Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Clarification of Free Parameters of State-dependent Coefficient Form: Effect on Solving State-dependent Riccati Inequality

Yoshihiro Sakayanagi * **Daisuke Nakayama** ** **Shigeki Nakaura** *** **Mitsuji Sampei** ****

* *Tokyo Institute of Technology, Tokyo, Japan (e-mail: yoshi@ctrl.titech.ac.jp)*
** *Tokyo Institute of Technology, Tokyo, Japan*
*** *Tokyo Institute of Technology, Tokyo, Japan (e-mail: nakaura@ctrl.titech.ac.jp)*
**** *Tokyo Institute of Technology, Tokyo, Japan (e-mail: sampei@ctrl.titech.ac.jp)*

Abstract: Recently, nonlinear H_∞ control theory has been paid attention for the powerful design method for a robust stabilization. The solvable condition of nonlinear H_∞ control problem is given by the Hamilton Jacobi Inequality (HJI). The HJI is the partial differential inequality which is quite difficult to solve, so some numerical approaches have been researched.
The approach to solve the HJI based on State-dependent Riccati Inequality (SDRI) is proposed. The SDRI is derived from the HJI with a State-dependent Coefficient form (SDC form) of a nonlinear system under an integrability constraint. Here, since the SDC form for a nonlinear system is not unique, free parameters of the SDC form can be considered.
In this paper, a new expression of free parameters to completely express the SDC form is proposed. Using free parameters, a desirable numerical solution of the SDRI can be calculated. We focus on a constant solution of the SDRI because the integrability constraint can be neglected. Finally, numerical examples to verify the advantage of the free parameters of SDC form are given.

Keywords: Robust control of nonlinear systems,LMIs.

1. INTRODUCTION

Recently, Nonlinear H_∞ Control Theory has become a remarkably popular tool in engineering applications (van der Schaft (1992))(Imura et al. (1993a))(Imura et al. (1993b)). It is known that in order to solve Nonlinear H_∞ Control Problems, we have to deal with a kind of partial differential inequality called Hamillton-Jacobi Inequality (HJI). For Linear H_∞ Control Problems, we can derive the linear H_∞ controller easily by solving a familiar Algebraic Riccati Inequality (ARI), but it turns out to be much more complicate to derive nonlinear H_∞ controller due to a necessity on dealing with the HJI. Since HJI is a partial differential inequality, it is quite hard to solve HJI analytically.

Numerical solutions of HJI have been researched. One of the researches is about approximate solution of HJI using Taylor Expansion around a equilibrium point (Patpong et al. (1996)). This approximate solution shows a good behavior around the equilibrium point, but does not guarantee a global characteristic. On the other hand, there is a way using nonlinear matrix inequality which is so-called State-dependent Riccati Inequality(SDRI) (Lu and Doyle (1995))(Erdem and Alleyne (1999))(Cloutier and Stansbery (1999)). For a nonlinear system(4), the SDRI(10) is derived from State-dependent Coefficient (SDC) Form of the nonlinear system(7). Lu and Doyle (1995) showed about SDRI issues. Roughly concluding, if there exists a positive definite matrix $P(x)$ which is a solution of SDRI and also positive definite scalar function $V(x)$ satisfying $\partial V/\partial x = 2x^T P(x)$, then such the $V(x)$ is a positive definite solution of the HJI. By solving the point-wise ARIs via LMI problem formulation, they got a set of point-wise solutions and also an approximate continuous solution $P(x)$.

For these methods which use SDRI to solve HJI, there is a problem that SDC form of nonlinear system is not unique. This problem means that there are many representations of $A(x)$ satisfying $f(x) = A(x)x$. In other words, free parameters can be considered in SDC form(Shamma and Cloutier (2003))(Jaganath et al. (2005)). Since the solution of SDRI depends on $A(x)$, it is important to choose a good representation of $A(x)$. In other words, the solution of SDRI $P(x)$ can be adjusted by using different $A(x)$.

