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Asymptotic ε-Nash Equilibrium for 2nd Order Two-Player Nonzero-sum Singular LQ Games with Decentralized Control

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Abstract: We propose a new equilibrium concept: asymptotic ε-Nash equilibrium for 2nd order two-player nonzero-sum games where each player has a control-free cost functional quadratic in the system states over an infinite horizon and each player's control strategy is constrained to be continuous linear state feedback. Based on each player's singular control problem, the asymptotic ε-Nash equilibrium implemented by partial state feedback is constructed and the feedback gains can be found by solving a group of algebraic equations which involves the system coefficients and weighting matrices in the cost functionals. As an illustration of the theories discussed in this paper, a numerical example is given where the partial state feedback gains can be found explicitly in terms of the system coefficients and weighting matrices in the cost functionals.

1. INTRODUCTION

For dynamic optimal control systems (where the system evolution can be described by differential or difference equations), we will have singular optimal control problems if optimal solutions cannot be decided from a minimum or a maximum of the Hamiltonians associated with the minimum principle for dynamic systems. Obviously, when the Hamiltonian of an optimal control problem is linear in the control variables and if there are no bounds on the control variables, the optimal control problem is singular. In terms of the classical linear quadratic regulator (LQR) problem which is described by (1) (where x is the system state, u is the control variable, and state weighting matrices P and Q in the cost functional are positive semi-definite),

$$\begin{aligned} &\dot{x} = Ax + Bu \qquad (x(t_0) = x_0) \\ &J(x_0, u) = x_{t_f}^{\mathrm{T}} P x_{t_f} + \int_{t_0}^{t_f} \left(x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u \right) dt \end{aligned} \tag{1}$$

if the control weighting matrix R in the cost functional is positive definite, (1) is a regular optimal control problem; if R is positive semi-definite, (1) is a singular optimal control problem. An extreme case of singular LQR problem is that when R=0, i.e. the cost functional is control-free. In singular optimal control problems, Hamiltonians may have non-unique optima and we cannot find the possible optimal control candidates directly from the Weierstrass necessary condition, that is, the partial derivative of the Hamiltonian with respect to the control variable should be zero. A differential game is singular if some players in the game have singular optimal control problems. Singular problems are very difficult to solve and there are few publications contributed to this topic.

Based on the theory of characteristics, Kelley (1965) introduced a non-singular transformation which is now well known as Kelley's transformation for Mayer problems linear in a single control variable. The result of the transformation is that the original singular optimization problem is transformed into a regular optimization problem of reduced dimension so that the Legendre-Clebsch condition can be applied. Butman (1968) dealt with optimization problems where the cost function does not depend on the control vector. The coefficient matrix of controls in the system model is state-independent and has full column rank so that the state space at every time slot can be decomposed into two orthogonal subspace: one is spanned by the columns of control coefficient matrix and the other one is just the complement of the first subspace. As in Kelley's paper, an optimal problem of lower dimension with new defined control variables appearing in the performance index was studied for the original problem. Because the optimal state trajectory of singular control problems (which is often called singular arc) can only exist within some subspace of state space, if the control variable is not constrained within some compact set, impulses had to be used in controls at the initial time in order to bring the state trajectory immediately onto the singular arc. Speyer and Jacobson (1971) studied the optimization problem when the control entering the integral of the cost functional appears linearly. Kelley's transformation was applied to the accessory minimum problem (finding the control deviation to minimize the second variation of the augmented cost functional) to obtain a lower dimension regular optimal control problem. Ho (1972) applied the above techniques to a singular stochastic optimal control problem. In the above transformed lower dimension regular optimal control problems, the states with non-singular weighting matrices in performance indices were regarded as control variables.

In the scenario of non-cooperative games, all players' strategies should be derived simultaneously to obtain the Nash equilibrium solution. So each player is sensitive to changes in the other players' strategies. Thus in singular games there are special phenomena and properties that cannot be found in singular control problems. Amato and Pironti (1994) studied a two-player linear quadratic game where the singularity arose because of semi-definiteness of one player's weighting matrix in the cost functional. The transformation method in Butman (1968) was applied to obtain a reduced order non-singular game. Melikyan (2001) applied the method of singular characteristics to study equivocal surfaces which only exist in singular zero-sum games due to particular convex-concave Hamiltonians. Kamneva (2003) studied an optimum-time differential game where the evolution of system states was affected by two separate parts which were controlled by two players respectively. The singular surfaces contained in this optimum-time game were dispersal, equivocal and switching surfaces.

Comparing optimal control and nonzero-sum differential games, Starr and Ho (1969 a, b) pointed out the special difficulties in the latter: 1) the necessary conditions that must be satisfied by Nash equilibrium solutions are a set of partial differential equations while the counterpart in optimal control are a set of ordinary differential equations; 2) the relationship between the open-loop and closed-loop solution in optimal control is no longer guaranteed in game problems. Sarma and Prasad (1972) discussed methods to categorize and construct different kinds of switching surfaces in N-person nonzero-sum differential games and pointed out that the dispersal surfaces in nonzero-sum games are more complicated than those in zero-sum games. Olsder (2001) studied nonzero-sum differential games with saturation constraints in the control variables. Open- and closed-loop bang-bang control were provided and applied to nonlinear and linear examples. Singular surfaces as a by-product were also discussed for a better understanding of the properties of value functions.

