eta_1 &\le \frac{\epsilon}{4}\|\tilde\chi\|^2 + \epsilon h_3\|\eta\|^2 \\ er &\le \frac{1}{4}e^2 + r^2 \\ 2\epsilon\tilde\chi^T P_2\bar d r &\le \frac{\epsilon}{4}\|\tilde\chi\|^2 + \epsilon h_4 r^2 \end{aligned} \quad (21)$$

with $\{h_1, h_2, h_3, h_4\} = \max_{\substack{\alpha\in\Omega_1\\ \theta\in\Omega_2}} \{(\beta + 2\|P_1 D\|)^2, \frac{1}{\epsilon}(1 + 2\epsilon\|P - 2f_1\|)^2, 4\|P_2 f_1\|^2, 4\|P_2\bar d\|^2\}$, recalling property (a4) of Proj, from (20) and (21), we obtain

$$\begin{aligned}\dot V \le & -e^2\left(k - c_1 - h_1 - h_2 - \frac{1}{4}\right) - \|\eta\|^2\left(1 - \frac{1}{4} - \epsilon h_3\right) \\ & -\frac{1}{4}\epsilon\|\tilde\chi\|^2 + r^2(1 + \epsilon h_4) - \|\tilde\theta\|^2 + \|\tilde\theta\|^2 \end{aligned} \quad (22)$$

so that if

$$\epsilon < \frac{3}{4h_3},\ k > c_{1M} + h_1 + h_2 + \frac{1}{4} \quad (23)$$

with $c_{1M} = \max_{\alpha\in\Omega_1}(c_1)$, the inequality (22) implies

$$\dot V \le -c_v V + (1 + \epsilon h_4)r^2 + \|\tilde\theta\|^2 \quad (24)$$

where $c_v > 0$ is a suitable real. Equation (24) in turn, since $r(t)$ and $\tilde\theta(t)$ are bounded [recall property (a1) of Proj], implies that $e(t)$, $\|\eta(t)\|$, $\|\tilde\chi(t)\|$ are bounded. Consequently, $\|\tilde x(t)\|$ is also bounded. With reference to (12), we note that $\mu_i(t)$ may be equivalently generated by the following filters with proper initial conditions ($\bar\xi_i[j](t) \in \mathbb{R}^{2\bar m+1}$, $1 \le i \le \bar m$, $j = 1, 2$)

$$\begin{aligned} \dot{\bar\xi}_i[1] &= D\bar\xi_i[1] + \beta(E_{2i-1} - E_{2i}(b_2 + a_1))e - \beta E_{2i}\eta_1 \\ \mu_i[1] &= [\,1\ 0\ \cdots\ 0\,]\,\bar\xi_i[1], \quad 1 \le i \le \bar m \\ \dot{\bar\xi}_i[2] &= D\bar\xi_i[2] + E_{2i}u_r \\ \mu_i[2] &= [\,1\ 0\ \cdots\ 0\,]\,\bar\xi_i[2], \quad 1 \le i \le \bar m \\ \mu_i &= \mu_i[1] + \mu_i[2], \quad 1 \le i \le \bar m \end{aligned} \quad (25)$$

by means of the relations

$$\bar\xi_i[1] + \bar\xi_i[2] = \xi_i - \beta E_{2i}e, \qquad 1 \le i \le \bar m\ . \quad (26)$$

Since $u_r(t) = \bar w_{c1}(t) + r(t)$ and $\bar m < m$, $u_r(t)$ is a bounded persistently exciting signal of order $\bar m$ and (see Sastry and Bodson (1989)) $\mu[2] = [\mu_1[2], \ldots, \mu_{\bar m}[2]]^T$ is a bounded persistently exciting vector of order $\bar m$. This fact implies that (see Marino et al. (2001)) the solution of the matrix differential equation

$$\dot Q = -Q + \mu[2]\mu^T[2], \quad Q(0) = e^{-T_p}k_p I \quad (27)$$

with T_p and k_p positive reals satisfying

$$\int_t^{t+T_p} \mu[2](\tau)\mu^T[2](\tau)d\tau \ge k_p I, \quad \forall t \ge 0$$

is such that

$$\sup_{t\ge 0}\|\mu[2](t)\|^2 I \ge Q(t) \ge k_p e^{-2T_p} I, \quad \forall t \ge 0\ . \quad (28)$$