In this paper we focus on the representation of SDC form. As we saw in this introduction, it is important to consider about the SDC form because the SDC form to a nonlinear system is not unique. This time we introduce a perfect representation of SDC form so that an appropriate solution of the SDRI can be calculated.

2. H_∞ CONTROL PROBLEM

2.1 Linear H_∞ Control Problem

Let us consider the following linear system S_l

$$S_l \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \end{cases} \tag{1}$$

where w is an unknown disturbance, z is a controlled output, u is a control input to be chosen. Objectives of linear H_∞ control is to find a controller that achieves closed-loop stability and makes L_2-gain of system S_l less than or equal to a prior $\gamma > 0$. For an easy formulation of control input, let us assume $C_1^T D_{12} = 0,\ D_{12}^T D_{12} = I$. Then the control input is given by

$$u = -B_2^T P x \tag{2}$$

where P is a positive definite matrix which satisfies following ARI

$$PA + A^T P + \frac{1}{\gamma^2} P B_1 B_1^T P - P B_2 B_2^T P + C_1^T C_1 < 0. \tag{3}$$

2.2 Nonlinear H_∞ Problem

Let us consider the following nonlinear system S_{nl}

$$S_{nl} \begin{cases} \dot{x} &= f(x) + g_1(x) w + g_2(x) u \\ z &= h_1(x) + j_{12}(x) u \end{cases} \tag{4}$$

where w, z, u is same as (1). And objectives of nonlinear H_∞ problem is also same as linear one. Refer to (van der Schaft (1992)), under standard assumptions $h_1^T j_{12} = 0$ and $j_{12}^T j_{12} = I$, an optimal feedback control law is given by

$$u(x) = -\frac{1}{2} g_2^T(x) \frac{\partial V}{\partial x} \tag{5}$$

where $V(x)$ is a positive definite solution of HJI

$$\frac{\partial V}{\partial x^T} f + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x^T} g_1 g_1^T \frac{\partial V}{\partial x} - \frac{1}{4} \frac{\partial V}{\partial x^T} g_2 g_2^T \frac{\partial V}{\partial x} + h_1^T h_1 + \varepsilon x^T x \le 0 \tag{6}$$

for some positive ε.

2.3 State-dependent Coefficient Form

Let us define as follows

$$f(x) = A(x)x,\ g_1(x) = B_1(x),\ g_2(x) = B_2(x)$$
$$h_1(x) = C_1(x)x,\ j_{12}(x) = D_{12}(x),$$

then the nonlinear control system S_{nl} can be transformed into SDC form

$$S_{nl} \begin{cases} \dot{x} = A(x)x + B_1(x) w + B_2(x) u \\ z = C_1(x)x + D_{12}(x) u \end{cases}. \tag{7}$$

With assumption

$$\frac{\partial V}{\partial x} = 2P(x)x \tag{8}$$

the HJI of the nonlinear system S_{nl} becomes

$$x^T \big(P(x)A(x) + A^T(x)P(x) + \frac{1}{\gamma^2} P(x)B_1(x)B_1^T(x)P(x) - P(x)B_2(x)B_2^T(x)P(x) + C_1^T(x)C_1(x) \big) x < 0. \tag{9}$$

Finally we get a SDRI

$$P(x)A(x) + A^T(x)P(x) + \frac{1}{\gamma^2} P(x)B_1(x)B_1^T(x)P(x) - P(x)B_2(x)B_2^T(x)P(x) + C_1^T(x)C_1(x) < 0. \tag{10}$$

For this SDRI a nonlinear H_∞ control input u is given by

$$u = -\frac{1}{2} g_2^T(x) \frac{\partial V}{\partial x} = -B_2^T(x)P(x)x. \tag{11}$$

3. FREE PARAMETERS OF STATE-DEPENDENT COEFFICIENT FORM

Since it is not unique to decide a state-dependent coefficient form to a general nonlinear system, free parameters of the SDC form can be considered. In this section, we clarify these free parameters, and consider them to find a suitable solution for HJI.

