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Control of Oriented Mechanical systems: A Method Based on Dual Quaternion

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Abstract: This paper focuses on a new type of control laws for oriented mechanical systems. The starting point is dual quaternion and its properties. Logarithm of dual quaternion is defined, based on which control laws are developed, tackling both regulation and tracking problems using logarithmic feedback. The control laws are shown to have several merits, including global asymptotically convergence, computational efficiency, and proper handling of the coupling between rotation and translation. Simulation results validate our design.

Keywords: Mechanical systems control, Geometric Control, Nonlinear control, PD control, Dual quaternion.

1. INTRODUCTION

Mechanical systems have always been a source of challenges for control theory. The abundant characteristics of mechanical systems bring various difficulties, one of which is control of the orientation. As the rotation group has a topology different from Euclidean space, the orientation of a rigid body cannot be obtained through integration of the angular velocity. For this reason attitude control has been studied for years, see Wen and Delgado (1991), Arambel and Manikonda (2000).

In more complicated applications, such as task-space control of manipulators, control of mobile robots or underwater vehicles, rotation and translation of a mechanism are controlled simultaneously. In Bullo et al. (2000) configuration of these mechanisms is formulated as a Riemannian manifold. In many cases the manifold is actually the Special Euclidean group $SE(3)$. Control laws are designed on Riemannian manifold and $SE(3)$ in Bullo et al. (2000) and Bullo and Murray (1995) respectively. A typical result is the generalized proportional-derivative(PD) laws using logarithmic feedback developed on $SE(3)$ and its subgroups.

A conventional way to perform control on $SE(3)$, called double geodesic control law, is also addressed in Bullo and Murray (1995). Its main idea is to control translation and rotation separately. Compared to the geometric method, the conventional method is less attractive, because the former can take the coupling between translation and rotation into consideration, making the resulting trajectory more natural. However, using matrices to describe rigid transformations leads to complex computation, which prevents the existing geometric control laws from practical application.

Fortunately matrix is not the only tool to describe rigid transformation. Dual quaternion is a good substitution. It has been shown that dual quaternion is a compact and efficient tool for representation and computation(Wu et al. (2005)). This paper will rebuild the control laws using dual quaternion. The new control laws, which are also geometric, are more accessible than the former results.

Like the role of $se(3)$ in Bullo and Murray (1995), the logarithm of dual quaternion, which is defined using the logarithm of unit quaternion and the logarithm of dual number, is of great importance in our design. The newly proposed definition, which embodies the geometric structure of dual quaternion, is of importance itself, having the potential to be used in related problems such as task space trajectory planning.

Since its invention, dual quaternion has been applied in various fields, such as inertial navigation, mechanical design, robotics, and other related problems(Dooley and McCarthy (1993), Goddard (1997), Daniilidis (1999), Perez and McCarthy (2004), Wu et al. (2005)). To summarize, dual quaternion is mostly used for kinematic analysis, except for Dooley and McCarthy (1991), where dynamics of spatial rigid body is formulated in dual quaternion. It is well known that quaternion contributes a lot to attitude control. As a generalization of quaternion(Hsia and Yang (1981)), dual quaternion seldom plays a role in the control of oriented mechanisms. This paper is also an effort to fill this gap.

This paper is organized as follows. Necessary mathematical preliminaries are given in Section II, where logarithm of dual quaternion is defined and discussed in detail. Control laws using logarithmic feedback are developed in Section III. Section IV shows simulation results and Section V concludes the paper.

2. MATHEMATICAL PRELIMINARIES

This section presents a brief review of unit quaternion and dual quaternion. Readers are referred to Bottema (1979) and Wu et al. (2005) for more details.

2.1 Unit Quaternion

A unit quaternion usually has the form

$$q = (q_s, q_{\mathbf{v}})$$

where q_s is a scalar, $q_{\mathbf{v}}$ is a vector.

Basic operations of quaternions are listed as below:

$$q^* = (q_s, -\mathbf{q_v}) \tag{1}$$
$$\| q \|^2 = qq^*$$
$$q^{-1} = q^* / \| q \|^2$$
$$\frac{q_1}{q_2} = \frac{q_1 q_2^*}{\|q_2\|^2} \tag{2}$$

Rotation about a unit axis $\mathbf{n}$ with angle ϕ is expressed as

$$q = (\cos(\phi/2), \sin(\phi/2)\mathbf{n}) \tag{3}$$

q obtained in this way is naturally a unit quaternion. On the contrary, any unit quaternion can be expressed using (3). In the following a symbol Q_u is used to represent all unit quaternions.

