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Performance Analysis of Kalman Filter Based Hybrid Estimation Algorithms

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Abstract: Kalman filter based hybrid estimation algorithms have been used in many applications. However, performance analysis of these algorithms is difficult because many of these algorithms use a set of Kalman filters that are coupled with each other. We present an algorithm to compute the means and cross-covariances of the residuals of the set of coupling (or interacting) Kalman filters. Specifically, we derive the cross-covariances, each of which is the covariance of the residuals of two interacting Kalman filters, to account for the mutual interactions between the Kalman filters. From the means and cross-covariances of the Kalman filter residuals, we then compute the means of the likelihood functions and the mean-squared estimation errors as performance measures of hybrid estimation algorithms. We consider the Interacting Multiple Model algorithm as an example in this paper. In general, the proposed algorithm could be applicable to various Kalman filter based hybrid estimation algorithms.

1. INTRODUCTION

Hybrid estimation (or multiple model) algorithms have been used in various applications, such as target tracking [Bar-Shalom et al. (2001); Maybeck (1982)] and fault diagnosis [Zhang and Li (1998)]. In these applications, the system dynamics can be modeled as a stochastic hybrid system consisting of both continuous state evolutions and discrete state (or mode) transitions. Hybrid estimation (HE) involves estimating both the continuous state and the mode of the system. A Kalman filter based HE algorithm typically consists of a bank of Kalman filters, each matched to a mode of the hybrid system. The HE algorithm uses the residuals from the Kalman filters to compute the respective mode probabilities (i.e. probabilities of the modes being the correct one.) The optimal solution to this problem has an exponentially increasing complexity with respect to time due to the need to consider all mode histories [Li and Bar-Shalom (1993a)]. Two well-known suboptimal (but practical) solutions are the Generalized Pseudo Bayesian algorithm of order n (GPBn) [Maybeck (1982)] and the Interacting Multiple Model algorithm [Blom and Bar-Shalom (1988)]. The GPBn algorithm considers r^n previous mode histories (or hypotheses) at each time step, where r is the number of modes. The IMM algorithm considers only r mode histories. However, it uses a more sophisticated hypothesis merging technique, known as "mixing" to achieve comparable performance as the GPB2 algorithm with a lower computational complexity. On the other hand, the "mixing" technique results in mutual interactions among the Kalman filters, and makes the analysis of the IMM algorithm difficult.

In most Kalman filter based HE algorithms, the Kalman filter residuals are the only information for mode probability updates. Hence, a good understanding of the characteristics of the Kalman filter residuals is important to the analysis of the algorithms' performances. Hanlon and Maybeck (2000) proposed a method to compute the mean and covariance of the residual of a single Kalman filter in the presence of model mismatches. This method may be used to analyze HE algorithms, such as GPB1, in which the set of Kalman filters run independently without mutual interactions. However, this method cannot be used in a HE algorithm that uses a set of interacting Kalman filters. In this paper, we consider a bank of interacting Kalman filters and characterize their residuals by computing (in addition to their respective means and covariances) their *cross-covariances*, where each cross-covariance is the covariance of the residuals of two interacting Kalman filters. This cross-covariance term, which is a novel idea of this paper, accounts for the coupling between the two Kalman filters. In addition, from the characterization of the Kalman filter residuals, we propose an algorithm to investigate the performance of the HE algorithm off-line, a task which otherwise has to be carried out by costly Monte Carlo simulations in most cases. Furthermore, our analytical approach provides different insights into the performance of the HE algorithm compared to the simulation approach. In this paper, we consider the performance analysis of the IMM algorithm as an example. The proposed algorithm could be applicable to various Kalman filter based HE algorithms in general. We would like to note that an algorithm for off-line performance predictions of the IMM algorithm has also been proposed by Li and Bar-Shalom (1993b). However, they do not explicitly compute the cross-covariances of the Kalman filter residuals to account for the mutual interactions between the various Kalman filters. Furthermore, their algorithm requires some restrictive assumptions, such that the various modes of the hybrid system are required to have the same state matrix and output matrix. Our proposed algorithm does not need these assumptions.

The rest of the paper is organized as follows: Section

2 presents some preliminary discussions, which includes a review of the IMM algorithm and a definition of the operating scenario under which the IMM algorithm is evaluated. The algorithm development is presented in Section 3. Section 4 presents some simulation results to illustrate the proposed algorithm. Conclusions are given in Section 5.

