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Limitations of nonlinear periodic sampled-data control for robust stabilization

Robert Schmid *

* *Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville 3010 Australia. (email: schmid@ee.unimelb.edu.au)*

Abstract: This paper presents a performance analysis of nonlinear periodically time varying sampled-data controllers acting upon a linear time invariant plant. Time invariant controllers are distinguished from strictly periodically time varying controllers. For a given nonlinear strictly periodic controller, a time invariant controller is constructed. Necessary and sufficient conditions are given under which the time invariant controller gives strictly better control performance than the time invariant controller from which it was obtained, for the robust stabilization of L_p unstructured perturbations, for all $p \in [1, \infty]$.

Keywords: Robust stabilization, L_p space, Nonlinear Periodic systems, Sampled-data systems

1. INTRODUCTION

Time-varying and nonlinear feedback control is often applied to systems for which conventional linear time invariant control cannot achieve the desired system performance. The use of periodic linear and nonlinear control to achieve particular performance specifications has been actively studied for the last two decades. Periodic control has been shown to have advantages over time-invariant control in a number of areas, including simultaneous stabilization of a number of plants Das (2001), stabilization of nonholonomic systems Godhavn and Egeland (1997), and output feedback stabilization and pole placement Moreau and Aeyels (2004); Yen and Wu (1993).

Analyses of the limitations of time-varying linear and nonlinear control have also been done. A number of results have shown that time varying and nonlinear control provides no advantages over linear time-invariant control (LTI) for controlling LTI plants for disturbance rejection Dahleh and Shamma (1992); Chapellat and Dahleh (1992). In Schmid and Zhang (2003); Schmid (2006) it was shown that periodic control of LTI plants can give strictly worse disturbance rejection performance than time invariant control.

In this note, we analyze the performance of periodic nonlinear sampled-data controllers of continuous time LTI plants for the problem of robust stabilization. The problem of robust stabilization of an LTI plant involves considering a family of plants, and obtaining a controller which stabilizes the closed loop system for all plants in the family. In Khargonekar et al (1985) it was shown that for discrete controllers of LTI discrete plants, linear time varying controllers could provide improved gain and phase margins, relative to that achievable by LTI controllers. For problems involving unstructured perturbations of an nominal LTI plant, numerous authors have shown that linear and nonlinear time-varying controllers offer no advantages over LTI controllers for these problems (e.g. Chapellat and Dahleh (1992); Poolla and Ting (1987); Shamma and Dahleh (1991); Schmid (2006); Djouadi (2003)).

In our analysis, strictly nonlinear periodically time varying controllers will be distinguished from nonlinear time invariant controllers. For a given strictly periodically time varying sampled-data controller, we give necessary and sufficient conditions for the construction of a time invariant sampled-data controller that will give superior robust stabilization performance than the strictly periodic controller. Our results show that an optimal stabilizing controller will be time invariant. Hence the use of strictly periodically time varying controllers for achieving certain performance specifications may come at a price; a strictly periodic controller can give inferior performance for the robust stabilization of unstructured perturbations.

This paper is organized as follows: Section 2 provides some necessary mathematical preliminaries and formulates the problem of robust stabilization for unstructured perturbations by nonlinear periodic sampled-data control. Section 3 presents properties of nonlinear periodic systems to be used for the performance analysis. In Section 4, for a given periodic nonlinear sampled-data controller, we show how to construct a time invariant nonlinear sampled-data controller. Section 5 compares the robust stabilization performance of the nonlinear periodic sampled data controller with that of the constructed time invariant sampled-data controller.