The kinematic equation of rotation is given in Wu et al. (2005):

$$2\dot{q} = q \circ \omega^b \tag{4}$$

where ω^b represents the body angular velocity, explicitly expressed as

$$\omega^b = 2q^* \circ \dot{q} \tag{5}$$

2.2 Dual Vector

A dual vector is defined as

$$\hat{\mathbf{v}} = \mathbf{l} + \varepsilon\mathbf{m}, \quad \text{with } \varepsilon^2 = 0 \text{ but } \varepsilon \neq 0$$

where $\mathbf{l}$ and $\mathbf{m}$ are both 3-dimensional real vectors.

Given two dual vectors $\hat{\mathbf{V}}_1 = \mathbf{l}_1 + \varepsilon\mathbf{m}_1, \hat{\mathbf{V}}_2 = \mathbf{l}_2 + \varepsilon\mathbf{m}_2$, the dot product and the cross product can be defined:

$$\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2 = \mathbf{l}_1 \cdot \mathbf{l}_2 + \varepsilon(\mathbf{m}_1 \cdot \mathbf{l}_2 + \mathbf{l}_1 \cdot \mathbf{m}_2)$$
$$\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2 = \mathbf{l}_1 \times \mathbf{l}_2 + \varepsilon(\mathbf{l}_1 \times \mathbf{m}_2 + \mathbf{m}_1 \times \mathbf{l}_2)$$

For future design a new type of dot product is defined between a dual vector and a 6-dimensional real vector. Given a 6-dimensional real vector $\mathbf{K}$, it can be divided into two 3-dimensional real vectors $\mathbf{K}_a$, $\mathbf{K}_b$:

$$\mathbf{K} = (k_1, k_2, k_3, k_4, k_5, k_6) \triangleq (\mathbf{K}_a, \mathbf{K}_b)$$

Then the special dot product is defined as

$$\mathbf{K} \cdot \hat{\mathbf{V}} = \mathbf{K}_a \cdot \mathbf{l} + \varepsilon\mathbf{K}_b \cdot \mathbf{m} \tag{6}$$

2.3 Dual Quaternion

A dual quaternion is defined as

$$\hat{q} = (\hat{q}_s, \hat{\mathbf{q}}_{\mathbf{v}})$$

where $\hat{q}_s$ is a dual scalar, $\hat{\mathbf{q}}_{\mathbf{v}}$ is a dual vector.

Operations of dual quaternion are similar to that of quaternion, only adding ˆ to each elements:

$$\hat{q}^* = (\hat{q}_s, -\hat{\mathbf{q}}_{\mathbf{v}}) \tag{7}$$
$$\| \hat{q} \|^2 = \hat{q}\hat{q}^*$$

Note that the norm here is a dual number.

A dual quaternion can also be defined as

$$\hat{q} = q + \varepsilon q^o \tag{8}$$

where q and q^o are both quaternions.

Suppose there is a rotation q succeeded by a translation $\mathbf{p}$. The whole transformation can be represented using a dual quaternion(Wu et al. (2005)) with

$$q^o = \tfrac{1}{2} q \circ \mathbf{p} \tag{9}$$

For convenience the quaternion $(0, \mathbf{p})$ is identified with the vector $\mathbf{p}$. Kinematic equation of a rigid body expressed in dual quaternion is

$$2\dot{\hat{q}} = \hat{q} \circ \hat{\omega}^b \tag{10}$$

where

$$\hat{\omega}^b = 2\hat{q}^* \circ \dot{\hat{q}} = \omega^b + \varepsilon(\dot{\mathbf{p}} + \omega^b \times \mathbf{p}) \tag{11}$$

represents the generalized body velocity. $\hat{\omega}^b$ is also called the *twist*.

Given $\hat{q} = q + \varepsilon q^o$, if $q \cdot q^0 = 0$, $\hat{q}$ is said to be normalized(Ge and Raveni (1994)). It can be verified that, dual quaternion acquired through (9) is naturally normalized. Therefore, a definition is presented

$$DQ_u = \{\hat{q} | \hat{q} \text{ is normalized and has unit norm}\}$$

DQ_u is a manifold with 3 dual dimensions. In the rest of this paper, unless otherwise stated, by dual quaternion we mean an element in DQ_u. When $\mathbf{p} = 0$, DQ_u is simplified as Q_u.