2. PRELIMINARIES

2.1 Review of the IMM Algorithm

The Interactive Multiple Model Estimation (IMM) algorithm uses a bank of Kalman filters, each matched to a mode of the following stochastic hybrid system:

$$x(k) = A_{m(k)}x(k-1) + B_{m(k)}u(k) + w_{m(k)}(k) \tag{1}$$

$$z(k) = C_{m(k)}(k)x(k) + v_{m(k)}(k) \tag{2}$$

where $x(k)$ is the (discrete-time) continuous state; $u(k)$ is the input vector; $z(k)$ is the measurement vector; $m(k) \in \{1, 2, \ldots, r\}$ is the discrete state (or mode); $w_{m(k)}(k)$ and $v_{m(k)}(k)$ are uncorrelated Gaussian noise vectors with covariance matrices $Q_{m(k)}$ and $R_{m(k)}$ respectively. The matrices A_j, B_j, etc. model the system dynamics corresponding to mode $m(k) = j$. Denoting the event $\{m(k) = j\}$ as $m_j(k)$, the evolution of mode $m(k)$ is described by

$$\pi_{ij} = p[m_j(k)|m_i(k-1)] \quad \text{for} \quad i, j = 1, \ldots, r$$

where π_{ij} is a constant; $p[\cdot|\cdot]$ denotes a conditional probability. We assume that at time $k-1$, we have, from each Kalman filter i, the posterior mean $\hat{x}_i(k-1)$ and covariance $P_i(k-1)$ of the continuous state estimate; and the posterior mode probability (that the true mode is mode i) $\alpha_i(k-1)$. The IMM algorithm updates the mean $\hat{x}_j(k)$, covariance $P_j(k)$, and mode probability $\alpha_j(k)$ for each Kalman filter j recursively as follows:

(1) *Mixing/Interacting:* Compute mixing probability

$$\begin{aligned}\gamma_{ji}(k-1) &:= p[m_i(k-1)|m_j(k)] \\ &= \frac{1}{\sum_{l=1}^{r} \pi_{lj}\alpha_l(k-1)} \pi_{ij}\alpha_i(k-1)\end{aligned} \tag{3}$$

The initial conditions to Kalman filter j are given by

$$\hat{x}_{j0}(k-1) = \sum_{i=1}^{r} \gamma_{ji}(k-1)\hat{x}_i(k-1) \tag{4}$$

$$\begin{aligned}P_{j0}(k-1) = \sum_{i=1}^{r} \big\{ P_i(k-1) + [\hat{x}_i(k-1) - \\ \hat{x}_{j0}(k-1)][\hat{x}_i(k-1) - \hat{x}_{j0}(k-1)]^T \big\} \gamma_{ji}(k-1)\end{aligned} \tag{5}$$

(2) *Filtering:* Each Kalman filter j computes

$$\hat{x}_j^-(k) = A_j\hat{x}_{j0}(k-1) + B_ju(k) \tag{6}$$

$$\hat{x}_j(k) = \hat{x}_j^-(k) + K_j(k)r_j(k) \tag{7}$$

$$r_j(k) := z(k) - C_j\hat{x}_j^-(k) \tag{8}$$

$$K_j(k) = P_j^-(k)C_j^T S_j(k)^{-1} \tag{9}$$

$$P_j^-(k) = A_jP_{j0}(k-1)A_j^T + Q_j \tag{10}$$

$$S_j(k) = C_jP_j^-(k)C_j^T + R_j \tag{11}$$

$$P_j(k) = [I - K_j(k)C_j]P_j^-(k) \tag{12}$$

(3) *Mode Probability Update:* Compute the Likelihood function

$$\Lambda_j(k) := \mathcal{N}_q(r_j(k); 0, S_j(k)) \tag{13}$$

where q is the dimension of $r_j(k)$; $\mathcal{N}_q(\cdot; \mu, \Sigma)$ denote a q-dimensional multivariate Gaussian pdf with mean μ and covariance Σ. Compute the prior mode probability

$$\alpha_j^-(k) = \sum_{i=1}^{r} \pi_{ij}\alpha_i(k-1) \tag{14}$$

The posterior mode probability is given by

$$\alpha_j(k) = \frac{1}{\sum_{l=1}^{r} \Lambda_l(k)\alpha_l^-(k)} \Lambda_j(k)\alpha_j^-(k) \tag{15}$$