The exact optimal control/strategies in singular optimal control/game problems may contain discontinuities where optimal paths cross the switching surfaces and Hamiltonians are non-smooth. In this paper, we discuss a class of 2nd order two-player nonzero-sum linear quadratic games where each player has a singular optimal control problem. The only constraint on two players' strategies is that they must be continuous linear state feedback, which is practical from a point of view of real applications. Because of the exclusion of discontinuities in players' strategies, exact singular control for each player may not be fulfilled. Thus we propose a new equilibrium concept: asymptotic ε-Nash equilibrium. Note that the term 'asymptotic' here is in the sense of time and is different from that in Glizer (2000). The 'asymptotic' in Glizer (2000) is related to the expansion order of the approximate solution in the boundary layer method used to solve singularly perturbed Riccati equations. ε-optimality has been employed to construct equilibrium solutions in game problems. For example, in Fudenberg and Levine (1986), Zhukovskiy and Salukvadze (1994), Xu and Mizukami (1997) and Jimenez and Poznyak (2006), ε-equilibria and ε-saddle points were discussed for non-cooperative games.

In this paper the problem and the new equilibrium concept: of asymptotic ε-Nash equilibrium, are introduced in section 2. The singular control problem faced by each player is formulated in section 3.1. Based on section 3.1, the asymptotic ε-Nash equilibrium implemented by partial state feedback, which results in a decentralized system, is proposed in section 3.2. Concluding remarks are given in section 5 following a numerical example in section 4.

2. PROBLEM STATEMENT

A two-player linear system is described by (2)

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + b_1u_1 \qquad (x_1(0) = x_{10}) \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + b_2u_2 \qquad (x_2(0) = x_{20}) \end{aligned} \tag{2}$$

where the state $x_i \in \mathbb{R}$ and each player's control $u_i \in U_i \subset C[0, +\infty]$ (i=1,2). Here, $C[a, b]$ denotes the set of continuous scalar functions of an argument over $[a, b]$. a_{11}, $a_{12}\neq 0$, $a_{21}\neq 0$, a_{22}, $b_1\neq 0$, and $b_2\neq 0$ are constant real numbers. Player i (i=1,2) will choose his/her own control strategy u_i (i=1,2) independently such that his/her performance index (3) can be minimized.

$$J_i\left(x_0, u_1, u_2\right) = \int_0^\infty \left(q_{ij}x_j^2 + q_{ii}x_i^2\right)dt \quad (i, j = 1, 2; i \neq j) \tag{3}$$

The constant real numbers $q_{ii}>0$ and $q_{ij}\geq 0$ (i, j=1, 2; $i\neq j$). Besides the constraint of continuity, each player should construct his/her control strategy by linear state feedback method as described by (4)

$$u_i(t) = -K_i x(t) = -\left(k_{i1} \ \ k_{i2}\right)\left(x_1(t) \ \ x_2(t)\right)^{\mathrm{T}} \quad (i = 1, 2) \tag{4}$$

In (4), $()^{\mathrm{T}}$ means the transpose of a matrix. The two players have perfect knowledge about system state $x(t)$, system model and the structure of the other player's performance index in their strategy derivation. Denote the game described by (2)-(4) by G_Γ.

In the following part, the variables' subscripts i and j should be understood as i, j= 1, 2 and $i\neq j$. If there are exceptions, they will be stated directly.

Usually, the Nash equilibrium solution (u_1^*, u_2^*) (5) is desired for the game G_Γ.

$$\begin{aligned} J_1\left(x_0, u_1^*, u_2^*\right) &\leq J_1\left(x_0, u_1, u_2^*\right) \\ J_2\left(x_0, u_1^*, u_2^*\right) &\leq J_2\left(x_0, u_1^*, u_2\right) \end{aligned} \tag{5}$$

But note that each player's control does not appear in performance indices (3). Thus each player faces a singular control problem. Because each player's control is constrained to be continuous and impulsive control cannot be used, the exact optimal control for each player's singular optimal control problem may not be obtained. So we introduce the 'ε-Nash equilibrium' and 'asymptotic ε-Nash equilibrium' concepts based on the definition of ε-optimality. First define the truncated performance index as

$$J_i\left(t, x(t), u_1, u_2\right) = \int_t^\infty \left(q_{ij}x_j^2 + q_{ii}x_i^2\right)d\tau \quad \left(t \in [0, +\infty]\right) \tag{6}$$