2.4 Logarithm of Unit Quaternion

Given a unit quaternion expressed as (3), its logarithm was presented in Kim and Kim (1996). With some modification the formula is restated as

$$\log q = (0, \tfrac{\phi}{2}\mathbf{n}), 0 \leqslant \phi \leqslant 2\pi$$

It can be simplified as

$$\log q = \tfrac{1}{2}\phi\mathbf{n} \tag{12}$$

When $\phi = 0$, $q = (1, 0, 0, 0) \triangleq O$; When $\phi = 2\pi$, $q = -O$. Arbitrary $\mathbf{n}$ fits these two cases. Their logarithms are defined specially as $\log O = \log(-O) = (0, 0, 0)$. From the viewpoint of control, both O and $-O$ are equilibriums(Arambel and Manikonda (2000)).

Note that $\phi \pm 2k\pi$ also satisfies (12) for arbitrary integer k. Using $\vee^3$ to denote the 3-dimensional vector space, elements in Q_u are mapped into $\vee^3$ by the logarithmic operation.

2.5 Logarithm of Normalized Dual Quaternion

Given a dual quaternion, an interesting conclusion is given in Bottema (1979):

$$q + \varepsilon q^0 = q e^{\varepsilon\gamma}, \text{with } \gamma = q^0 / q \tag{13}$$

Substituting (9) one obtains

$$\gamma = \tfrac{1}{2}\mathbf{p}$$

Computing the logarithm on both sides of (13) gives

$$\log(q + \varepsilon q^0) = \log q + \varepsilon \tfrac{1}{2}\mathbf{p}$$

Let $p = \| \mathbf{p} \|$, $\mathbf{s} = \mathbf{p}/p$, it follows that

$$\log \hat{q} = \tfrac{1}{2}(\phi\mathbf{n} + \varepsilon p\mathbf{s}) \tag{14}$$

Logarithm of a dual quaternion is a vector with 3 dual dimensions. In future discussion, a symbol $\hat{\vee}^3$ is used to represent all 3-dimensional dual vectors.

Let $\hat{O} = (1,0,0,0) + \varepsilon(0,0,0,0)$. Just like the case of Q_u, the equilibriums of system (10) are $\hat{O}$ and $-\hat{O}$. Their logarithms are defined as $(0,0,0) + \varepsilon(0,0,0)$.

2.6 More Definitions and Discussions

Given $q \in Q_u, \hat{q} \in DQ_u$ and their logarithms, a reasonable definition of the inner products on $\vee^3, \hat{\vee}^3$ is

$$< \log q, \log q > = 4 \log q \cdot \log q^T$$
$$< \log \hat{q}, \log \hat{q} > = 4 \log \hat{q} \cdot (\log \hat{q})^T$$

With the inner products a new type of norm for dual quaternions can be defined:

$$\mathcal{R}(\hat{q}) = \sqrt{< \log(\hat{q}), \log(\hat{q}) >} \tag{15}$$

For all $\hat{q} \in DQ_u, \hat{v} \in \hat{\vee}^3$, adjoint mapping Ad is defined as

$$Ad_{\hat{q}}\hat{v} = \hat{q}\hat{v}\hat{q}^* \tag{16}$$

Given two dual quaternions $\hat{q}_1, \hat{q}_2$, their difference is evaluated by

$$\hat{e} = \hat{q}_1^* \circ \hat{q}_2 \tag{17}$$

Note that $\hat{q}_1 = \hat{q}_2$ implies $\hat{e} = \hat{O}$.

Let $\hat{q}_1 = q_1 + \frac{1}{2}q_1 \circ \mathbf{p}_1, \hat{q}_2 = q_2 + \frac{1}{2}q_2 \circ \mathbf{p}_2$, it can be calculated that

$$\begin{aligned} \hat{e} &= q_e + \tfrac{1}{2}q_e \circ \mathbf{p}_e \\ q_e &= q_1^* \circ q_2 \\ p_e &= \mathbf{p}_2 - Ad_{q_e^*}\mathbf{p}_1 \end{aligned} \tag{18}$$

Given a sequence of transformation $\hat{q}(t) \in DQ_u$,

$$\hat{\mathbf{r}}(t) = \log \hat{q}(t) \qquad \text{and} \qquad \hat{\omega}^b(t) = 2\hat{q}^* \circ \dot{\hat{q}}$$

can now be computed. A useful lemma follows.