(4) *Output:* The mean and covariance of the combined state estimate are

$$\hat{x}(k) = \sum_{j=1}^{r} \alpha_j(k)\hat{x}_j(k) \tag{16}$$

$$P(k) = \sum_{j=1}^{r} \big\{ P_j(k) + [\hat{x}_j(k) - \hat{x}(k)][\hat{x}_j(k) - \hat{x}(k)]^T \big\} \alpha_j(k) \tag{17}$$

2.2 Operating scenario

The performance of the IMM algorithm (as well as other HE algorithms in general) depends on the operating scenario [Li and Bar-Shalom (1993b)]. Mathematically, we describe an operating scenario in discrete time by the following dynamic system (which we call the true system):

$$x_T(k) = A_T(k)x_T(k-1) + B_T(k)u(k) + w_T(k) \tag{18}$$

$$z(k) = C_T(k)x_T(k) + v_T(k) \tag{19}$$

where $x_T(k)$ is the true system state vector; $w_T(k)$ and $v_T(k)$ are mutually uncorrelated white Gaussian noise vectors with covariance matrices $Q_T(k)$ and $R_T(k)$ respectively. We use $\mathcal{S}$ to denote the operating scenario defined above. Note that the system (18) (as well as (19)) is general in the sense that $A_T(k)$ may or may not belong to the mode set $\mathcal{A} = \{A_1, A_2, \ldots, A_r\}$ of the IMM algorithm. In other words, the true system, (18) and (19), could be more general than the hybrid system model, (1) and (2), assumed by the IMM algorithm. This would be useful if we want to analyze the performance of the IMM algorithm in the presence of mismatches in the Kalman filter models.

From (18), the true system state $x_T(k)$ is a stochastic process under the operating scenario $\mathcal{S}$. Our objective is to analyze the average performance of the IMM algorithm under $\mathcal{S}$. This scenario-dependent approach to performance analysis/predcition of HE algorithm has also been used by other authors, such as Li and Bar-Shalom (1993b); Hanlon and Maybeck (2000). The requirement to specify a specific operating scenario implicity implies that we need to specify the true mode sequence in our analysis. However, this does not necessarily limit the practical applicability of the performance analysis algorithm. For example, in target tracking applications, one may use the proposed algorithm to predict the tracking errors when the target performs certain maneuvers. Hanlon and Maybeck (2000) also used a similar scenario-dependent approach to characterize the residual of a Kalman filter when a fault of a system occurs.

As noted by Li and Bar-Shalom (1993b), the scenario-dependent approach average out the randomness due to uncertainties in the continuous subspace (of $x_T(k)$ and $z(k)$ in this case) whereas the essential information concerning the scenario is retained. In fact, if a Monte Carlo simulation was used to investigate the performance of the IMM algorithm, one would also need to define an operating scenario in order to set up the simulation. It is precisely because of the scenario dependency of the performances of HE algorithms that we need a more analytical approach to investigate them.

3. ALGORITHM DEVELOPMENT

In this section, we present an algorithm to characterize the Kalman filter residuals and to compute the means of likelihood functions and mean-squared estimation errors for the IMM algorithm. First, we compute the means and cross-covariances of the Kalman filter residuals under the operating scenario $\mathcal{S}$. Thus, at each time step k, we compute the mean of each Kalman filter residual $r_j(k)$ by taking the conditional expectation

$$\bar{r}_j(k) = \mathbb{E}\{r_j(k)|\mathcal{S}\}$$

Note that we use a bar over a vector to denote its mean under the conditional expectation.

The cross-covariance of two vectors X, Y is defined as

$$Cov(X,Y) := \mathbb{E}\{(X-\bar{X})(Y-\bar{Y})^T|\mathcal{S}\}$$

Thus, the cross-covariance of two Kalman filter residuals $r_i(k)$ and $r_j(k)$ is $Cov(r_i(k), r_j(k))$. The covariance of the residual $r_j(k)$ is $Cov(r_j(k), r_j(k))$.