2. PROBLEM FORMULATION

For any $p \in [1, \infty)$, let L_p^n be the space of all n-dimensional Lebesgue measurable vector functions $u : \mathbf{R}^+ \to \mathbf{R}^n$ with each $u \in L_p^n$ having bounded L_p^n norm $\|u\|_p = \left(\int_0^\infty |u(t)|_p^p \, dt\right)^{\frac{1}{p}}$, where $|\cdot|_p$ is the p-norm on $\mathbf{R}^n$, i.e. $|x|_p =$

$(\sum_{i=1}^{n} |x_i|^p)^{1/p}$. Also let L_∞^n be the space of functions with bounded L_∞^n norm $\|u\|_\infty = \text{ess sup}\{|u(t)|_\infty : t \in \mathbf{R}^+\}$. We denote $L_0^n = \{u \in L_\infty^n : \lim_{t\to\infty} \sup_{\tau \geq t} |u(\tau)| = 0\} \subseteq L_\infty^n$. We consider systems $G : L_p^n \to L_p^m$. G is assumed to be nonlinear, in the sense of not necessarily linear. We write the evaluation of a system G at a signal u as $G(u) = Gu$ and for all $t \in \mathbf{R}^+$, $(Gu)(t)$ denotes the value of the signal Gu at time t. We write the composition of systems F and G (assuming it exists) as $G \circ F = GF$. When composing three or more systems, we assume the order of operation is right-to-left: $HGF = H \circ (G \circ F)$. For any $\tau \in \mathbf{R}$ and $p \in [1, \infty]$, let $q^{-\tau} : L_p^n \to L_p^n$ be the back shift operator defined by $(q^{-\tau}u)(t) = u(t-\tau)$ for all $u \in L_p^n$, and $t \in \mathbf{R}^+$. Let $P_\tau : L_p^n \to L_p^n$ be the truncation operator defined by

$$(P_\tau u)(t) = \begin{cases} u(t) & \text{if } t \leq \tau \\ 0 & \text{elsewhere.} \end{cases} \quad (1)$$

G is *causal* if, for all $\tau \in \mathbf{R}^+$ and all $u \in L_p^n$, $P_\tau G u = P_\tau G P_\tau u$.

G has *pointwise finite memory* (Shamma and Zhao (1993)) if there exists a function $FM(\cdot,\cdot;G) : L_p^n \times \mathbf{R}^+ \to \mathbf{R}^+$ such that for all $u \in L_p^n$ and $t \in \mathbf{R}^+$,

(1) $FM(u, t; G) \geq t$,
(2) $FM(u, t; G) = FM(P_t u, t; G)$,
(3) $(I - P_{FM(u,t;G)})Gu = (I - P_{FM(u,t;G)})G(I - P_t)u$

G has *pointwise fading memory* if it can be approximated arbitrarily closely in norm by pointwise finite memory systems.

A system G is *time invariant*, if $G = q^\tau G q^{-\tau}$ for all $\tau \in \mathbf{R}^+$. A system G is *T-periodic*, $0 < T < \infty$, if $G = q^T G q^{-T}$ and $G \neq q^\tau G q^{-\tau}$ for all $\tau \in (0, T)$.

The L_p^m*-induced system norm* of G is given by

$$\|G\|_p = \sup\left\{\|Gu\|_p / \|u\|_p : \ u \in L_p^n, \ u \neq 0\right\} \quad (2)$$

G is a *stable* system if $\|G\|_p < \infty$. For stable G, there exists a sequence of non-zero signals $\{u_k\}_{k\in\mathbf{Z}^+} \subseteq L_p^n$ such that $\|G\|_p = \lim_{k\to\infty} \frac{\|Gu_k\|_p}{\|u_k\|_p}$; we say G *attains its norm* on the sequence. The L_p^m *incremental norm* of G is

$$\|G\|_p^{inc} = \sup\left\{\frac{\|Gu - Gv\|_p}{\|u - v\|_p} : \ u, v \in L_p^n, \ u - v \neq 0\right\}. \quad (3)$$

G is *incrementally* L_p *stable* if $\|G\|_p^{inc} < \infty$. If G is incrementally stable then it is also stable, and $\|G\|_p^{inc} = \|G\|_p$.