Definition 1: For $\forall \varepsilon > 0$, $(u_{1\varepsilon}^*, u_{2\varepsilon}^*)$ is called an **ε-Nash equilibrium** solution for game G_Γ if we have

$$\begin{aligned} J_1(x_0, u_{1\varepsilon}^*, u_{2\varepsilon}^*) &\leq J_1(x_0, u_1, u_{2\varepsilon}^*) + \varepsilon \qquad (\forall u_1 \in U_1) \\ J_2(x_0, u_{1\varepsilon}^*, u_{2\varepsilon}^*) &\leq J_2(x_0, u_{1\varepsilon}^*, u_2) + \varepsilon \qquad (\forall u_2 \in U_2) \end{aligned} \tag{7}$$

Definition 2: For $\forall \varepsilon > 0$, $(u_{1T_\varepsilon}^*, u_{2T_\varepsilon}^*)$ is called an **asymptotic ε-Nash equilibrium** solution for game G_Γ if there exists a finite number $T_\varepsilon \in [0, +\infty]$ such that for $\forall t \ge T_\varepsilon$ we have

$$\begin{aligned} &J_1(t, x_0, u_{1T_\varepsilon}^*, u_{2T_\varepsilon}^*) \le J_1(t, x_0, u_1, u_{2T_\varepsilon}^*) + \varepsilon \quad (\forall u_1 \in U_1) \\ &J_2(t, x_0, u_{1T_\varepsilon}^*, u_{2T_\varepsilon}^*) \le J_2(t, x_0, u_{1T_\varepsilon}^*, u_2) + \varepsilon \quad (\forall u_2 \in U_2) \end{aligned} \tag{8}$$

Definition 2 focuses on what will happen to game G_Γ from time point T_ε on.

Remark 1: **1)** If ε=0 in (7), then ε-Nash equilibrium (7) is consistent with the ordinary Nash equilibrium (5). **2)** If T_ε=0 in (8), then the asymptotic ε-Nash equilibrium (8) is consistent with the ε-Nash equilibrium (7). **3)** If $T_\varepsilon = \varepsilon$=0 in (8), then the asymptotic ε-Nash equilibrium (8) is consistent with the ordinary Nash equilibrium (6).

We now try to find the ε-Nash equilibrium $(u_{1\varepsilon}^*, u_{2\varepsilon}^*)$ and asymptotic ε-Nash equilibrium $(u_{1T_\varepsilon}^*, u_{2T_\varepsilon}^*)$ for game G_Γ which is implemented by linear state feedback strategy (4).

3. AYMPTOTIC ε-NASH EQUILIBRIUM

Observing performance index (3), because of q_{ii}>0, we can regard x_i as the fake control for each player first to solve his/her singular optimal control problem as shown in section 3.1. Then the asymptotic ε-Nash equilibrium implemented by decentralized control strategy is obtained in section 3.2. At the same time, the asymptotic stability of the closed system and the value of T_ε are also provided.

3.1 Singular Optimal Control Problem for Each Player

When discussing the singular optimal control problem P_i (9) faced by each player *i*, let us just temporarily fix the other player's control u_j.

$$P_i: \begin{cases} \dot{x}_j = a_{ji}x_i + a_{jj}x_j + b_j u_j \quad (x_j(0) = x_{j0}) \\ J_i(x_0, u_1, u_2) = \int_0^\infty \left(q_{ij}x_j^2 + q_{ii}x_i^2 \right) dt \end{cases} \tag{9-1, 2}$$

By the dynamic programming method, find player *i*'s optimal control x_i. First construct the Hamilton-Jacobi-Bellman equation (10) for player *i*.

$$\min_{x_i}\{H_i\} = 0 \tag{10-1, 2}$$

where the Hamiltonian H_i is

$$H_i = (\partial V_i / \partial x_j)\left(a_{ji}x_i + a_{jj}x_j + b_j u_j \right) + q_{ij}x_j^2 + q_{ii}x_i^2$$

Assume the scalar function $V_i(t, x)$ determined by (10) has the form $V_i = \theta_i x_j^2$ with θ_i a positive real number.

It can be shown that $\partial^2 H_i / \partial x_i^2 = 2q_{ii} > 0$, i.e. H_i is strictly convex with respect to x_i. This implies that there exists a unique control strategy x_i such that H_i can be minimized. By the necessary condition (it is also sufficient because of the strict convexity of H_i with respect to x_i): the first order derivative of Hamiltonian H_i with respect to x_i equals zero, i.e. $\partial H_i / \partial x_i = 2\left(a_{ji} + (\partial u_j / \partial x_i) b_j \right)\theta_i x_j + 2q_{ii}x_i = 0$, then the pseudo control strategy x_i to minimize H_i should be of the following form

$$x_i = -q_{ii}^{-1}\left(a_{ji} + (\partial u_j / \partial x_i) b_j \right)\theta_i x_j \tag{11-1, 2}$$