We define the state estimation error for Kalman filter j

$$e_j(k-1) := x_T(k-1) - \hat{x}_j(k-1) \tag{20}$$

and the error for the mixed initial condition

$$\begin{aligned} e_{j0}(k-1) &:= x_T(k-1) - \hat{x}_{j0}(k-1) \\ &= \sum_{i=1}^{r} \gamma_{ji}(k-1)e_i(k-1) \end{aligned} \tag{21}$$

3.1 Characterization of Residuals

This subsection presents a recursive algorithm to compute the means and cross-covariances of the Kalman filter residuals. In the derivations, we shall omit the time index k for the matrices $A_T(k)$, $\Delta A_j(k)$, $K_j(k)$, etc, for simplicity.

Substituting (6) and (19) into (8), we have

$$r_j(k) = C_T x_T(k) + v_T(k) - C_j[A_j\hat{x}_{j0}(k-1) + B_j u(k)] \tag{22}$$

Substituting (18) and (21) into (22),

$$\begin{aligned} r_j(k) =& C_jA_je_{j0}(k-1) + [C_TA_T - C_jA_j] \\ & x_T(k-1) + [C_TB_T - C_jB_j]u(k) \\ & + C_Tw_T(k) + v_T(k) \end{aligned} \tag{23}$$

Similarly, using (18), (6), (7) and (20),

$$\begin{aligned} e_j(k) =& [I - K_jC_j]A_je_{j0}(k-1) + [A_T - A_j - \\ & K_j(C_TA_T - C_jA_j)]x_T(k-1) + \\ & [B_T - B_j - K_j(C_TB_T - C_jB_j)]u(k) \\ & + [I - K_jC_T]w_T(k) - K_jv_T(k) \end{aligned} \tag{24}$$

Taking the conditional expectation on (23) and (24) respectively, we have

$$\begin{aligned} \bar{r}_j(k) =& C_jA_j\bar{e}_{j0}(k-1) + [C_TA_T - C_jA_j] \\ & \bar{x}_T(k-1) + [C_TB_T - C_jB_j]u(k) \end{aligned} \tag{25}$$

$$\begin{aligned} \bar{e}_j(k) =& [I - K_jC_j]A_j\bar{e}_{j0}(k-1) + [A_T - A_j - \\ & K_j(C_TA_T - C_jA_j)]\bar{x}_T(k-1) + [B_T \\ & - B_j - K_j(C_TB_T - C_jB_j)]u(k) \end{aligned} \tag{26}$$

where, from (21),

$$\bar{e}_{j0}(k-1) = \sum_{i=1}^{r} \gamma_{ji}(k-1)\bar{e}_i(k-1) \tag{27}$$

and from (18),

$$\bar{x}_T(k) = A_T\bar{x}_T(k-1) + B_Tu(k) \tag{28}$$

Thus, the means of the residual and the state estimation error for Kalman filter j are computed recursively from (25)-(28).

Next, to derive the cross-covariance of the residuals, we subtract (25) from (23), yielding

$$\begin{aligned} r_j(k) - \bar{r}_j(k) =& C_jA_j[e_{j0}(k-1) - \bar{e}_{j0}(k-1)] + \\ & [C_TA_T - C_jA_j](x_T(k-1) - \bar{x}_T(k-1)) \\ & + C_Tw_T(k) + v_T(k) \end{aligned}$$

Then, the cross-covariance of the residuals is

$$\begin{aligned} & Cov(r_i(k), r_j(k)) = \mathbb{E}\{[r_i(k) - \bar{r}_j(k)][r_i(k) - \bar{r}_j(k)]^T|\mathcal{S}\} \\ & = C_iA_iCov(e_{i0}(k-1), e_{j0}(k-1))A_j^TC_j^T + [C_TA_T \\ & \quad - C_iA_i]Cov(x_T(k-1), x_T(k-1))[C_TA_T - C_jA_j]^T \\ & \quad + C_iA_iCov(e_{i0}(k-1), x_T(k-1))[C_TA_T - C_jA_j]^T \\ & \quad + [C_TA_T - C_iA_i]Cov(x_T(k-1), e_{j0}(k-1))A_j^TC_j^T \\ & \quad + C_TQ_TC_T^T + R_T \end{aligned} \tag{29}$$

where, using (18), (21) and (24),

$$\begin{aligned} & Cov(x_T(k), x_T(k)) = \\ & \qquad A_TCov(x_T(k-1), x_T(k-1))A_T^T + Q_T \end{aligned} \tag{30}$$