We consider the closed-loop sampled-data control system $\Phi(P_0, \hat{K}_h)$ in Figure 1, where P_0 is an LTI nth order continuous time plant, and $u_1(t), y_2(t), e_1(t) \in \mathbf{R}^n, u_2(t), y_1(t), e_2(t) \in \mathbf{R}^m$, S is a sampler and H is a zero-order hold which are synchronized with a sampling period h; thus $\hat{y}_2(k) = (Sy_2)(k) = y_2(kh)$ and $u(t) = (H\hat{u}_2)(t) = \hat{u}_2([t/h])$, where $[t/h]$ denotes the integer part of t/h. $\hat{K}_h$ is a n-input m-output nonlinear N-periodic discrete time controller. If $\hat{K}_h$ has integer period $N \geq 2$, we say that $\hat{K}_h$ is *strictly periodic*; if $N = 1$, we say it is *time invariant.* This feedback system is said to be *well posed* if given input signals $(u_1, u_2) \in L_p^{n+m}$ there exist unique $(e_1, e_2) \in L_p^{n+m}$ satisfying

$$e_1 = u_1 - Ke_2, \ e_2 = u_2 + P_0 e_1 \quad (4)$$

such that the input-output mapping $\Phi(P_0, \hat{K}_h) : (u_1, u_2) \mapsto (e_1, e_2)$ is causal. $\Phi(P_0, \hat{K}_h)$ is Nh-periodic due to the h-periodic sampling process. A controller $\hat{K}_h$ is said to *incrementally pointwise fading memory* L_p *stabilize* P_0 if $\Phi(P_0, \hat{K}_h))$ is incrementally L_p stable with pointwise fading memory.

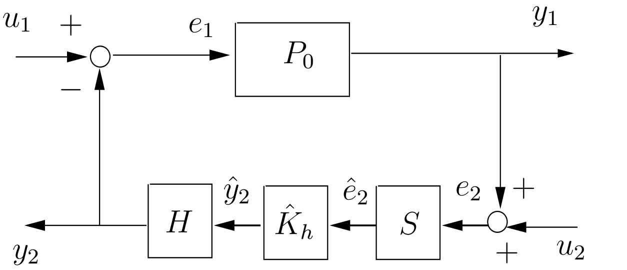

Fig. 1. The closed loop sampled-data control system $\Phi(P, \hat{K}_h)$

A commonly considered robust stabilization problem (Poolla and Ting (1987); Schmid (2006); Shamma and Dahleh (1991); Shamma and Zhao (1993)) involves additive unstructured uncertainties. We consider a family of plants

$$\mathcal{P}_{add} = \{P : P = P_0 + \Delta W\} \quad (5)$$

where P_0 is a "nominal" strictly causal LTI plant, $\Delta : L_p^n \to L_p^m$ is a stable nonlinear time varying perturbation, and $W : L_p^n \to L_p^n$ is a stable LTI system with stable inverse (weighting function). We assume that all plants in $\mathcal{P}_{add}$ result in a well-posed feedback system. A discrete controller $\hat{K}_h$ is said to *robustly* L_p *stabilize* $\mathcal{P}_{add}$ if it stabilizes $\Phi(P, \hat{K}_h)$ for every $P \in \mathcal{P}_{add}$. For a given plant P_0 and controller $\hat{K}_h$, and for each $p \in [1, \infty)$, we define the *additive* L_p *robust stability margin* of $\hat{K}_h$ to be the largest value of r such that $\mathcal{P}_{add}$ is robustly L_p stabilized by $\hat{K}_h$ for all nonlinear time varying perturbations $\Delta : L_p^n \to L_p^m$ satisfying $\|\Delta\|_p < r$. For $p = \infty$ we consider perturbations $\Delta : L_0^n \to L_0^m$ satisfying $\|\Delta\|_\infty < r$. We denote this robust stability margin by $r_p(\hat{K}_h)$. When comparing two controllers, the one that achieves the greater stability margin is said to give superior robust stabilization performance.