Consider the function

$$W = V + p\|Q\tilde\theta - \mu[2]\beta e\|^2 + p_1\sum_{i=1}^{\bar m}\bar\xi_i^T[1]P_2\bar\xi_i[1] \quad (29)$$

where p and p_1 are suitable positive reals yet to be defined. Recalling (15), (18), (22), (24), (25) and (27), and since (recall that $\mu[1]$, $\mu[2]$ and $\dot\mu[2]$ are bounded)

$$\begin{aligned} & 2p[Q\tilde\theta - \mu[2]\beta e]^T[(-Qg\mathrm{Proj}(\mu e, \hat\theta) - \beta\dot\mu[2]e - \mu[2](c_1 \\ & -k + \beta)e - \mu[2]\beta\eta_1 - \mu[2]\tilde\chi_1] \\ & \le \frac{p}{4}\|Q\tilde\theta - \mu[2]\beta e\|^2 + 4ph_5\left\|\begin{bmatrix} e\\ \eta\\ \tilde\chi\end{bmatrix}\right\|^2 \\ & 2p[Q\tilde\theta - \mu[2]\beta e]^T\mu[2]r \le \frac{p}{4}\|Q\tilde\theta - \mu[2]\beta e\|^2 + 4ph_6 r^2 \\ & 2p[Q\tilde\theta - \mu[2]\beta e]^T\mu[2]\mu^T[1]\tilde\theta \\ & \le \frac{p}{4}\|Q\tilde\theta - \mu[2]\beta e\|^2 + 4ph_7\|\mu[1]\|^2 \\ & 2p_1\sum_{i=1}^{\bar m}\bar\xi_i^T[1]P_2[\beta(E_{2i-1} - E_{2i}(b_2 + a_1))e - \beta E_{2i}\eta_1] \\ & \le \frac{p_1}{2}\sum_{i=1}^{\bar m}\|\bar\xi_i[1]\|^2 + 2p_1h_8\left\|\begin{bmatrix} e\\ \eta_1\end{bmatrix}\right\|^2 \end{aligned}$$

with $h_5 \ge \sup_{t\ge 0}\{\|g\|Q\|\mu(t) + \beta\dot\mu[2](t) + \mu[2](t)(c_1 - k + \beta)\|^2, \|\mu[2](t)\beta\|^2, \|\mu[2](t)\|^2\}$, $h_6 \ge \sup_{t\ge 0}\{\|\mu[2](t)\|^2\}$, $h_7 \ge \sup_{t\ge 0}\{\|\mu[2](t)\tilde\theta(t)\|^2\}$,

$$h_8 \ \ge\ \|P_2\|^2\beta^2\left\|\begin{bmatrix} E_1 - E_2(b_2 + a_1) & -E_2 \\ \vdots & \vdots \\ E_{2\bar m-1} - E_{2\bar m}(b_2 + a_1) & -E_{2\bar m}\end{bmatrix}\right\|^2,$$ its time derivative is such that

$$\begin{aligned}\dot W \le & -\left\|\begin{bmatrix} e\\ \eta\\ \tilde\chi\end{bmatrix}\right\|^2(c_v - 4ph_5 - 2p_1h_8) - [Q\tilde\theta - \mu[2]\beta e]^2\frac{5}{4}p \\ & -\sum_{i=1}^{\bar m}\|\bar\xi_i[1]\|^2(4ph_7 + \frac{p_1}{2}) + (1 + \epsilon h_4 + 4ph_6)r^2 \end{aligned} \quad (30)$$

By choosing p and p_1 such that

$$p_1 = 16ph_7,\ p < \frac{c_v}{4(h_5 + 8h_7h_8)} \quad (31)$$

from (30) we have, for suitable positive reals c_{w_a}, c_{w_b},

$$W(t) \le W(0)e^{-c_{w_a}t} + \frac{c_{w_b}}{c_{w_a}}\sup_{\tau\ge 0}\{r^2(\tau)\}\ . \quad (32)$$