$$Cov(e_{j0}(k), x_T(k)) = \sum_{i=1}^{r} \gamma_{ji}Cov(e_i(k), x_T(k)) \tag{31}$$

$$\begin{aligned} & Cov(e_j(k), x_T(k)) = [I - K_jC_j]A_j \\ & \quad Cov(e_{j0}(k-1), x_T(k-1))A_T^T + [A_T - A_j - \\ & \quad K_j(C_TA_T - C_jA_j)]Cov(x_T(k-1), x_T(k-1))A_T^T \\ & \quad + [I - K_jC_T]Q_T \end{aligned} \tag{32}$$

In deriving (29)-(32), we use the fact that $w(k)$ and $v(k)$ are mutually uncorrelated white noise vectors.

Subtracting (26) from (24),

$$\begin{aligned} & e_j(k) - \bar{e}_j(k) = (I - K_jC_j)A_j[e_{j0}(k-1) - \bar{e}_{j0}(k-1)] + \\ & [A_T - A_j - K_j(C_TA_T - C_jA_j)][x_T(k-1) - \bar{x}_T(k-1)] \\ & \quad + (I - K_jC_T)w_T(k) - K_jv_T(k) \end{aligned}$$

Then, similar to (29), the cross-covariance of the state estimation error is

$$
\begin{aligned}
&Cov\{e_i(k), e_j(k)\} = (I - K_iC_i)A_i \\
&\quad Cov\{e_{i0}(k-1), e_{j0}(k-1)\}A_j^T(I - K_jC_j)^T \\
&\quad [A_T - A_i - K_i(C_TA_T - C_iA_i)]Cov(x_T(k-1), \\
&\quad x_T(k-1))[A_T - A_j - K_j(C_TA_T - C_jA_j)]^T \\
&\quad + [I - K_iC_i]A_iCov(e_{i0}(k-1), x_T(k-1)) \\
&\quad [A_T - A_j - K_j(C_TA_T - C_jA_j)]^T \\
&\quad + [A_T - A_i - K_i(C_TA_T - C_iA_i)] \\
&\quad Cov(x_T(k-1), e_{i0}(k-1))A_j^T[I - K_jC_j]^T \\
&\quad + (I - K_iC_T)Q_T(I - K_jC_T)^T + K_iR_TK_j^T
\end{aligned} \tag{33}
$$

Similarly, subtract (27) from (21), and then substitute the result into the following cross-covariance:

$$
\begin{aligned}
&Cov\{e_{i0}(k-1), e_{j0}(k-1)\} = \mathbb{E}\Big\{ \sum_{l=1}^{r} \gamma_{il}(k-1)[e_l(k-1) \\
&- \bar{e}_l(k-1)] \sum_{s=1}^{r} \gamma_{js}(k-1)[e_s(k-1) - \bar{e}_s(k-1)]|\mathcal{S}\Big\} \\
&= \sum_{l=1}^{r}\sum_{s=1}^{r} \gamma_{il}(k-1)\gamma_{js}(k-1)Cov\{e_l(k-1), e_s(k-1)\}
\end{aligned} \tag{34}
$$

Thus, the cross-covariances of the residuals and the state estimation errors are computed recursively from (29)-(34).

From the means and cross-covariances of the Kalman filter residuals, we can compute two performance measures of the IMM algorithm as presented below.

3.2 Mean of Likelihood Function

The likelihood functions are often used to determine the mode probabilities in HE algorithms. We compute their mean values as a performance measure of the mode estimation accuracy. The likelihood function $\Lambda_j(k)$, defined in (13), depends on the Kalman filter residual $r_j(k)$, which has a Gaussian pdf given by

$$p[r_j(k)|\mathcal{S}] = \mathcal{N}_q(r_j(k); \bar{r}_j(k), V(k))$$

From Section 3.1, the mean $\bar{r}_j(k)$ is given by (25) and the covariance $V(k) = Cov(r_j(k), r_j(k))$ is given by (29). Thus, the mean of the likelihood function for mode j is given by

$$\bar{\Lambda}_j(k) = \int_{\mathbb{R}^q} \mathcal{N}_q(r_j; 0, S_j(k))\mathcal{N}_q(r_j; \bar{r}_j(k), V(k))dr_j \tag{35}$$