3.1 Existence of Free Parameters of SDC Form

Since the choices of $B_1(x)$, $B_2(x)$ and $D_{12}(x)$ are unique, we will only focus on $A(x)$ and $C_1(x)$.

Example 1. Consider nonlinear system

$$S_{nl} \begin{cases} \dot{x} = \begin{bmatrix} -5\sin(2x_1) \\ x_1 - 2x_2 - 3x_1^3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ z = \begin{bmatrix} u \\ 5x_1 \\ 10x_2 \end{bmatrix} \end{cases} \tag{12}$$

where $x = [x_1\ \ x_2]^T$. Two different SDC forms for this $f(x)$ can be chosen as

$$f(x) = A_1(x)x = \begin{bmatrix} -5\dfrac{\sin(2x_1)}{x_1} & 0 \\ 1 - 3x_1^2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{13}$$

$$f(x) = A_2(x)x = \begin{bmatrix} 0 & -5\dfrac{\sin(2x_1)}{x_2} \\ -3x_1^2 & -2 + \dfrac{x_1}{x_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{14}$$

It is easy to see that the value of $A(x)$ will be different depends on SDC form. Since the solution of SDRI depends on $A(x)$, it is quite important to deal with the free parameters of SDC form. □

3.2 Representation of Free Parameters of SDC Form

We can represent $f(x)$ as below.

$$f(x) = A(x)x = (A(x) + E(x))\, x \tag{15}$$

$E(x)$ is any matrix that satisfies

$$E(x)x = 0. \tag{16}$$

$E(x) \in \mathbb{R}^{n\times n}$ include $n \times (n-1)$ free parameters, since each row of $E(x)$ include $n-1$ independent vectors which intersect perpendicularly to x.

Example 2. Let us think the same $A_1(x)$,$A_2(x)$ as previous example and

$$E(x) = \begin{bmatrix} 5\dfrac{\sin(2x_1)}{x_1} & -5\dfrac{\sin(2x_1)}{x_2} \\ -1 & \dfrac{x_1}{x_2} \end{bmatrix}.$$

$E(x)$ satisfies $E(x)x = 0$. And $A_2(x) = A_1(x) + E(x)$, so $A_2(x)$ can be represented with $A_1(x)$ and $E(x)$. □

We introduce the new representation which claify free parameters without less and surplus.

Theorem 1. Let $A_0(x)$ be one of a state-dependent coefficient matrix of $f(x)$, $x \in \mathbb{R}^n$ such that

$$f(x) = A_0(x)x \tag{17}$$

Arbitrary $A(x)$ which satisfies

$$\forall x \neq 0,\ f(x) = A(x)x \tag{18}$$

can be represented by

$$A(x) = A_0(x) + M_a(x)\Theta(x) \tag{19}$$

The first column of $M_a(x) \in \mathbb{R}^{n\times n}$ must be 0. Anather $n \times (n-1)$ elements are free parameters. And $\Theta(x)$ is combined rotation matrices which rotate x_1 axis to the direction of x. (Obviously the first row of Θ is $x^T/\|x\|$.) $\|x\| = \sqrt{\sum_{j=1}^n x_j^2}$ is Euclid norm of x. (For detail of $\Theta(x)$,see the appendix A.) □

Proof. $2-n$th rows of $\Theta(x)$ intersect perpendicularly to x, and first column of $M_a(x)$ is 0. So $M_a(x)\Theta(x)x = 0$ and each row of $M_a(x)\Theta(x)$ is linear combination of independent vectors. It satisfies condition of $E(x)$. And this representation clarify just enough $n \times (n-1)$ free parameters.