3.2 Asymptotic ε-Nash Equilibrium by Decentralized Control

Substituting (11) into (10), we have

$$\begin{aligned} &2x_j\theta_i\left[a_{jj}x_j - a_{ji}q_{ii}^{-1}\left(a_{ji} + (\partial u_j / \partial x_i) b_j \right)\theta_i x_j + b_j u_j \right] \\ &+x_j\theta_i\left(a_{ji} + b_j(\partial u_j / \partial x_i) \right) q_{ii}^{-1}\left(a_{ji} + (\partial u_j / \partial x_i) b_j \right)\theta_i x_j \\ &+q_{ij}x_j^2 = 0 \end{aligned} \tag{12-1, 2}$$

Because (11) is the desired state relationship for each player in order to minimize his/her own performance index, now define two new variables

$$s_i = x_i + q_{ii}^{-1}\left(a_{ji} + (\partial u_j / \partial x_i) b_j \right)\theta_i x_j \tag{13}$$

If s_i tends to zero, then (11) can be satisfied. For two real positive numbers l_i, which will determined later, let

$$\dot{s}_i = -l_i s_i \tag{14-1, 2}$$

i.e.

$$\begin{aligned} &\dot{x}_i + q_{ii}^{-1}\left(a_{ji} + (\partial u_j / \partial x_i) b_j \right)\theta_i \dot{x}_j \quad (i, j = 1, 2; i \ne j) \\ &= \left(a_{ii}x_i + a_{ij}x_j + b_i u_i \right) \\ &\quad + q_{ii}^{-1}\left(a_{ji} + (\partial u_j / \partial x_i) b_j \right)\theta_i\left(a_{jj}x_j + a_{ji}x_i + b_j u_j \right) \\ &= -l_i\left\{ x_i + q_{ii}^{-1}\left(a_{ji} + (\partial u_j / \partial x_i) b_j \right)\theta_i x_j \right\} \end{aligned} \tag{15-1, 2}$$

System (14) is exponentially stable. Depending on the eigenvalues $-l_i$'s, s_i will converge to the equilibrium (the origin) with convergence rate $e^{-l_i t}$. From (12) and (15), we need to find θ_1, θ_2, u_1 and u_2. Observing (12), we know that if each player applies partial state feedback strategy, i.e.

$$u_i(t) = -k_{ii}x_i(t) \tag{16}$$

then (12) will be the equation of θ_i which only involves x_j. After factoring out x_j, (12) is equivalent to Riccati equation (17). Note that, from (16), we have $\partial u_i / \partial x_j = 0$.

$$2\theta_1\left(a_{22} - b_2 k_{22} \right) - q_{11}^{-1}a_{21}^2\theta_1^2 + q_{12} = 0 \tag{17-1}$$

$$2\theta_2\left(a_{11} - b_1 k_{11} \right) - q_{22}^{-1}a_{12}^2\theta_2^2 + q_{21} = 0 \tag{17-2}$$

At the same time, under assumption (16), (15) becomes (18)

$$\begin{aligned} &\left(a_{11} - b_1 k_{11} + q_{11}^{-1}a_{21}\theta_1 a_{21} + l_1 \right)x_1 \\ &+\left[a_{12} + q_{11}^{-1}a_{21}\theta_1\left(a_{22} - b_2 k_{22} \right) + l_1 q_{11}^{-1}a_{21}\theta_1 \right]x_2 = 0 \end{aligned} \tag{18-1}$$

$$\begin{aligned} &\left[a_{21} + q_{22}^{-1}a_{12}\theta_2\left(a_{11} - b_1 k_{11} \right) + l_2 q_{22}^{-1}a_{12}^{\mathrm{T}}\theta_2 \right]x_1 \\ &+\left(a_{22} - b_2 k_{22} + q_{22}^{-1}a_{12}^{\mathrm{T}}\theta_2 a_{12} + l_2 \right)x_2 = 0 \end{aligned} \tag{18-2}$$

The necessary and sufficient condition such that (15) can be satisfied for any x_1 and x_2 is that the coefficients of every state in (18) is zero, i.e. (19) and (20)

$$a_{11} - b_1 k_{11} + q_{11}^{-1} a_{21} \theta_1 a_{21} + l_1 = 0 \quad (19\text{-}1)$$

$$a_{12} + q_{11}^{-1} a_{21} \theta_1 \left(a_{22} - b_2 k_{22}\right) + l_1 q_{11}^{-1} a_{21} \theta_1 = 0 \quad (19\text{-}2)$$

$$a_{21} + q_{22}^{-1} a_{12} \theta_2 \left(a_{11} - b_1 k_{11}\right) + l_2 q_{22}^{-1} a_{12} \theta_2 = 0 \quad (20\text{-}1)$$

$$a_{22} - b_2 k_{22} + q_{22}^{-1} a_{12} \theta_2 a_{12} + l_2 = 0 \quad (20\text{-}2)$$

From (17), (19) and (20), we need to find $l_1>0$, $l_2>0$, $\theta_1>0$, $\theta_2>0$, k_{11} and k_{22}. For this problem, we have the following theorem

Theorem: If there exists a solution $l_1>0$, $l_2>0$, $\theta_1>0$, $\theta_2>0$, k_{11} and k_{22} to algebraic equations (17), (19) and (20), then decentralized control strategy (16) constitutes an asymptotic ε-Nash equilibrium for Game G_Γ. Meanwhile the closed-loop system is stable and T_ε can be found by (43).