It can be verified that for any functions $\mathcal{N}_q(x; \mu_1, \Sigma_1)$ and $\mathcal{N}_q(x; \mu_2, \Sigma_2)$ [Li and Bar-Shalom (1993b)]

$$\mathcal{N}_q(x; \mu_1, \Sigma_1)\mathcal{N}_q(x; \mu_2, \Sigma_2) = \kappa\mathcal{N}_q(x; \mu_{12}, \Sigma_{12}) \tag{36}$$

where

$$\kappa = \frac{|\Sigma_{12}|^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}|\Sigma_1|^{\frac{1}{2}}|\Sigma_2|^{\frac{1}{2}}} e^{-0.5\left(\mu_i^T\Sigma_1^{-1}\mu_1 + \mu_2^T\Sigma_2^{-1}\mu_2 - \mu_{12}^T\Sigma_{12}^{-1}\mu_{12}\right)} \tag{37}$$

$$\Sigma_{12}^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1} \tag{38}$$

$$\mu_{12} = \Sigma_{12}\left(\Sigma_1^{-1}\mu_1 + \Sigma_2^{-1}\mu_2\right) \tag{39}$$

Using (36)-(39), (35) becomes

$$\bar{\Lambda}_j(k) = \frac{|(S_j(k)^{-1} + V(k)^{-1})^{-1}|^{\frac{1}{2}}}{(2\pi)^{\frac{q}{2}}|S_j(k)|^{\frac{1}{2}}|V(k)|^{\frac{1}{2}}} e^{-0.5\left\{\bar{r}_j(k)^T\left[V^{-1}(k) - V^{-1}(k)(S_j^{-1}(k) + V^{-1}(k))V^{-1}(k)\right]\bar{r}_j(k)\right\}}$$

Using the matrix inversion lemma , we have [Householder (1965)]

$$\bar{\Lambda}_j(k) = \frac{|(S_j(k)^{-1} + V(k)^{-1})^{-1}|^{\frac{1}{2}}}{(2\pi)^{\frac{q}{2}}|S_j(k)|^{\frac{1}{2}}|V(k)|^{\frac{1}{2}}} e^{-0.5\left\{\bar{r}_j(k)^T[V(k) + S_j(k)]^{-1}\bar{r}_j(k)\right\}} \tag{40}$$

3.3 Mean-squared Error of Estimation

The mean-squared error of the continuous state estimate for Kalman filter j is

$$
\begin{aligned}
F_j(k) &:= \mathbb{E}\big\{[x_T(k) - \hat{x}_j(k)][x_T(k) - \hat{x}_j(k)]^T|\mathcal{S}\big\} \\
&= \mathbb{E}\big\{e_j(k)e_j(k)^T|\mathcal{S}\big\} \\
&= \mathbb{E}\big\{[e_j(k) - \bar{e}_j(k)][e_j(k) - \bar{e}_j(k)]^T|\mathcal{S}\big\} + \bar{e}_j(k)\bar{e}_j(k)^T \\
&= Cov\big\{e_j(k), e_j(k)\big\} + \bar{e}_j(k)\bar{e}_j(k)^T
\end{aligned} \tag{41}
$$

Note that (41) can be evaluated using (26) and (33).

The mean-squared error of the combined state estimate (i.e. the output (16) of the IMM algorithm) is

$$F(k) := \mathbb{E}\{[x_T(k) - \hat{x}(k)][x_T(k) - \hat{x}(k)]^T|\mathcal{S}\}$$

Using (16) and the fact that $\sum_{i=1}^{r}\alpha_i = 1$,

$$
\begin{aligned}
F(k) &= \mathbb{E}\Big\{\sum_{i=1}^{r}\alpha_i[x_T(k) - \hat{x}_i(k)] \\
&\qquad \sum_{j=1}^{r}\alpha_j[x_T(k) - \hat{x}_j(k)]^T\Big|\mathcal{S}\Big\} \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\mathbb{E}\big\{e_i(k)e_j(k)^T\big|\mathcal{S}\big\}
\end{aligned} \tag{42}
$$

where, similar to (41),

$$\mathbb{E}\big\{e_i(k)e_j(k)^T|\mathcal{S}\big\} = Cov\big\{e_i(k), e_j(k)\big\} + \bar{e}_i(k)\bar{e}_j(k)^T$$

3.4 Summary of the Proposed Algorithm

A cycle of the recursive algorithm for characterizing the Kalman filter residuals and computing the means of likelihood functions and the mean-squared estimation errors of the IMM algorithm is as follows.