Lemma 1.

$$\tfrac{1}{2}\tfrac{d}{dt}\mathcal{R}^2(\hat{q}) = 2 < \hat{\mathbf{r}}, \hat{\omega}^b > \tag{19}$$

Proof is given in Appendix A.

For the special case of Q_u, equation (19) is simplified as

$$\tfrac{1}{2}\tfrac{d}{dt}\mathcal{R}^2(q) = 2 < \mathbf{r}, \omega^b > \tag{20}$$

For Q_u there is another lemma.

Lemma 2.

$$< 2\dot{\mathbf{r}}, \omega^b > \quad \leqslant \quad < \omega^b, \omega^b > \tag{21}$$

Proof is given in Appendix B.

Remark: The relation between DQ_u and $\hat{\vee}^3$(or Q_u and $\vee^3$) deserves further research. Their relation has in common with the relation between a Lie group and its Lie algebra. Therefore, DQ_u and $\hat{\vee}^3$ together embodies the geometric structure of dual quaternion.

3. CONTROL LAW DESIGN

A unified model for all oriented mechanisms can be written as

$$\begin{aligned} 2\dot{\hat{q}} &= \hat{q} \circ \hat{\omega}^b \\ \dot{\hat{\omega}}^b &= f(\hat{q}, \hat{\omega}^b) + \hat{U} \end{aligned} \tag{22}$$

$f(\hat{q}, \hat{\omega}^b)$ and $\hat{U}$ are dynamics related terms, varying with different mechanisms. Differentiating (11) yields

$$\dot{\hat{\omega}}^b = \dot{\omega}^b + \varepsilon(\ddot{\mathbf{p}} + \omega^b \times \dot{\mathbf{p}} + \dot{\omega}^b \times \mathbf{p}) \tag{23}$$

Then

$$f(\hat{q}, \hat{\omega}^b) = \varepsilon\omega^b \times \dot{\mathbf{p}} \tag{24}$$

$$\hat{U} = \dot{\omega}^b + \varepsilon(\ddot{\mathbf{p}} + \dot{\omega}^b \times \mathbf{p}) \triangleq U + \varepsilon U^o \tag{25}$$

As $\dot{\omega}^b$ and $\ddot{\mathbf{p}}$ are caused by torques and forces respectively, the vector U corresponds to the acting torques on a moving rigid body, while U^o corresponds to both forces and torques. In this paper we take $\hat{U}$ as the input variable to be designed. To execute a control law, one need to solve the forces and torques from $\hat{U}$. This procedure is possible as long as the mechanism is fully-actuated. Now we are on the way to design control laws. The procedure starts from the kinematic model (10).

3.1 The Regulation Problem

To stabilize system (10), a control law using logarithmic feedback is presented:

$$\hat{\omega}^b = -2k_p \log \hat{q} \tag{26}$$

To prove the stability, consider the candidate Lyapunov function:

$$W(\hat{q}) = \tfrac{1}{2}\mathcal{R}^2(\hat{q}) \tag{27}$$

Differentiating (27) and substituting (19)(26) gives

$$\dot{W}(\hat{q}) = < 2\log\hat{q}, -2k_p \log \hat{q} > = -k_p W$$

Thus the logarithmic control law ensures exponential stability.

Here "exponential stability" means that $\log \hat{q}$ converges to zero exponentially. That is, (26) will drive any initial posture to $\hat{O}$. Note that there is an exception. If the initial posture is $-\hat{O}$, by (26) $\hat{q}$ will stay at $-\hat{O}$.

Actually $\hat{O}$ and $-\hat{O}$ are physically identical. When the initial posture $\hat{q}_0$ is near $-\hat{O}$, that is to say, when the scalar part of $\hat{q}_0$'s quaternion part is negative, it is more reasonable to take $-\hat{O}$ as the equilibrium; otherwise the system will follow a "longer" trajectory leading to $\hat{O}$. To handle this multi-equilibrium problem, for a given initial posture

$$\hat{q}(0) = (q_{s0}, q_{\mathbf{v}0}) + \varepsilon(q_{s0}^0, q_{\mathbf{v}0}^0)$$

a parameter λ is introduced:

$$\lambda = \begin{cases} 1, & \text{if } q_{s0} \geqslant 0 \\ -1, & \text{otherwise} \end{cases} \tag{28}$$

Then a revision of (26)

$$\hat{\omega}^b = -2k_p\lambda \log(\lambda\hat{q}) \tag{29}$$

will drive the system to $\hat{O}$ or $-\hat{O}$ as demanded.