3. ANALYSIS OF PERIODIC SYSTEM NORMS

In this section, $G : L_p^n \to L_p^m$ is a stable periodic system with period $T = Nh$, where $N, \ n, \ m \in \mathbf{Z}^+$, $N \geq 2$, $h > 0$ and $p \in [1, \infty]$. Also i is an integer with $0 \leq i \leq N - 1$. For brevity we introduce $G_{ih} : L_p^n \to L_p^m$ with $G_{ih} = q^{ih} G q^{-ih}$.

Definition 1. Define $G_{TI} : L_p^n \to L_p^m$ with

$$G_{TI} = \frac{1}{N} \sum_{i=0}^{N-1} G_{ih} \quad (6)$$

Then G_{TI} is a stable periodic system with period $T = h$ and satisfies $\|G_{TI}\|_p \leq \|G\|_p$ for all $p \in [1, \infty]$, by the triangle inequality. We will establish, for each $p \in [1, \infty]$, conditions under which $\|G_{TI}\|_p < \|G\|_p$.

For each $1 \leq i \leq N-1$, there must exist at least one signal $w \in L_p^n$ such that $Gw \neq q^{ih}Gq^{-ih}w$, else G would be have period less than Nh. However, there may be one or more signals $w \in L_p^n$ for which $Gw = q^{ih}Gq^{-ih}w$, for all $1 \leq i \leq N-1$. To characterize this situation precisely, we will introduce the notion of L_p *h-periodicity.*

Definition 2. For any $p \in [1, \infty]$, G is L_p *norm h-periodic to an input signal* $u \in L_p^n$ if, for all $0 \leq i \leq N-1$, $\|Gu\|_p = \|G_{ih}u\|_p$, and G is L_p *norm h-periodic to a sequence of input signals* $\{u_k\}_{k\in\mathbf{Z}^+} \subseteq L_p^n$ if, for all $0 \leq i \leq N-1$,

$$\lim_{k\to\infty} \frac{\|Gu_k\|_p}{\|u_k\|_p} = \lim_{k\to\infty} \frac{\|G_{ih}u_k\|_p}{\|u_k\|_p} \tag{7}$$

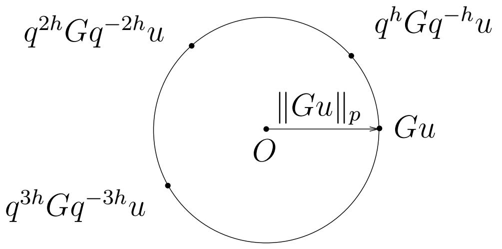

Fig. 2. G is L_p norm h-periodic to $u \in L_p^n$

Definition 3. 1. $p = 1$ G is L_1 *h-periodic to* $u \in L_1^n$ if (a) G is L_1 norm h-periodic to u, and (b) the set

$$M_1 = \{t \in \mathbf{R}^+ : |\sum_{i=0}^{N-1} G_{ih}u(t)|_1 < \sum_{i=0}^{N-1} |G_{ih}u(t)|_1\} \tag{8}$$

has measure zero.

2. $p \in (1, \infty)$ G is L_p *h-periodic to* $u \in L_p^n$ if $Gu = G_{ih}u$ for all $0 \leq i \leq N-1$.

3. $p = \infty$ G is L_∞ *h-periodic to* $u \in L_\infty^n$ if (a) G is L_∞ norm h-periodic to u, and (b) $\|\sum_{i=0}^{N-1} G_{ih}u\|_\infty = \sum_{i=0}^{N-1} \|G_{ih}u\|_\infty$.