Proof:

Suppose that the conditions in the theorem are satisfied. Define a Lyapunov function candidate as

$$V(t) = \theta_2 x_1^2 + \theta_1 x_2^2 \quad (21)$$

Because of the assumptions $\theta_1>0$ and $\theta_2>0$, we know that (21) is a Lyapunov function. Considering (17), by some manipulation we can get its derivative with respect to time

$$\dot{V}(t) = \underbrace{\left(-q_{21} - q_{22}^{-1} a_{12}^2 \theta_2^2\right)}_{\le 0} x_1^2 + \underbrace{\left(-q_{12} - q_{11}^{-1} a_{21}^2 \theta_1^2\right)}_{\le 0} x_2^2 + s_2 a_{12} \theta_2 x_1 + x_1 \theta_2 a_{12} s_2 + s_1 a_{21} \theta_1 x_2 + x_2 \theta_1 a_{21} s_1 \quad (22)$$

Define the neighbourhood $\Omega_{|s_i| \le \delta_i}$ ($\delta_i>0$) of the hyper-plane $s_i=0$ as

$$\Omega_{|s_i| \le \delta_i} \triangleq \left\{ (x_1, x_2) : |s_i| = \left| x_i + q_{ii}^{-1} a_{ji} \theta_i x_j \right| \le \delta_i \right\} \quad (23)$$

It can be proved that $\Omega_{|s_i| \le \delta_i}$ is invariant under the conditions in the theorem. Also define several neighbourhoods around the origin as

$$\mathrm{N}_x \triangleq \left\{ (x_1, x_2) : |x| \le \overline{\varepsilon}, \overline{\varepsilon} > 0 \right\} \quad (24)$$

$$\mathrm{N}_x^{'} \triangleq \left\{ (x_1, x_2) : |x_1| \le \delta, |x_2| \le \delta, \delta > 0 \right\} \quad (25)$$

$$\mathrm{N}_{x_1} \triangleq \left\{ (x_1, x_2) : |x_1| \le \delta, \delta > 0 \right\} \quad (26)$$

$$\mathrm{N}_{x_2} \triangleq \left\{ (x_1, x_2) : |x_2| \le \delta, \delta > 0 \right\} \quad (27)$$

In (24), $|v|$ denotes the L^2-norm of a vector v. Select

$$\delta = \min \left\{ \frac{\sqrt{2}}{2} \overline{\varepsilon}, \frac{\sqrt{2\overline{\varepsilon}}}{3 q_{11}^{-1} \theta_1 |a_{21}|}, \frac{\sqrt{2\overline{\varepsilon}}}{3 q_{22}^{-1} \theta_2 |a_{12}|} \right\} \quad (28)$$

$$\delta_i \le q_{ii}^{-1} \theta_i |a_{ji}| \delta / 2 \quad (29)$$

Then we have

$$\mathrm{N}_x^{'} \in \mathrm{N}_x \quad (30)$$

Considering (13), the closed-loop system under (16) is

$$\begin{aligned} \dot{x}_1 &= \left(a_{11} - b_1 k_{11}\right) x_1 + a_{12} \left(s_2 - q_{22}^{-1} a_{12} \theta_2 x_1\right) \\ \dot{x}_2 &= a_{21} \left(s_1 - q_{11}^{-1} a_{21}^{\mathrm{T}} \theta_1 x_2\right) + \left(a_{22} - b_2 k_{22}\right) x_2 \end{aligned} \quad (31)$$

By (14) we have

$$s_i(t) = e^{-L_i t} s_i(0) \quad \left(t \ge 0; s_i(0) = x_i(0) + q_{ii}^{-1} a_{ji} \theta_i x_j(0) \right) \quad (32)$$

So there must exist a time T such that the state trajectory of (31) satisfies (33)

$$x(t) \in \bigcap_{i=1}^{2} \Omega_{|s_i| \le \delta_i} \quad (t \ge T) \quad (33)$$

Actually, we can select T as

$$T = \max_{i=1,2} \{T_i\} \quad (34)$$

where $T_i = \begin{cases} \left(\ln\left(\delta_i / |s_i(0)| \right) \right) \Big/ \left(\min\limits_{i=1,2} l_i \right), & |s_i(0)| \ne 0 \\ 0, & \text{else} \end{cases}$.

We consider three situations which can cover the whole state space.