To control (22), both $\mathcal{R}(\hat{q})$ and $\hat{\omega}^b$ should be driven to zero. The following control law is a possible choice:

$$\hat{U} = -2k_p \log(\hat{q}) - k_d\hat{\omega}^b - f(\hat{q}, \hat{\omega}^b) \tag{30}$$

Formula (30) can be divided into two parts: the real part and the dual part. The real part leads to stability of pure rotation control. By the dual part stability of translation will be guaranteed.

The real part of (30) is written as

$$U = -2k_p \log q - k_d\omega^b - f(q, \omega^b)$$

It implies that

$$\dot{\omega}^b = -2k_p \log q - k_d\omega^b \tag{31}$$

Let $\mathbf{r} = \log q$. Borrowing the skill used in Wen and Bayard (1988), a candidate Lyapunov function is constructed by introducing a cross term:

$$\begin{aligned} W_\epsilon = &\tfrac{1}{2}(< 2\mathbf{r}, 2\mathbf{r} > + \tfrac{1}{k_p} < \omega^b, \omega^b >) \\ &+ \epsilon < \mathbf{r}, \omega^b > \end{aligned} \tag{32}$$

where ϵ is a small positive scalar.

Differentiating (32) and applying (20) yields

$$\begin{aligned} \tfrac{d}{dt}W_\epsilon = &2 < \mathbf{r}, \omega^b > + \tfrac{1}{k_p} < \omega^b, \dot{\omega}^b > \\ &+ \epsilon < \mathbf{r}, \dot{\omega}^b > + \epsilon < \dot{\mathbf{r}}, \omega^b > \end{aligned}$$

Substituting (31) and (21) gives

$$\begin{aligned} \tfrac{d}{dt}W_\epsilon \leqslant &- \tfrac{k_d}{k_p} < \omega^b, \omega^b > -2k_p\epsilon < \mathbf{r}, \mathbf{r} > \\ &- k_d\epsilon < \mathbf{r}, \omega^b > + \tfrac{\epsilon}{2} < \omega^b, \omega^b > \end{aligned}$$

Rearranging the terms yields

$$\begin{aligned} \tfrac{d}{dt}W_\epsilon \leqslant &- \tfrac{k_p}{2}\epsilon W_\epsilon \\ &+ (\epsilon - \tfrac{k_d}{k_p})(< \omega^b, \omega^b > + k_p\epsilon < \mathbf{r}, \omega^b >) \end{aligned}$$

Choosing ϵ to be small enough so that $\epsilon - \frac{k_d}{k_p} < 0$ holds, it follows that

$$\tfrac{d}{dt}W_\epsilon \leqslant -\tfrac{k_p}{2}\epsilon W_\epsilon \tag{33}$$

Thus the rotational part is asymptotically stable. When $\epsilon > 0$, exponential convergence is achieved.

Applying the dual part of (30) and substituting (11)(23) yields

$$\ddot{\mathbf{p}} = -k_p\mathbf{p} - k_d\dot{\mathbf{p}} + k_p\mathbf{r} \times \mathbf{p} - \omega^b \times \dot{\mathbf{p}} \tag{34}$$

As the rotational part is stabilized, ω^b and $\mathbf{r}$ approach zeroes as time passes by. Then (34) implies the convergence of $\mathbf{p}$.

To summarize, if the initial posture is not $-\hat{O}$, formula (30) will drive the dynamic system (22) to $\hat{O}$ asymptotically. Thus asymptotical stability is proved.

Similar skill as in (29) is used to handle the multi-equilibrium problem:

$$\hat{U} = -2k_p\lambda \log(\lambda\hat{q}) - k_d\hat{\omega}^b - f(\hat{q}, \hat{\omega}^b) \tag{35}$$

It is well known that at least 4 parameters are needed to get a description of orientation without singularity. Though the logarithm of a unit quaternion has only three elements, the control laws are still singularity-free due to the use of the indicator λ.

3.2 The Tracking Problem

Given a reference trajectory $\hat{q}_d(t)$, the tracking error is express as

$$\hat{e} = \hat{q}_d^* \circ \hat{q} \tag{36}$$

Let $\hat{\omega}_d = 2\hat{q}_d^* \circ \dot{\hat{q}}_d$, it is the demanded velocity. As $\hat{q}_d$ is normalized, it can be verified that $2\dot{\hat{q}}_d^* = -\hat{\omega}_d \circ \hat{q}_d^*$.