Example 4. Let $N = 2$, $m = 1$ and assume G and u are such that $(Gu)(t) = e^{-t^2}$ and $(G_hu)(t) = -e^{-t^2}$. Then G is L_1 norm h-periodic to u, but $M_1 = \mathbf{R}^+$ has positive measure, so G is not L_1 h-periodic to u. As $Gu \neq G_hu$, G is not L_p h-periodic to u for all $p \in (1, \infty)$. G is L_∞ norm h-periodic to u, but G is not L_∞ h-periodic to u.

Definition 5. For any $p \in (1, \infty)$, G is L_p h-periodic to a sequence of input signals $\{u_k\}_{k\in\mathbf{Z}^+} \subseteq L_p^n$ if, for all $0 \leq i \leq N-1$,

$$\lim_{k\to\infty} \frac{\|(G - G_{ih})u_k\|_p}{\|u_k\|_p} = 0 \tag{9}$$

Definition 6. For any $p \in \{1, \infty\}$, G is L_p h-periodic to a sequence of input signals $\{u_k\}_{k\in\mathbf{Z}^+} \subseteq L_p^n$ if (a) G is L_p norm h-periodic to $\{u_k\}$, and (b) if the sequences $\{G_{ih}u_k/\|u_k\|_p\}_{k\in\mathbf{Z}^+}$ are convergent with limits $y_i \in L_p^m$ for each $0 \leq i \leq N-1$, then,

1. $p = 1$ the set

$$M_1 = \{t \in \mathbf{R} : |\sum_{i=0}^{N-1} y_i(t)|_1 < \sum_{i=0}^{N-1} |y_i(t)|_1\} \tag{10}$$

has measure zero.

2. $p = \infty$ $\|\sum_{i=0}^{N-1} y_i\|_\infty = \sum_{i=0}^{N-1} \|y_i\|_\infty$.

We will use the phrase "in the absence of h-periodicity" to describe the situation where G is not L_p h-periodic to a sequence of signals upon which it attains its norm. The next theorem states that, in the absence of h-periodicity, $\|G_{TI}\|_p < \|G\|_p$. For $p \in (1, \infty)$, this condition is both necessary and sufficient. For $p \in \{1, \infty\}$, it is sufficient; necessity can be shown with the addition of a further assumption. When h-periodicity occurs $\|G\|_p = \|G_{TI}\|_p$.

Theorem 7. For any $p \in [1, \infty]$, the h-periodic system in (6) satisfies

$$\|G_{TI}\|_p < \|G\|_p \tag{11}$$

if G is not L_p h-periodic to any sequence $\{w_k\}_{k\in\mathbf{Z}^+} \subseteq L_p^n$ of inputs on which G attains its L_p norm.

For $p \in (1, \infty)$, if G is L_p h-periodic to a sequence $\{w_k\}_{k\in\mathbf{Z}^+} \subseteq L_p^n$ of inputs on which G attains its L_p norm, then

$$\|G_{TI}\|_p = \|G\|_p \tag{12}$$

For $p \in \{1, \infty\}$, if G is L_p h-periodic to a sequence $\{w_k\}_{k\in\mathbf{Z}^+} \subseteq L_p^n$ of inputs on which G attains its L_p norm, and the output sequences $\{G_{ih}w_k/\|w_k\|_p\}_{k\in\mathbf{Z}^+}$ are convergent in L_p^m, for all $0 \leq i \leq N-1$, then

$$\|G_{TI}\|_p = \|G\|_p \tag{13}$$

Proof See Theorem 5.2 in Schmid and Zhang (2003)

A natural question is to ask how likely an Nh-periodic system is to be h-periodic to one or more input signals. We define

Definition 8. Let $G : L_p^n \to L_p^m$ be Nh-periodic. For each $p \in [1, \infty]$, we define the L_p h-periodic set of G to be

$$M_p(G) = \{w \in L_p^n : G \text{ is } L_p \text{ } h\text{-periodic to w }\} \tag{14}$$

Our next result shows that a linear system will be h-periodic to few input signals:

Lemma 9. Assume that G is linear and that $p \in (1, \infty)$. Then the L_p h-periodic set of G has empty interior.