Since $r(t)$ may be expressed as $r(t) = \sum_{i=1}^{m-\bar{m}} r_i(t)$ with $r_i(t)$ being the contributions of the $m - \bar{m}$ unmodeled exosystem modes, from (32) we obtain

$$e(t) \leq \sqrt{\frac{2}{\beta} W(0)} e^{-\frac{c_{w_a}}{2}} t + \sqrt{\frac{2c_{w_b}}{\beta c_{w_a}}(m-\bar{m})} \sum_{i=1}^{m-\bar{m}} \sup_{\tau \geq 0} |r_i(\tau)|$$

which implies (iii) in Definition 2.1. Now, consider the case $\rho > 1$ and let system (1) be written as

$$\begin{aligned} \dot{\bar{x}} &= A_c \bar{x} + ay + \frac{1}{\beta} \bar{b} u + Pw \\ y &= C_c \bar{x} \end{aligned} \tag{33}$$

with $\bar{b} = [0, \ldots, 0, 1, \bar{b}_{\rho+1}, \ldots, \bar{b}_n]^T$. Consider the filtered transformation ($\lambda_i > 0$, $1 \leq i \leq \rho - 1$)

$$\dot{\phi} = \begin{bmatrix} -\lambda_1 & 1 & 0 \cdots & 0 \\ 0 & -\lambda_2 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots \cdots & \vdots \\ 0 & 0 & 0 \cdots & -\lambda_{\rho-1} \end{bmatrix} \phi + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$x_1 = \bar{x}_1, \; x_i = \bar{x}_i - \frac{1}{\beta} \sum_{j=2}^{\rho} b_i[j] \phi_{j-1}, \; 2 \leq i \leq n. \tag{34}$$

The vectors $b[j] = [b_1[j], \ldots, b_n[j]]^T$ are recursively given by

$$\begin{aligned} b[\rho] &= \bar{b} \\ b[j-1] &= A_c b[j] + \lambda_{j-1} b[j], \; \rho \geq j \geq 2 \\ b[1] &= b = [1, b_2, \ldots, b_n]^T \end{aligned} \tag{35}$$

with b_i, $2 \leq i \leq n$ solutions of $s^{n-1} + b_2 s^{n-2} + \cdots + b_n = (s^{n-\rho} + \bar{b}_{\rho+1} s^{n-\rho-1} + \cdots + \bar{b}_n) \prod_{i=1}^{\rho-1}(s + \lambda_i)$. From (33) and (34), we obtain

$$\begin{aligned} \dot{x} &= A_c x + ay + \frac{1}{\beta} b \phi_1 + Pw \\ y &= C_c x \; . \end{aligned} \tag{36}$$

In this case there exists a solution (π, c) to the regulator equations

$$\begin{aligned} \pi R &= A_c \pi + aq + \frac{1}{\beta} bc + P \\ q &= C_c \pi \end{aligned} \tag{37}$$

so that defining $x_r = \pi w$, $\phi_{1r} = cw$, we can write

$$\begin{aligned} \dot{x}_r &= A_c x_r + a y_r + \frac{1}{\beta} b \phi_{1r} + Pw \\ y_r &= C_c x_r \end{aligned} \tag{38}$$

and the regulator error equations are

$$\begin{aligned} \dot{\tilde{x}} &= A_c \tilde{x} + ae + \frac{1}{\beta} b(\phi_1 - cw) \\ \dot{w} &= Rw, \quad w \in \mathbb{R}^{2m+1} \\ e &= C_c \tilde{x} \end{aligned} \tag{39}$$

which may be equivalently written as

$$\begin{aligned} \dot{\tilde{x}} &= A_c \tilde{x} + ae + \frac{1}{\beta} b(\phi_1 - \bar{w}_{c1} - r) \\ \dot{\bar{w}}_c &= R_c \bar{w}_c, \quad \bar{w}_c \in \mathbb{R}^{2\bar{m}+1} \\ e &= C_c \tilde{x} \; . \end{aligned} \tag{40}$$

By considering ϕ_1 as an input, system (40) is in the form (5) and, therefore, we can follow the same steps of the relative-degree-one case (using ϕ_1 in place of u in the filters (12)) to obtain the ideal control

$$\begin{aligned} \phi_1^* &= -\hat{\chi}_1 - \mu^T \hat{\theta} - ke \\ \dot{\hat{\chi}} &= D\hat{\chi} - \bar{d}\phi_1 \\ \dot{\xi}_i &= D\xi_i + [\, 0 \;\, I_{2\bar{m}+1} \,] E_{2i+1} \phi_1 \\ \mu_i &= [\, 1 \; 0 \; \cdots \; 0 \,] \xi_i, \; 1 \leq i \leq \bar{m} \; . \end{aligned} \tag{41}$$