Consequently $A(x)$ is a state-dependent coefficient matrix.

$$A(x)x = A_0(x)x + M_a(x)\Theta(x)x = A_0(x)x = f(x) \tag{20}$$

Furthermore we check whether all of $A(x)$ are represented by M_a. Let us think about $A_1(x)$. To represent $A_1(x)$, there should be $M_a(x)$ such that

$$M_a(x) = (A_1(x) - A_0(x))\,\Theta^T(x). \tag{21}$$

A first column of $(A_1(x) - A_0(x))\,\Theta^T(x)$ is 0 , because a first column of $\Theta^T(x)$ is $x/\|x\|$ and $A_1(x)x - A_0(x)x = 0$. Since inverse matrix of rotation matrix is transpose matrix of it, we have

$$A(x) = \left(A_0(x) + (A_1(x) - A_0(x))\,\Theta^T(x)\Theta(x)\right) = A_1(x). \tag{22}$$

So we can represent all of $A(x)$ by using $A_0(x) + M_a(x)\Theta(x)$. □

Example 3. Let us think the same $A_1(x)$,$A_2(x)$ as previous example and

$$\Theta(x) = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \quad cos\theta = x_1/\|x\|, sin\theta = x_2/\|x\|.$$

As we saw in the previous proof, if we choose

$$\begin{aligned} M_a(x) &= (A_2(x) - A_1(x))\Theta^T(x) \\ &= \frac{1}{\|x\|}\begin{bmatrix} 5\dfrac{\sin(2x_1)}{x_1} & -5\dfrac{\sin(2x_1)}{x_2} \\ -1 & x_1/x_2 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} \\ &= \|x\|\begin{bmatrix} 0 & -5\dfrac{\sin(2x_1)}{x_1x_2} \\ 0 & 1/x_2 \end{bmatrix} \end{aligned}$$

then we get

$$\begin{aligned} &A_1(x) + M_a(x)\Theta(x) \\ &= \begin{bmatrix} -5\dfrac{\sin(2x_1)}{x_1} & 0 \\ -1-3x_1^2 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -5\dfrac{\sin(2x_1)}{x_1x_2} \\ 0 & 1/x_2 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -5\dfrac{\sin(2x_1)}{x_2} \\ -3x_1^2 & -2+x_1/x_2 \end{bmatrix} = A_2(x) \end{aligned}$$

□

Remark 1. Although $C_1(x)$ can be represented with free parameter $M_c(x)$ as well as $M_a(x)$, the nonlinearity of $C_1(x)$ can be transformed into $A(x)$ by using coordinate transformation.

Let us consider coordinate transformation

$$\tilde{x} = T(x) = \begin{bmatrix} h_1^\perp(x) \\ h_1(x) \end{bmatrix}. \tag{23}$$

$h^\perp(x)$ is any function which is independent of h(x).In other words, the rank of $\partial T(x)/\partial x$ should be n. And $x = T^{-1}(\tilde{x})$. By using coordinate transformation, nonlinear system(4) transformed into

$$S_{nl}\begin{cases} \dot{\tilde{x}} &= \dfrac{\partial T(x)}{\partial x}\left(f(x) + g_1(x)w + g_2(x)u\right) \\ &:= \tilde{f}(\tilde{x}) + \tilde{g_1}(\tilde{x})w + \tilde{g_2}(\tilde{x})u \\ z &= \begin{bmatrix} 0 & I \end{bmatrix}\tilde{x} + j_{12}(x)u \\ &:= C_1\tilde{x} + \tilde{j_{12}}(\tilde{x})u \end{cases} \tag{24}$$

As we can see, the controlled output z is represented in a linear expression with this transformed system. And the $f(x)$ is transformed into $\tilde{f}(\tilde{x})$ which includes the nonlinearity of z. From now on, we only focus on $M_a(x)$ to describe free parameters of the SDC form. □

3.3 Solving SDRI via LMI

When SDRI(10) is fixed with a state x, it is a same inequality as ARI(3) with variable P. To solve this matrix inequality, (3) can be transformed into LMI. By using Schur Complement and a variable transformation $X = P^{-1}$, Riccati Inequality becomes

$$\begin{bmatrix} AX + XA^T + \dfrac{1}{\gamma^2}B_1B_1^T - B_2B_2^T & XC_1^T \\ C_1X & -I \end{bmatrix} < 0 \tag{25}$$

which is a LMI with respect to a variable X. By solving this LMI, we will get $P = X^{-1}$ as a constant solution of SDRI.