Proof: Define for each $1 \leq i \leq N-1$, the Nh-periodic system $F_i : L_p^n \to L_p^m$ with $F_i = G - G_{ih}$. From Definition 3.2, $w \in M_p(G)$ if and only if $F_iw = 0$ for all $1 \leq i \leq N-1$. Thus $M_p(G) = \bigcap_{1\leq i\leq N-1} ker(F_i)$, where $ker(F_i) = \{w \in L_p^n : F_iw = 0\}$. As G is linear, each G_{ih} is linear and hence each F_i is linear. Thus each $ker(F_i)$ is a proper subspace of L_p^n and hence has empty interior. Thus $M_p(G)$ also has empty interior.□

As $M_p(G)$ is the intersection of $N-1$ sets, the likelihood of time invariance reduces with increasing N.

4. CONSTRUCTION OF A DISCRETE TIME INVARIANT CONTROLLER

In this section, $\hat{K}_h$ is a given discrete strictly periodic stabilizing controller for an LTI plant P_0 in Figure 1, with period $N \geq 2$. Theorem 10 introduces a time invariant discrete stabilizing controller $\hat{K}_{TI}$.

Theorem 10. Let $\hat{K}_h$ be any strictly periodic discrete controller with period $N \geq 2$ that incrementally pointwise fading memory stabilizes P_0. Let N, M, $\tilde{M}$, $\tilde{N}$, X, Y, $\tilde{X}$, $\tilde{Y}$ be stable, causal LTI discrete systems such that

$$SP_0H = NM^{-1} = \tilde{M}^{-1}\tilde{N},$$
$$I = \tilde{X}M - \tilde{Y}N = \tilde{M}X - \tilde{N}Y \qquad (15)$$

Then $\hat{K}_h$ may be parameterized by

$$\hat{K}_h = (Y - M\hat{Q}_h)(X - N\hat{Q}_h)^{-1} \qquad (16)$$
$$\hat{Q}_h = (\hat{K}_hN - M)^{-1}(\hat{K}_hX - Y) \qquad (17)$$

where $\hat{Q}_h$ is a strictly periodic stable causal discrete system with period N. Define the discrete control parameter $\hat{Q}_{TI}$ and discrete controller $\hat{K}_{TI}$ by

$$\hat{Q}_{TI} = \frac{1}{N}\sum_{i=0}^{N-1} q^i\hat{Q}_hq^{-i} \qquad (18)$$
$$\hat{K}_{TI} = (Y - M\hat{Q}_{TI})(X - N\hat{Q}_{TI})^{-1}. \qquad (19)$$

Then $\hat{K}_{TI}$ is time invariant and incrementally pointwise fading memory stabilizes P_0.

Proof: The existence of the parameterizations (16) - (17) follows from Theorem 2.7 in Poolla and Ting (1987). As $\hat{K}_h$ incrementally pointwise fading memory stabilizes P_0, $\hat{Q}_h$ is incrementally stable with pointwise fading memory (Theorem 5.2 in Shamma and Zhao (1993)). Hence by $\hat{Q}_{TI}$ is also incrementally stable with pointwise fading memory, and by Theorem 5.2 in Shamma and Zhao (1993) again, $\hat{K}_{TI}$ incrementally pointwise fading memory stabilizes P_0.

For two periodic operators A and B with periods T_A and T_B respectively, the product operator AB has period given by the least common multiple of T_A and T_B. The inverse of a periodic operator A has the same period as A. As $\hat{K}_h$ has period N, the parameter $\hat{Q}_h$ also has period N, because all the other systems in (17) are LTI. Hence $\hat{Q}_{TI}$ is a time invariant system, because

$$q^1\hat{Q}_{TI}q^{-1} = \frac{1}{N}\sum_{i=0}^{N-1} q^i\hat{Q}_hq^{-i} \qquad (20)$$
$$= \frac{1}{N}\sum_{i=1}^{N}\hat{Q}_h$$
$$= \hat{Q}_{TI}. \qquad (21)$$

Thus $\hat{K}_{TI}$ is also time invariant, because all the other systems in (19) are LTI. □

Our next Lemmas provide a formula for the robust stability margin of a controller $\hat{K}_h$ for the plant P_0.