Differentiating (36) yields

$$2\dot{\hat{e}} = \hat{e} \circ \hat{\omega}_e \tag{37}$$

$$\hat{\omega}_e = \hat{\omega}^b - Ad_{\hat{e}^*}\hat{\omega}_d \tag{38}$$

Given the facts

$$2\dot{\hat{e}}^* = -\hat{\omega}_e \circ \hat{e}^* \ \text{ and } \ \hat{\omega}_e \times Ad_{\hat{e}^*}\hat{\omega}_d = -Ad_{\hat{e}^*}\hat{\omega}_d \times \hat{\omega}_e$$

The term $Ad_{\hat{e}^*}\hat{\omega}_d$ can be differentiated as

$$(Ad_{\hat{e}^*}\hat{\omega}_d)' = Ad_{\hat{e}^*}\hat{\Omega}_d + Ad_{\hat{e}^*}\hat{\omega}_d \times \hat{\omega}_e \triangleq \hat{U}_{tr} \tag{39}$$

where $\hat{\Omega}_d$ is the demanded acceleration. Here $\hat{U}_{tr}$ represents the compensation brought by the reference trajectory.

Differentiating (38) yields

$$\dot{\hat{\omega}}_e = f(\hat{q}, \hat{\omega}) - \hat{U}_{tr} + \hat{U} \tag{40}$$

Equation (37) together with (40) constitute the dynamic error system. Applying (30) yields the tracking control law:

$$\hat{U} = -2k_p \log(\hat{e}) - k_d\hat{\omega}_e + \hat{U}_{tr} - f(\hat{q}, \hat{\omega}^b) \tag{41}$$

In the control laws designed, k_p and k_d are positive scalars. Thanks to the special inner product defined by (6), the parameters can be replaced by 6-dimensional vectors K_p and K_d with positive elements.

The newly built control laws are similar to the results developed in Bullo and Murray (1995). With the discussion about Lie group and Lie algebra being avoided, our result is more accessible. Moreover, in the formulas vectors are used instead of matrices, and matrix multiplications are replaced by cross products between vectors, making the control laws computationally more efficient. As logarithm is not defined for some matrices, control laws derived in Bullo and Murray (1995) demands k_p to have a lower bound to avoid singularities. The restriction is unnecessary here, as logarithm is properly defined everywhere on DQ_u. Such improvements make the development in this paper more than a simplified version of the former geometric design.

4. SIMULATION RESULTS

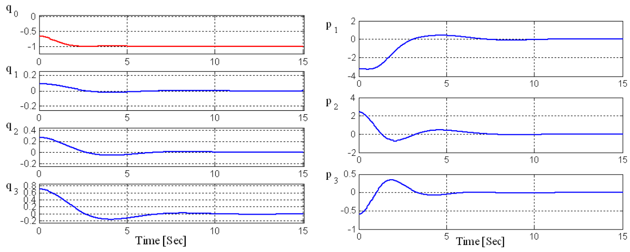
Firstly the regulation law (35) will be tested. Taking $k_p = 1.5, k_d = 0.8$, starting with zero velocity and

$$\begin{aligned} \hat{q}(0) = &(-0.6628, 0.0885, 0.2654, 0.6946) \\ &+\varepsilon(0.2079, 1.9701, 0.2111, -0.5300) \end{aligned}$$

The simulation results are shown in Fig. 1(a) and Fig. 1(b). Notice that q_0 converges to -1 monotonically as expected.

To test the tracking law (41), a reference trajectory $\hat{q}_{dl}(t)$ is selected as

$$\begin{aligned} \hat{q}_{dl}(t) = &(0.7775, 0.3631, 0.3631, 0.3631) \\ &+\varepsilon(-0.10890.0777, 0.0777, 0.0777)t \end{aligned}$$



(a) The quaternion part (b) The translation part

Fig. 1. Convergence of the posture $\hat{q}(t) = q(t) + q^0(t)$. On the left is the quaternion part $q(t)$, on the right is the translation part $\mathbf{p} = 2q^0(t) \circ q^*(t)$

which actually represents a line, expressed in Cartesian coordination as

$$\begin{aligned} x &= 0.2t & \theta_1 &= \tfrac{\pi}{4} \\ y &= 0.2t & \theta_2 &= \tfrac{\pi}{4} \\ z &= 0.2t & \theta_3 &= \tfrac{\pi}{4} \end{aligned}$$

where $(\theta_1, \theta_2, \theta_3) = 2 \log q$.