3.4 Solving SDRI with M_a via LMI

By substituting $A = A_0 + M_a\Theta$ into (3), Riccati Inequality becomes

$$\begin{aligned} &PA_0 + PM_a\Theta + A_0^TP + \Theta^TM_a^TP \\ &\quad + \frac{1}{\gamma^2}PB_1B_1^TP - PB_2B_2^TP + C_1C_1^T < 0. \end{aligned} \tag{26}$$

For this inequality, the same procedure as the case of linear Riccati Inequality is not suitable for getting LMI because there are extra terms $PM_a\Theta$ and $\Theta^TM_a^TP$ which includes the free parameters M_a and Θ.

To obtain the LMI from SDRI with M_a, we consider the following procedure. Since $-PB_2B_2^TP \leq 0,\ \forall P$, we will get

$$PA_0 + PM_a\Theta + A_0^TP + \Theta^TM_a^TP + \frac{1}{\gamma^2}PB_1B_1^TP + C_1C_1^T < 0. \quad (27)$$

By using Schur Complement, the matrix inequality (27) can be transformed into following LMI

$$\begin{bmatrix} PA_0 + G\Theta + A_0^TP + \Theta^TG^T + C_1^TC_1 & \frac{1}{\gamma}PB_1 \\ \frac{1}{\gamma}B_1^TP & -I \end{bmatrix} < 0 \quad (28)$$

with respect to variables P and $G = PM_a$.The first column of G should be 0. By solving this LMI, we will get a constant solution P and a free parameter to SDC form $M_a = P^{-1}G$.

3.5 Solving SDRI with M_a via BMI

By substituting $A = A_0 + M_a\Theta$ into (25), we get a matrix inequality

$$\begin{bmatrix} \begin{pmatrix} A_0X + XA_0^T + M_a\Theta X + X\Theta^TM_a^T \\ +\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T \end{pmatrix} & XC_1^T \\ C_1X & -I \end{bmatrix} < 0. \quad (29)$$

This is a Bilinear Matrix Inequality (BMI) which has a bilinear term $M_a\Theta X$. The variables to be calculated are M_a and X. Generally BMI does not have a formulation to calculate global solution. In this paper we have solved a BMI with iterating method. By fixing one of the variables BMI becomes LMI. Then BMI can be solved as LMI with fixing one variable and local solution is given by iterating step. The following steps describe the iterating method shortly.

(1) As a first step find P which is the solution of the Normal LMI (25).
(2) Solve LMI which is made by substituting P (found in step 1) into BMI (29) and get M_a as a solution of LMI.
(3) Solve LMI which is made by substituting M_a (found in step 2) into BMI (29) and get P as a solution of LMI.
(4) Iterate Step 2 and Step 3 until P converges.

By using this iterating method we get local solutions P and M_a of BMI.

4. COMPUTATION OF A CONSTANT SOLUTION

Generally the SDRI has a state-dependent solution $P(x)$. One way to find a state-dependent solution is to calculate point-wise solution of SDRI at each state. Or it is possible to consider the constant solution P which satisfies SDRI at local area.

As we saw in a previous section, the assumption which is so-called a integrability condition (8) is needed when we construct a SDRI from a HJI. In this section, a constant solution P is considered so that the integrability condition can be neglected and the advantage of free parameters of SDC form will be shown.