1) If $x_1(t) \notin \mathrm{N}_{x_1}$ and $x_2(t) \notin \mathrm{N}_{x_2}$ when $t \ge T$, by (29) we know that

$$s_2 a_{12} \theta_2 x_1 + x_1 \theta_2 a_{12} s_2 - q_{22}^{-1} a_{12}^2 \theta_2^2 x_1^2 < 0$$

$$s_1 a_{21} \theta_1 x_2 + x_2 \theta_1 a_{21} s_1 - q_{11}^{-1} a_{21}^2 \theta_1^2 x_{21}^2 < 0$$

Observing (22), we have

$$\dot{V}(t) < 0 \quad \left(x_1(t) \notin \mathrm{N}_{x_1}, x_2(t) \notin \mathrm{N}_{x_2}, t \ge T \right)$$

Under this situation, state trajectory of (31) approaches the origin. And once $x(t)$ enters the neighborhood N_x of the origin, $x(t)$ will remain within N_x because $\Omega_{|s_i| \le \delta_i}$ (i=1,2) is invariant.

2) If $x_1(t) \in \mathrm{N}_{x_1}$ and $x_2(t) \notin \mathrm{N}_{x_2}$ when $t \ge T$

$$|s_2| = \left| x_2 + q_{22}^{-1} a_{12} \theta_2 x_1 \right| \le \delta_2$$

$$\Rightarrow |x_2| \le q_{22}^{-1} \theta_2 |a_{12}| \delta + \delta_2 \le 3 q_{22}^{-1} \theta_2 |a_{12}| \delta / 2 \le \sqrt{2\overline{\varepsilon}} / 2$$

$$\Rightarrow |x| \in \mathrm{N}_x$$

Under this situation, state trajectory of (31) is already within the neighbourhood N_x of the origin.

3) Similarly, if $x_1(t) \notin \mathrm{N}_{x_1}$ and $x_2(t) \in \mathrm{N}_{x_2}$ when $t \ge T$

$$|s_1| = \left| x_1 + q_{11}^{-1} a_{21} \theta_1 x_2 \right| \le \delta_1$$

$$\Rightarrow |x_1| \le q_{11}^{-1} \theta_1 |a_{21}| \delta + \delta_1 \le 3 q_{11}^{-1} \theta_1 |a_{21}| \delta / 2 \le \sqrt{2\overline{\varepsilon}} / 2$$

$$\Rightarrow |x| \in \mathrm{N}_x$$

State trajectory of (31) is within the neighbourhood N_x of the origin.

Thus, closed-loop system (31) is stable under control strategy pair (16).

Now, let us find T_ε. By (13) and (32), we have

$$\begin{aligned} x_i(t) &= -q_{ii}^{-1} a_{ji} \theta_i x_j(t) + s_i(t) \\ &= -q_{ii}^{-1} a_{ji} \theta_i x_j(t) + e^{-L_i t} s_i(0) \quad (t \ge 0) \end{aligned} \quad (35)$$

Then truncated version of the cost functional in (9) becomes (36)

$$J_i(T_{\varepsilon i},x_0,u_1,u_2)=\int_{T_{\varepsilon i}}^{\infty}\left(q_{ij}x_j^2+q_{ii}x_i^2\right)dt \qquad (i=1,2)$$
$$=\int_{T_{\varepsilon i}}^{\infty}\left[q_{ij}x_j^2+q_{ii}\left(s_i-q_{ii}^{-1}a_{ji}\theta_i x_j\right)^2\right]dt \qquad (36)$$
$$=\underbrace{\int_{T_{\varepsilon i}}^{\infty}\left[\left(q_{ij}+q_{ii}^{-1}a_{ji}^2\theta_i^2\right)x_j^2-2a_{ji}\theta_i s_i x_j\right]dt}_{\text{part 1}}+\underbrace{q_{ii}\left(\int_{T_{\varepsilon i}}^{\infty}s_i^2(t)dt\right)}_{\text{part 2}}$$

Define $W_i=\theta_i x_j^2$. And by (35) and the first equation in (17), we have

$$\dot{W}_i=2\theta_i x_j\dot{x}_j=2\left(-q_{ji}-q_{ii}^{-1}\theta_i^2 a_{ji}^2\right)\theta_i x_j^2+2\theta_i a_{ji}s_i x_j \qquad (37)$$

Note that $x_j(\infty)=0$. Integrating both sides of (37) from $T_{\varepsilon i}$ to ∞, we have

$$0-\theta_i x_j^2(T_{\varepsilon i})=-\int_{T_{\varepsilon i}}^{\infty}\left[\left(q_{ij}+q_{ii}^{-1}a_{ji}^2\theta_i^2\right)x_j^2-2a_{ji}\theta_i s_i x_j\right]dt$$

Thus we find the value of part 1 in (36), i.e.

$$\text{part 1}=\int_{T_{\varepsilon i}}^{\infty}\left[\left(q_{ij}+q_{ii}^{-1}a_{ji}^2\theta_i^2\right)x_j^2-2a_{ji}\theta_i s_i x_j\right]dt=\theta_i x_j^2(T_{\varepsilon i}) \quad (38)$$