Defining $\tilde{\phi}_1 = \phi_1 - \phi_1^*$, the error dynamics are given by

$$\begin{aligned} \beta \dot{e} &= \tilde{\chi}_1 + c_1 e + \beta \eta_1 + \mu^T \tilde{\theta} - ke + \tilde{\phi}_1 - r(t) \\ \dot{\tilde{\chi}} &= D\tilde{\chi} + f_1 e + f_2 \eta_1 + \bar{d} r(t) \\ \dot{\eta} &= \Gamma \eta + de \\ \dot{\tilde{\phi}}_1 &= -\lambda_1 \phi_1 + \phi_2 - \dot{\phi}_1^* \\ &= -\lambda_1 \phi_1 + \phi_2 + \dot{\hat{\chi}}_1 + \mu^T \dot{\hat{\theta}} + \dot{\mu}^T \hat{\theta} + \frac{k}{\beta}(\tilde{\chi}_1 + c_1 e \\ &\quad + \beta \eta_1 + \mu^T \tilde{\theta} - ke + \tilde{\phi}_1 - r) \\ \dot{\phi}_2 &= -\lambda_2 \phi_2 + \phi_3 \\ &\;\; \vdots \\ \dot{\phi}_{\rho-1} &= -\lambda_{\rho-1} \phi_{\rho-1} + u \; . \end{aligned} \tag{42}$$

If $\rho = 2$, we define $\phi_2 = u$ as

$$\begin{aligned} u &= \lambda_1 \phi_1^* - \dot{\hat{\chi}}_1 - \dot{\mu}^T \hat{\theta} - \mu^T \dot{\hat{\theta}} \\ \dot{\hat{\theta}} &= g \mathrm{Proj}((e + k\tilde{\phi}_1)\mu, \hat{\theta}) \end{aligned} \tag{43}$$

so that by using the function $V_1 = V + \frac{1}{2}\beta \tilde{\phi}_1^2$ along with the boundedness of $\tilde{\theta}(t)$, we can prove that all the signals in the closed loop system are bounded, $\forall t \geq 0$. Define the filters to generate $\bar{\xi}_i[j]$, $j = 1, 2$, as in (25) with ϕ_{1r} in place of u_r. Then, by using the function

$$W_1 = V_1 + p \| Q\tilde{\theta} - \mu[2]\beta e \|^2 + p_1 \sum_{i=1}^{\bar{m}} \bar{\xi}_i^T[1] P_2 \bar{\xi}_i[1] \tag{44}$$

and choosing ϵ, k satisfying (23), p and p_1 satisfying (31) and λ_1 sufficiently large, we have

$$\dot{W}_1 \leq -c_{w_{1a}} W_1 + c_{w_{1b}} r^2 \tag{45}$$

for suitable positive reals $c_{w_{1a}}$ and $c_{w_{1b}}$, from which property (iii) in Definition 2.1 may be obtained as in the case $\rho = 1$. If $\rho > 2$, we can iterate the previous arguments by using the techniques introduced in Marino and Tomei (2000) to avoid the differentation of the operator Proj. Now, assume that $m = \bar{m}$ so that $r(t) = 0$ but $u_r(t)$ or $\phi_{1r}(t)$ in (25) (for $\rho = 1$ and $\rho > 1$, respectively) are still persistently exciting signals. This fact implies that exponential convergence to zero of $e(t)$, $\eta(t)$, $\tilde{\chi}(t)$, $\tilde{\phi}_i(t)$ and $\tilde{\theta}(t)$ is obtained through (32) and (45). If $m < \bar{m}$, then $r(t) = 0$ but $u_r(t)$ or $\phi_{1r}(t)$ are no longer persistently exciting signals. The convergence to zero of $e(t), \eta(t), \tilde{\phi}_i(t)$ and $\tilde{\chi}(t)$ can be proved by using V, V_1 (for $\rho = 1$ and $\rho = 2$, respectively) and their time derivatives along with Barbalat Lemma (see Marino and Tomei (1995)). □