4.1 Considered System

Let us consider the nonlinear system (12) which is shown in previous section. The SDC form is selected as

$$S_{nl} \begin{cases} \dot{x} = (A_0(x)x + M_a(x)\Theta(x)) + B_1(x)w + B_2(x)u \\ z = C_1(x)x + D_{12}(x)u \end{cases}$$

$$A_0(x) = \begin{cases} \begin{bmatrix} -5\dfrac{\sin(2x_1)}{x_1} & 0 \\ 1-3x_1^2 & -2 \end{bmatrix} & (x \neq 0) \\ \begin{bmatrix} -10 & 0 \\ 1 & -2 \end{bmatrix} & (x = 0) \end{cases},$$

$$\Theta(x) = \begin{cases} \dfrac{1}{\|x\|}\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} & (x \neq 0) \\ \mathbf{0} & (x = 0) \end{cases}$$

$$,\quad B_1(x) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\quad B_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\quad C_1(x) = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}.$$

Parameters which are needed for calculation are $w = 15\sin(30t)$, $\gamma = 6.5$ and initial state $x_0 = [-4.0,\ 1.0]^T$.

Since a constant solution of SDRI is considered, it is important to check an area where P satisfies HJI. In this simulation, we checked P in a set which has 1681 states x

$$\{0.2[i_1,\ i_2]^T \mid -80 \leq i_1 \leq 80,\ -80 \leq i_2 \leq 80\} \quad (30)$$

where $i_1,\ i_2$ are integers.

4.2 Normal LMI at the Origin

Let us solve SDRI at the origin via normal LMI (25). The constant solution P is obtained as

$$P = \begin{bmatrix} 1.3637 & 2.0555 \\ 2.555 & 27.737 \end{bmatrix}.$$

The area which P satisfies HJI is shown in Fig. 1(a). In this case constant solution P satisfies 912/1681 states. The simulation results of states x_1, x_2 and input u are shown in Fig. 2(a), Fig. 2(b) and Fig. 2(c) respectively. As we can see, the area which constant solution P satisfies HJI is very small and the simulation with initial state x_0 can not be converged.

4.3 Normal LMI at Several States

Let us solve SDRI via normal LMI (25). This time we consider two states $x = [0,\ 0]^T$ and $x = [-1.0,\ 1.0]^T$ so that constant solution P satisfies a normal LMI at both states.

The constant solution P is

$$P = \begin{bmatrix} 10.184 & -3.4584 \\ -3.4584 & 47.406 \end{bmatrix}.$$

The area where P satisfies HJI is shown in Fig. 1(b). In this case constant solution P satisfies 1058/1681 states. The simulation results are shown in Fig. 2. As we can see, the area where constant solution P satisfies HJI becomes slightly larger than the one of normal LMI at the origin. And the system with initial state x_0 converges to the origin somehow.

4.4 LMI with $M_a(x)$ at Several States

Let us solve SDRI with $M_a(x)$ via LMI (28). This time also we consider same two states.

The constant solution P and a free parameter $M_a(x)$ are

$$P = \begin{bmatrix} 42.416 & 10.286 \\ 10.286 & 30.053 \end{bmatrix}$$

$$M_a([-1.0,\ 1.0]^T) = \begin{bmatrix} 0 & 4747.2 \\ 0 & 7725.2 \end{bmatrix}.$$

The area where P satisfies HJI is shown in Fig. 1(c). In this case constant solution P satisfies 1176/1681 states. The simulation results are shown in Fig. 2. As we can see, the area where constant solution P satisfies HJI becomes larger than the one of normal LMI at several states. And the system with initial state x_0 converges to the origin faster than Normal LMI.

4.5 BMI with $M_a(x)$ at Several States

Let us solve SDRI with $M_a(x)$ via BMI (29). This time also we consider same two states.

The constant solution P and a free parameter $M_a(x)$ are

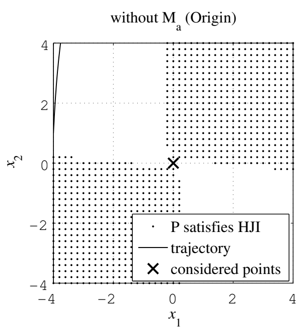
$$P = \begin{bmatrix} 124.81 & -1.7444 \\ -1.7444 & 34.677 \end{bmatrix}$$

$$M_a([-1.0,\ 1.0]^T) = \begin{bmatrix} 0 & 0.17508 \\ 0 & 5.2015 \end{bmatrix}.$$