Lemma 11. Sastry (1999) Let $\hat{K}_h$ be an incrementally pointwise fading memory stabilizing controller for P_0 as in Figure 1. Let

$$T(\hat{K}_h) = WH\hat{K}_hS(I + P_0H\hat{K}_hS)^{-1} \qquad (22)$$

The stability of closed loop systems in Figures 1 and 3 are equivalent.

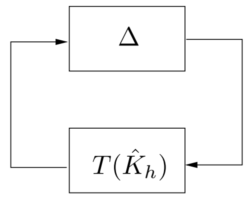

Fig. 3. Redrawn Block Diagram for robust stabilization

Lemma 12. For any $p \in [1, \infty]$, assume $\hat{K}_h$ incrementally point wise fading memory stabilizes P_0. The robust stability margin of K for $\mathcal{P}_{add}$ is

$$r_p(\hat{K}_h) = \frac{1}{\|T(\hat{K}_h)\|_p} \qquad (23)$$

Proof: By definition,

$$r_p(\hat{K}_h) = \sup\{\alpha > 0 : \text{ if } \|\Delta\|_p < \alpha, \text{ then } P_0 + \Delta W \text{ is stabilized by } \hat{K}_h\} \qquad (24)$$

Let $\|\Delta\|_p$ be an admissible disturbance satisfying $\|\Delta\|_p < \frac{1}{\|T(\hat{K}_h)\|_p}$; then $\|\Delta\|_p\|T(\hat{K}_h)\|_p < 1$, and so by the Small Gain Theorem, $P_0 + \Delta W$ is stabilized by $\hat{K}_hK$. Thus

$$\frac{1}{\|T(\hat{K}_h)\|_p} \leq r_p(\hat{K}_h) \qquad (25)$$

Next let Δ be an admissible disturbance with $\|\Delta\|_p < r_p(\hat{K}_h)$; then by Theorem 5.1 in Shamma and Zhao (1993) (noting that periodic systems satisfy the UINE property), $\hat{K}_h$ stabilizes $P_0 + \Delta W$ and

$$\|T(\hat{K}_h)\|_p \leq \frac{1}{r_p(\hat{K}_h)} \qquad (26)$$

□

5. L_P PERFORMANCE ANALYSIS OF PERIODIC SAMPLED-DATA CONTROLLERS

We now present our main result comparing the additive robust stability margins of the strictly periodic and time invariant controllers $\hat{K}_h$ and $\hat{K}_{TI}$. Theorem 13 shows that, in the absence of h-periodicity, $\hat{K}_{TI}$ will give strictly better robust stability performance than $\hat{K}_h$.

Theorem 13. For any $p \in [1, \infty]$, let $\hat{K}_h$ be any strictly periodic controller with parameterization (16)-(17) and period $N \geq 2$ that incrementally pointwise fading memory L_p stabilizes P_0. The time invariant stabilizing controller $\hat{K}_{TI}$ defined in (19) also incrementally pointwise fading memory L_p stabilizes P_0, and gives strictly better L_p robust stabilization than $\hat{K}_h$ in the sense that

$$r_p(\hat{K}_{TI}) > r_p(\hat{K}_h), \quad (27)$$

if the closed loop system $T(\hat{K}_h)$ with

$$T(\hat{K}_h) = WH\hat{K}_h S(I + P_0 H\hat{K}_h S)^{-1} \quad (28)$$

is not L_p h-periodic to any sequence $\{w_k\}_{k\in\mathbf{Z}^+} \subseteq L_p^n$ of inputs on which $T(\hat{K}_h)$ attains its L_p norm.