Starting from the initial posture

$$\hat{q}_0 = (1, 0, 0, 0) + \varepsilon(0, -2, 0, 0)$$

with zero initial velocity, formula (41) is applied to track $\hat{q}_{dl}(t)$. The tracking error $\hat{q}_e$ is given in Fig. 2. When $\hat{q}_e$ converges to $\hat{O}$, or $\log \hat{q}_e$ converges to zero, tracking is accomplished.

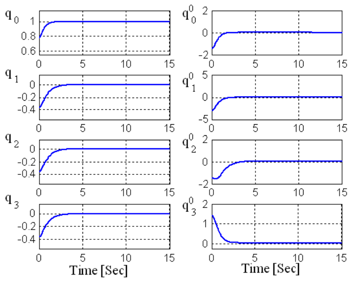


Fig. 2. The tracking error $\hat{q}_e = q + \varepsilon q^0$

To give an intuitionistic result, we consider an omnidirectional robot moving on the x-y plane while rotating about the z-axis. The posture of the robot is described by $\hat{q} = q + \frac{1}{2} q \circ \mathbf{p}$ with $\mathbf{p} = (x, y, 0)$ and $2 \log q = (0, 0, \theta_3)$. Actually the transformations constitute $SE(2)$.

Firstly (35) is applied to regulating the robot. Initially the robot is at the positions

$$\{(x_i, y_i) | x_i = \pm 1 \text{ or } 0, x_i^2 + y_i^2 = 1\}$$

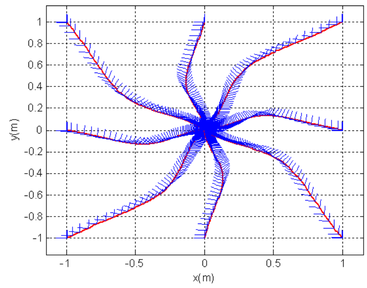
with $\theta_3 = \frac{\pi}{2}$ and with zero velocities. The gains k_p, k_d are chosen as

$$k_p = (1, 1, 1, 8, 8, 8)$$
$$k_d = (0.5, 0.5, 0.5, 4, 4, 4)$$

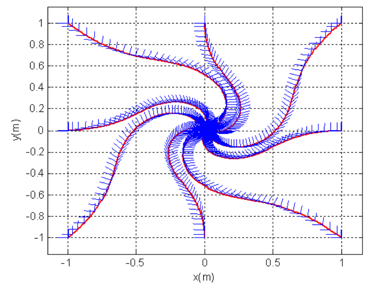
Using two perpendicular bars to indicate the heading angle θ_3, simulation results are shown in Fig. 3(a).

The double geodesic regulation law using equivalent gains, is also applied to do the simulations, and the results are given in Fig. 3(b).

Compared to the new regulation law, the double-geodesic law generates a spiraling (x, y) motion. Recall that all the control laws are built from (11). In conventional methods the term $\omega^b \times \mathbf{p}$ is omitted. That is to say, the rotation's influence on translation is not taken into account. However, the influence is not always neglectable. For example, in task space control, while the end-effector of a manipulator moves in the task space, all its configurations form $SE(3)$. A notation *swept volume* is used to denote the space covered by the end-effector when it turns and twists(Abrams and Allen (1995)). If applied to task space control of manipulators, the new method can result in less swept volume, which is desired in applications such as collision avoidance in a clattered environment.



(a) Results of the new regulation law



(b) Results of the double geodesic law

Fig. 3. Comparison of the new regulation law and the double geodesic law. A rigid body moving on the x-y plane is driven to the origin. The frames attached to the curve indicate the heading angle.

5. CONCLUDING REMARKS

Based on dual quaternion a new method for the control of oriented mechanisms is developed. Compared to the conventional method, the control laws can handle the coupling inside a motion, achieve harmony between rotation and translation. Compared to the preceding geometric results, complicated discussion on Lie group and Lie algebra is avoided and global convergence is achieved.

Due to its computational efficiency, dual quaternion is a wonderful algebraic tool for motion design. This paper reveals the geometric structure of dual quaternion and presents new geometric control laws. Combining dual-quaternion based trajectory planning schemes with these control laws will make a prospective solution for many applications, such as industrial robots and cooperative manipulators, which will be covered by future research.