3. EXAMPLE

Consider the system $\dot{x}_1 = x_2 + a_1 x_1 + \frac{1}{\beta} u + w_1$, $\dot{x}_2 = \frac{1}{\beta} b_2 u$, $e = x_1 + w_2$, in which:

$$w_1(t) = \begin{cases} \sin(\theta_1 t) + 0.2\sin(\theta_3 t), & t < 50 \text{ s} \\ \sin(\theta_1 t), & t \geq 50 \text{ s} \end{cases}$$

$$w_2(t) = \begin{cases} \sin(\theta_2 t), & t < 100 \text{ s} \\ 0, & t \geq 100 \text{ s} \end{cases} .$$

The parameters $\theta_i > 0$, $i = 1, 2$, a_1, $b_2 > 0$, $\beta > 0$ are unknown constants with known bounds, while for the unknown frequencies θ_i, $1 \leq i \leq 3$, we use an estimated exosystem of dimension 4. Following the design outlined in Section II (with slight modifications due to the fact that the spectrum of R is in this case $\{\pm j\omega_1, \pm j\omega_2\}$, i.e. zero is not included), the resulting control is given by

$$\begin{aligned} \dot{\xi}_1 &= D\xi_1 + [\,0\ \ u\ \ 0\ \ 0\,]^T, \quad \mu_1 = \xi_{11} \\ \dot{\xi}_2 &= D\xi_2 + [\,0\ \ 0\ \ 0\ \ u\,]^T, \quad \mu_2 = \xi_{21} \\ \dot{\hat{\chi}} &= D\hat{\chi} - \bar{d}u \\ \dot{\hat{\theta}}_1 &= g\text{Proj}(\mu_1 e, \hat{\theta}_1), \ \ \dot{\hat{\theta}}_2 = g\text{Proj}(\mu_2 e, \hat{\theta}_2) \\ u &= -ke - \hat{\chi}_1 - \mu_1\hat{\theta}_1 - \mu_2\hat{\theta}_2 \end{aligned} \tag{46}$$

with

$$D = \begin{bmatrix} -d_2 & 1 & 0 & 0 \\ -d_3 & 0 & 1 & 0 \\ -d_4 & 0 & 0 & 1 \\ -d_5 & 0 & 0 & 0 \end{bmatrix}, \qquad \bar{d} = [\,d_2\ \ d_3\ \ d_4\ \ d_5\,]^T .$$

Note that for $0 \leq t < 50$ s, the true exosystem is of $6th$ order so that the controller undermodels the exosystem; for $50 \leq t < 100$ s, the true exosystem is of dimension 4 as the estimated one; for $t > 100$ s, the true exosystem is of $2nd$ order so that the controller overmodels the exosystem. The unknown parameters of the system are: $a_1 = 1$, $b_2 = 0.5$, $\beta = 0.5$, while the parameters of the controller have been chosen as: $k = 5$, $g = 1000$, $d_2 = 6$, $d_3 = 13$, $d_4 = 12$, $d_5 = 4$. All initial conditions of the system and of the controller have been set to zero. The results are illustrated in Figs. 1 and 2. In Fig. 1 are reported the time histories of the output regulation error $e(t)$, the control input $u(t)$ and the estimates of the parameters $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$, while in Fig. 2 are represented the state variables $[x_1(t), x_2(t)]$, the disturbance $w_1(t)$ and the output reference $-w_2(t)$.

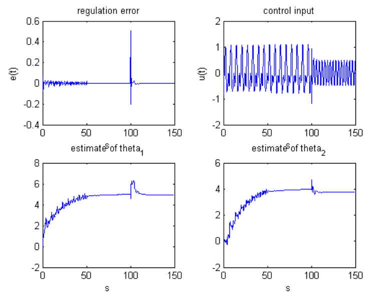

Fig. 1. Error, input and parameters estimates

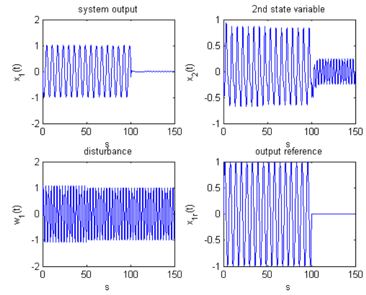

Fig. 2. State variables, disturbance and reference

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