Proof: Define $T(\hat{K}_{TI})$ with

$$T(\hat{K}_{TI}) = WH\hat{K}_{TI} S(I + P_0 H\hat{K}_{TI} S)^{-1}. \quad (29)$$

Then $T(\hat{K}_h)$ and $T(\hat{K}_{TI})$ have affine representations

$$T(\hat{K}_h) = S(Y\tilde{M} - M\hat{Q}_{TV}\tilde{M})H, \quad (30)$$

$$T(\hat{K}_{TI}) = S(Y\tilde{M} - M\hat{Q}_{TI}\tilde{M})H \quad (31)$$

Substituting for (18), we obtain

$$T(\hat{K}_{TI}) = S\left(\frac{1}{N}\sum_{i=0}^{N-1} q^{ih} T(\hat{K}_h) q^{-ih}\right) H \quad (32)$$

and so $T(\hat{K}_h)$ and $T(\hat{K}_{TI})$ are related by (6). Hence by Theorem 7,

$$\|T(\hat{K}_{TI})\|_p < \|T(\hat{K}_h)\|_p \quad (33)$$

Applying Lemma 12 yields (27). □

Since $\|T(\hat{K}_{TI})\|_p \leq \|T(\hat{K}_h)\|_p$, the stability margin of $\hat{K}_{TI}$ at least matches that of $\hat{K}_h$, and hence the time invariant controller $\hat{K}_{TI}$ provides at least equal, and in the absence of h-periodicity, superior robust stabilization than the strictly periodic controller from which it was derived.

We have assumed $\hat{K}_h$ incrementally pointwise fading memory stabilizes P_0 because this allows us to use Theorem 5.1 from Shamma and Zhao (1993) on the necessity of the small gain theorem. For the case $p = \infty$, Theorem 5.1 from that paper applies only to nonlinear time varying perturbations $\Delta : L_0^n \to L_0^m$, and this is why the definitions of robust stability margin given above differ for the cases $p \in [1, \infty)$ and $p = \infty$.

Recently Schmid (2006) showed the superior performance of discrete nonlinear time invariant controllers, relative to discrete periodic controllers, for the robust stabilization of an LTI discrete time plant. The above result extends the result of that paper to show the superior performance of nonlinear time invariant sampled-data controllers, for the robust stabilization of an LTI continuous time plant.

If $\hat{K}_h$ in Theorem 13 is linear strictly periodic, then $\hat{K}_{TI}$ is LTI, and hence specializing Theorem 13 to linear controllers reveals the superior performance of linear time invariant sampled-data controllers in comparison with linear strictly periodic sampled-data controllers. Shamma and Dahleh (1991) considered discrete systems (discrete LTI plant with a discrete controller), and showed that the for l_∞ signals, the robust stability performance of an linear time varying controller could be equalled by an LTI controller. This result is similarly extended to sampled-data systems. Moreover, in this paper we have explicitly constructed the LTI controller, and shown that in the absence of h-periodicity it provides strictly superior performance than the linear strictly periodic controller from which it was obtained.

Corresponding versions of Theorem 13 can be obtained for other plant families, for example multiplicative uncertainties Khargonekar et al (1987); Djouadi (2003)

$$\mathcal{P}_{mult} = \{P : P = (I + \Delta W)P_0\} \quad (34)$$

The relevant closed loop system response is

$$T_{mult}(\hat{K}_h) = WP_0 H\hat{K}_h S(I + P_0 H\hat{K}_h S)^{-1} \quad (35)$$

6. CONCLUSION

We have investigated the performance of strictly periodic sampled-data controllers for the stabilization of an LTI continuous time plant. The results imply that the use of periodic controllers to achieve performance specifications such as improvement in output feedback and pole placement may come at the price of inferior performance with respect to robust stabilization, relative to that achievable by time invariant controllers. The results give new insights into the limitations of periodic feedback control.

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