Appendix A. PROOF OF LEMMA 1

Starting from (15), standard calculation gives

$$\frac{1}{2}\frac{d}{dt}\mathcal{R}^2(\hat{q}) = < 2\log(\hat{q}), 2\frac{d}{dt}\log(\hat{q}) >$$
$$2\frac{d}{dt}\log(\hat{q}) = (\dot{\phi}\mathbf{n} + \varepsilon\dot{p}\mathbf{s}) + (\phi\dot{\mathbf{n}} + \varepsilon\phi\dot{\mathbf{n}})$$
$$\triangleq C_{\shortparallel} + C_{\perp}$$

As $\mathbf{n}$ and $\mathbf{s}$ are both unit vectors,

$$\mathbf{n}\cdot\dot{\mathbf{n}} = \mathbf{s}\cdot\dot{\mathbf{s}} = 0$$

Together with (14) it follows that

$$< 2\log(\hat{q}), C_{\perp} >= 0$$
$$\frac{1}{2}\frac{d}{dt}\mathcal{R}^2(\hat{q}) =< 2\log(\hat{q}), C_{\shortparallel} >$$

Following (3)(5)(11), $\hat{\omega}^b$ is calculated as

$$\begin{aligned}\hat{\omega}^b &= (\dot{\phi}\mathbf{n} + \varepsilon\dot{p}\mathbf{s}) + (\sin\phi\dot{\mathbf{n}} + 2\sin^2\tfrac{\phi}{2}\dot{\mathbf{n}}\times\mathbf{n}) \\ &\quad +\varepsilon(p\dot{\mathbf{s}} + \dot{p}\dot{\phi}\mathbf{n}\times\mathbf{s} + p\dot{\phi}\mathbf{n}\times\dot{\mathbf{s}} + \phi\dot{p}\dot{\mathbf{n}}\times\mathbf{s}) \\ &\quad +\varepsilon(\phi p\dot{\mathbf{n}}\times\dot{\mathbf{s}} + 2\sin^2\tfrac{\phi}{2}\dot{\mathbf{n}}\times\mathbf{s}\times\dot{\mathbf{s}}) \\ &\triangleq C_{\shortparallel} + \bar{C}_{\perp}\end{aligned}$$

Obviously

$$< 2\log(\hat{q}), \bar{C}_{\perp} >= 0$$

Thus

$$< 2\log(\hat{q}), \hat{\omega}^b >=< 2\log(\hat{q}), C_{\shortparallel} >$$

Then (19) follows.

Appendix B. PROOF OF LEMMA 2

Given $q = [\cos(\phi/2), \sin(\phi/2)\mathbf{n}]$,

$$2\mathbf{r} = 2\log q = \phi\mathbf{n}$$

It follows that

$$2\dot{\mathbf{r}} = \dot{\phi}\mathbf{n} + \phi\dot{\mathbf{n}} \triangleq D_{\shortparallel} + D_{\perp}$$
$$\omega^b = \dot{\phi}\mathbf{n} + \left[\sin\phi\dot{\mathbf{n}} + 2\sin^2\tfrac{\phi}{2}\dot{\mathbf{n}}\times\mathbf{n}\right]$$
$$\triangleq D_{\shortparallel} + \bar{D}_{\perp}$$

As $\mathbf{n}$ is a unit vector, $\dot{\mathbf{n}}\cdot\mathbf{n} = 0$ holds. Moreover,

$$< \dot{\mathbf{n}}\times\mathbf{n}, \dot{\mathbf{n}}\times\mathbf{n} >=< \dot{\mathbf{n}}, \dot{\mathbf{n}} >$$

Therefore,

$$< 2\dot{\mathbf{r}}, \omega^b > = < D_{\shortparallel}, D_{\shortparallel} > + < D_{\perp}, \bar{D}_{\perp} >$$
$$< \omega^b, \omega^b > = < D_{\shortparallel}, D_{\shortparallel} > + < \bar{D}_{\perp}, \bar{D}_{\perp} >$$
$$< D_{\perp}, \bar{D}_{\perp} > - < \bar{D}_{\perp}, \bar{D}_{\perp} >$$
$$= (\phi\sin\phi - \sin^2\phi - 4\sin^4\tfrac{\phi}{2}) < \dot{\mathbf{n}}, \dot{\mathbf{n}} >$$

When $-2\pi < \phi \leqslant 2\pi$, it can be verified that

$$\phi\sin\phi - \sin^2\phi - 4\sin^4\tfrac{\phi}{2} \leqslant 0$$

So

$$< 2\dot{\mathbf{r}}, \omega^b > \quad \leqslant \quad < \omega^b, \omega^b >$$

Now (21) is proved.

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