Also

$$\text{part 2}=\underbrace{q_{ii}s_i^2(0)\left(\int_{T_{\varepsilon i}}^{\infty}e^{-2l_i t}dt\right)}_{\text{part 2}}=\left(q_{ii}s_i^2(0)e^{-2l_iT_{\varepsilon i}}\right)\big/\left(2l_i\right) \quad (39)$$

Thus (36) becomes

$$\begin{aligned}J_i(T_{\varepsilon i},x_0,u_1,u_2)&=\int_{T_{\varepsilon i}}^{\infty}\left(q_{ij}x_j^2+q_{ii}x_i^2\right)dt\\&=\theta_i x_j^2(T_{\varepsilon i})+\left(q_{ii}s_i^2(0)e^{-2l_iT_{\varepsilon i}}\right)\big/\left(2l_i\right)\quad(i=1,2)\end{aligned} \qquad (40)$$

Define

$$J_{\varepsilon i}(T_{\varepsilon i},x_0,u_1,u_2)=\text{part 2}=\left(q_{ii}s_i^2(0)e^{-2l_iT_{\varepsilon i}}\right)\big/\left(2l_i\right)\quad(i=1,2)$$

In order to obtain

$$J_{\varepsilon i}(x_0,u_1,u_2)\le\varepsilon \quad (i=1,2) \qquad (41)$$

we need $T_{\varepsilon i}\ge 0.5l_i^{-1}\ln\left(\left(q_{ii}s_i^2(0)\right)\big/\left(2l_i\varepsilon\right)\right)$. So we can select $T_{\varepsilon i}$ as

$$T_{\varepsilon i}=\max\left\{0,\ 0.5l_i^{-1}\ln\left(\left(q_{ii}s_i^2(0)\right)\big/\left(2l_i\varepsilon\right)\right)\right\} \qquad (42)$$

Then T_ε can be selected as (43)

$$T_\varepsilon=\max(T_{\varepsilon 1},T_{\varepsilon 2}) \qquad (43)$$

The proof for inequalities (8) can be conducted like this. If (16) constitutes an asymptotic ε–Nash equilibrium for Game G$_\Gamma$, then we have

$$\begin{aligned}\theta_i x_j^2(T_\varepsilon)&\le J_i(T_\varepsilon,x_0,u_1=-k_{11}x_1,u_2=-k_{22}x_2)\\&\le\theta_i x_j^2(T_\varepsilon)+\varepsilon \qquad (i=1,2)\end{aligned} \qquad (44)$$

Take player 1 for example. If player 1 applies a partial state feedback gain $\bar{k}_{11}$ and $\bar{k}_{11}\ne k_{11}$. But player 2 still sticks to k_{22}. The closed-loop system under $(\bar{k}_{11},k_{22})$ is,

$$\begin{aligned}\dot{\bar{x}}_1&=\left(a_{11}-b_1\bar{k}_{11}\right)\bar{x}_1+a_{12}\bar{x}_2,\quad\left(\bar{x}_1(T_\varepsilon)=x_1(T_\varepsilon)\right)\\\dot{\bar{x}}_2&=a_{21}\bar{x}_1+\left(a_{22}-b_2k_{22}\right)\bar{x}_2,\quad\left(\bar{x}_2(T_\varepsilon)=x_2(T_\varepsilon)\right)\end{aligned} \qquad (45)$$

Suppose for $(\bar{k}_{11},k_{22})$, we could not find qualified solutions θ_i's and L_i's to algebraic equations (17), (19), and (20). But observing (9), we find that the singular control problem faced by player 1 remains the same and the solutions θ_1's to the first equation in (17) for these two different gain pairs are the same. So

$$\theta_1 x_2^2(T_\varepsilon)+\varepsilon\le J_1(T_\varepsilon,x_0,u_1=-\bar{k}_{11}\bar{x}_1,u_2=-k_{22}\bar{x}_2)+\varepsilon \quad (46)$$

Comparing (44) and (46), we verified (8). Thus, Theorem is proved. ▪

Remark 2: **1)** Under the Nash equilibrium solution (16), the closed-loop system is a decentralized system, which will greatly reduce the system complexity. **2)** If

$$0<\left|s_i(0)\right|\le\sqrt{2l_i\varepsilon/q_{ii}}$$

and the conditions in the theorem are satisfied, then decentralized control strategy (16) constitutes an ε-Nash equilibrium for Game G$_\Gamma$. **3)** If $\left|s_i(0)\right|=0$ $(i=1,2)$ and the conditions in the theorem are satisfied, then decentralized control strategy (16) constitutes an Nash equilibrium for Game G$_\Gamma$ and $J_i\left(x_0,u_1^*,u_2^*\right)=\theta_i x_j^2(0)$.

4. NUMERICAL EXAMPLE

For a 2nd order system described in the corollary, we can explicitly find the solution to (17), (19) and (20).