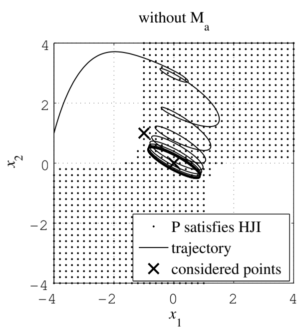
The area where P satisfies HJI is shown in Fig. 1(d). In this case constant solution P satisfies 1681/1681 states. The simulation results are shown in Fig. 2. As we can see, the constant solution P satisfies HJI at all states which we considered. And the system with initial state x_0 converges to the origin faster than LMI with $M_a(x)$.

5. CONCLUSION

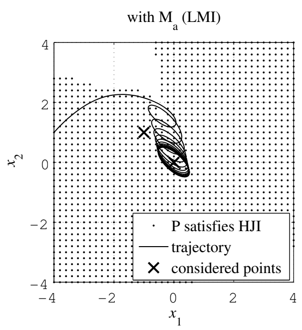
In this paper the solution of SDRI is considered as one of the solution of HJI related to nonlinear H_∞ control problem. It has been known that it is not unique to decide a SDC form of nonlinear system. To solve this problem we



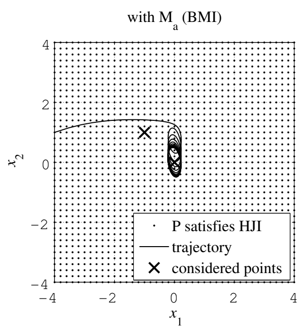
(a) Normal LMI at Origin



(b) Normal LMI at Several States

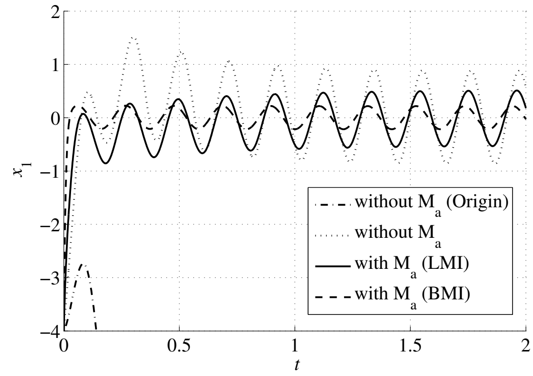


(c) LMI with $M_a(x)$ at Several States

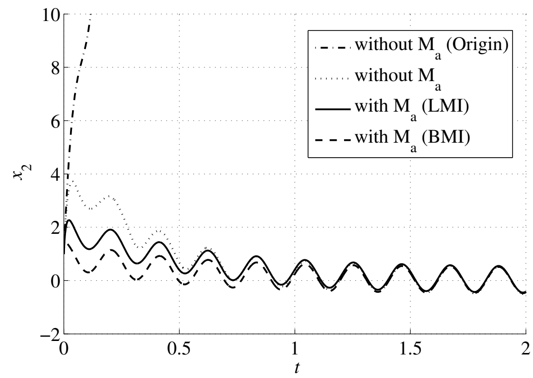


(d) BMI with $M_a(x)$ at Several States

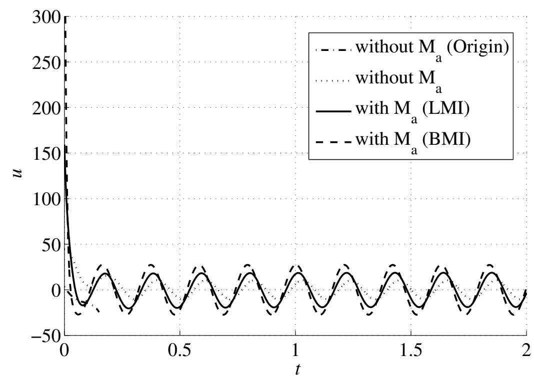
Fig. 1. The Area P Satisfies HJI and State Trajectory



(a) State x_1



(b) State x_2



(c) Input u

Fig. 2. Simulation Results of State and Input

have introduced the complete representation of the SDC form which includes free parameters. The effectiveness of free parameters are confirmed with numerical examples.