Corollary: If $n_1=n_2=m_1=m_2=1$, $a_{12}=a_{21}\ne0$, $q_{11}=q_{22}>0$, $q_{12}=q_{21}>0$ and $q_{21}/q_{22}>3$, then the unique solution to (17), (19) and (20) is

$$\begin{aligned}&\theta_1=\theta_2=q_{11}/\left|a_{21}\right|>0\\&l_1=l_2=\left(q_{21}/q_{11}-3\right)\left|a_{21}\right|/2>0\\&k_{ii}=\left[2a_{ii}q_{11}+\left(q_{21}-q_{11}\right)\left|a_{21}\right|\right]\big/\left(2b_iq_{11}\right)\end{aligned} \qquad (47)$$

The proof of the corollary just involves some algebra and is omitted here.

Example: The system parameters are: a_{11}=0.2, a_{22}=0.3, a_{21}=a_{12}=3, b_1=10, b_2=4; the parameters in the performance indices are: q_{11}=q_{22}=0.1, q_{12}=q_{21}=4. Select ε =15. For the initial conditions x_{10}=100, x_{20}= -70, it can be calculated according to the corollary that the unique solution to (17), (19) and (20) are θ_1=θ_2=0.0333, l_1=l_2=55.5, k_{11}=5.87, k_{22}=14.7 and T_ε=0. Correspondingly, the decentralized control strategy (16), which comprises the ε–Nash equilibrium, becomes

$$u_{1\varepsilon}(t)=-5.87x_1(t);\quad u_{2\varepsilon}(t)=-14.7x_2(t) \qquad (48)$$

Under (48), the two eigen-values of the closed-loop system are -61.5 and -55.5. The values of the two players' performance indices are respectively J_1=214.7 and J_2=335.1.

Figures 1-2 show the curves of system states and control inputs versus time. The parameter ε depicts the extent to which an ε–Nash equilibrium is close to the ordinary Nash equilibrium. Fixing the initial state conditions as x_{10}=100, x_{20}= -70, figure 3 shows the relationship of T_ε and ε for an asymptotic ε–Nash equilibrium. From figure 3 we know that the more we relax ε, the smaller T_ε we have, i.e. the faster we can reach an asymptotic ε–Nash equilibrium. Fixing ε =15, figure 4 shows the initial state conditions which guarantee an ε–Nash equilibrium, i.e. T_ε=0. The shaded area in figure 4 is symmetric about x_1=-x_2.

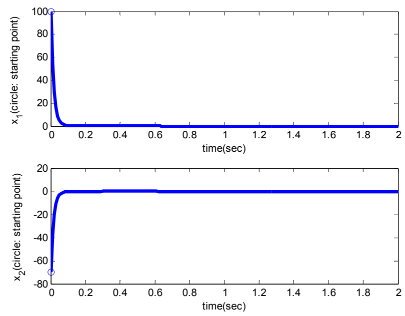

Fig. 1 System states versus time

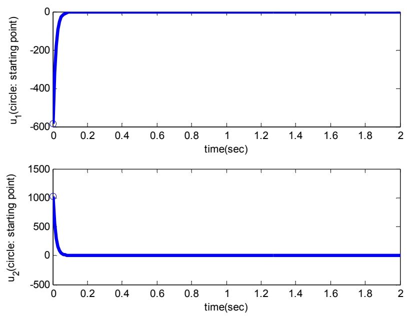

Fig. 2 Controls versus time

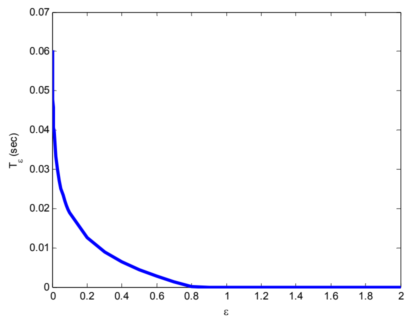

Fig.3 Relationship of T_ε and ε when fixing x_{10}=100, x_{20}= -70,

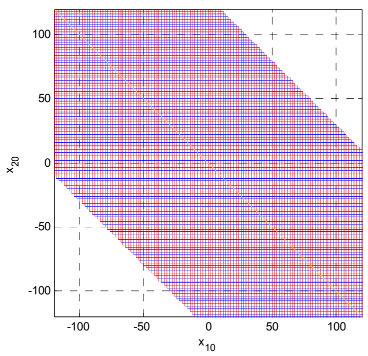

Fig. 4 Range of initial conditions obtaining ε–Nash equilibrium when fixing ε=15

6. CONCLUSIONS

For 2nd order two-player nonzero-sum games where each player has a control-free cost functional quadratic in system states, we proposed a new equilibrium concept: asymptotic ε-Nash equilibrium, to accommodate the constraint that each player's control strategy should be continuous linear state feedback. Based on each player's singular control problem, the asymptotic ε-Nash equilibrium, which is attained by partial state feedback and can also guarantee stability of the decentralized closed-loop system, was obtained from a group of algebraic equations. A numerical example illustrated the various aspects of the theories in this paper.

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