REFERENCES

James R. Cloutier and Donald T. Stansbery. Control of a continuously stirred tank reactor using an asymmetric solution of the state-dependent riccati equation. *IEEE CCA*, 1999.

E. B. Erdem and A. G. Alleyne. Globally stabilizing second order nonlinear systems by sdre control. *Proc. 1999 ACC*, pages 2501–2505, 1999.

Junichi Imura, Hiroshi Maeda, Toshiharu Sugie, and Tsuneo Yoshikawa. Robust stabilization of nonlinear systems by H_∞ state feedback. *32nd CDC*, pages 1650–1651, 1993a.

Junichi Imura, Toshiharu Sugie, and Tsuneo Yoshikawa. bounded real lemma of nonlinear systems and its application to H_∞ state feedback control. *32nd CDC*, pages 190–195, 1993b.

Chandrasekar Jaganath, Aaron Ridley, and Dennis. S. Bernstein. A sdre-based asymptotic observer for nonlinear discrete-time systems. *America Control Conference*, 2005.

Wei-Min Lu and John C. Doyle. H_∞ control of nonlinear systems: A convex characterization. *IEEE Trans. on Automatic Control*, 40(9):1668–1675, 1995.

L. Patpong, M. Sampei, M. Koga, and E. Shimizu. A numerical computational approach of hamilton-jacobi-isaacs equation in nonlinear H_∞ control problems. In *35th CDC*, 1996.

Jeff S. Shamma and James R. Cloutier. Existence of sdre stabilizing feedback. *IEEE TRANSACTIONS ON AUTOMATIC CONTROL*, 48(3), 2003.

A. J. van der Schaft. L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control. *IEEE Trans. on Automatic Control*, 37(6):770–784, 1992.

APPENDIX A

Hereunder the structure of combined rotation matrices are preseted.

$$\Theta(x) = \prod_{i=2}^{n} \Theta_i$$

Θ_i is a rotation matrix at $x_1 - x_i$ plane.

$$\Theta_i = \begin{bmatrix} \cos\theta_i & \mathbf{0} & \sin\theta_i & \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ -\sin\theta_i & \mathbf{0} & \cos\theta_i & \\ & \mathbf{0} & & I \end{bmatrix}$$

Let us choose trigonometric function as below.

$$(\cos\theta_i, \sin\theta_i) := \begin{cases} (t_{i+1}/t_i, x_i/t_i) & t_i \neq 0 \\ (1, 0) & t_i = 0 \end{cases}$$

$$t_i := \begin{cases} \sqrt{x_1^2 + \sum_{k=i}^{n} x_k^2} & 2 \le i \le n \\ x_1 & i = n+1 \end{cases}$$

The first row of $\Theta(x)$ is calculated to

$$\vartheta_{11} = \prod_{k=2}^{n} \cos\theta_k = \frac{t_3}{t_2}\frac{t_4}{t_3}\cdots\frac{t_{n+1}}{t_n} = \frac{t_{n+1}}{t_2} = \frac{x_1}{\|x\|}$$

$$\vartheta_{12} = \sin\theta_2 = \frac{x_2}{\|x\|}$$

$$\vartheta_{13} = \sin\theta_3\cos\theta_2 = \frac{x_3}{t_3}\frac{t_3}{t_2} = \frac{x_3}{\|x\|}$$

$$\vdots$$

$$\vartheta_{1i} = \sin(\theta_i)\prod_{k=2}^{i-1}\cos(\theta_k) = \frac{x_i}{t_i}\frac{t_3}{t_2}\cdots\frac{t_i}{t_{i-1}} = \frac{x_i}{\|x\|}.$$

So the first row of $\Theta(x)$ is $x^T/\|x\|$.

Other expressions of $\Theta(x)$ are exist.M_a include the degree of freedom of it. So we choose this simple one.