(1) Compute mixing probability $\gamma_{ij}(k-1)$ and the initial conditions $x_{j0}(k-1)$ and $P_{j0}(k-1)$, for $i, j = 1, 2, \ldots, r$, using (3)-(5).
(2) Compute the means of Kalman filter residuals $\bar{r}_j(k)$ and means of state estimation errors $\bar{e}_j(k)$ using (25)-(27). Also, compute the cross-covariances $Cov(r_i(k), r_j(k))$, $Cov(e_i(k), e_j(k))$ using (29)-(34).
(3) Update the estimated state for Kalman filter j as

$$\hat{x}_j^-(k) = A_j\hat{x}_{j0}(k-1) + B_ju(k)$$
$$\hat{x}_j(k) = \hat{x}_j^-(k) + K_j(k)\bar{r}_j(k)$$

Note that $\bar{r}_j(k)$, of which the randomness due to the process noise $w(k)$ and measurement noise $v(k)$ has been averaged out, is used to update the state vector. The gain $K_j(k)$ is updated using (9)-(11).
(4) Compute the mean of Likelihood function $\bar{\Lambda}_j(k)$ using (35). Then, compute $\alpha_j^-(k)$ using (14) and $\alpha_j(k)$ as

$$\alpha_j(k) \approx \frac{\bar{\Lambda}_j(k)\alpha_j^-(k)}{\sum_{l=1}^{r}\bar{\Lambda}_l(k)\alpha_l^-(k)}$$

(5) Compute the mean-squared errors of the state estimation $F_j(k)$, $F(k)$ using (41) and (42).

The above algorithm could be applicable to other Kalman filter based HE algorithms. For example, in the Generalized Pseudo Bayesian (GPB1) algorithm, instead of the mixing step (or step 1) of the IMM algorithm, it uses the combined estimate at the previous time-step as the initial conditions, i.e. instead of (4) and (5) the initial conditions are given by

$$x_{j0}(k-1) = \hat{x}(k-1) = \sum_{i=1}^{r} \alpha_i(k-1)\hat{x}_i(k-1)$$

$$P_{j0}(k-1) = P(k-1) = \sum_{i=1}^{r} \alpha_i(k-1)\{P_i(k-1) + \cdots\}$$

The rest of the filtering algorithm (steps 2-4) are the same as those of the IMM algorithm. Hence, the performance analysis algorithm presented above could be applicable to the GPB1 algorithm by replacing the terms $\gamma_{ji}(k-1)$ by $\alpha_i(k-1)$ in all equations.

4. SIMULATIONS

Scenario: We consider a two-dimensional aircraft tracking example in Air Traffic Control. The aircraft's position (ξ, η) is measured at $T_s = 5$ *sec* intervals with a standard deviation error of 100 *m* in each axis [Li and Bar-Shalom (1993a)]. The aircraft flies at constant velocity for 200 *sec*, then executes a coordinated turn at a constant turning rate of 2 *deg/s* for 45 *sec*, and finally flies at constant velocity for another 150 *sec*. Let $x_T = [\xi \;\; \dot{\xi} \;\; \eta \;\; \dot{\eta}]^T$ be the true state of the aircraft. The scenario is described by (18)-(19) with

$$A_T(k) = \begin{bmatrix} 1 & T_s & 0 & -0.5\dot{\psi}(k)T_s^2 \\ 0 & 1 & 0 & -\dot{\psi}(k)T_s \\ 0 & 0.5\dot{\psi}(k)T_s^2 & 1 & T_s \\ 0 & \dot{\psi}(k)T_s & 0 & 1 \end{bmatrix}$$

where $\dot{\psi}(k) = 2$ *deg/s* for $40 < k < 50$, $\dot{\psi}(k) = 0$ otherwise; $B_T(k) = 0$, $C_T(k) = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right]$ for all k. The noise covariances are given by

$$Q_T(k) = G \begin{bmatrix} w_\xi^2(k) & 0 \\ 0 & w_\eta^2(k) \end{bmatrix} G^T \qquad R_T(k) = \begin{bmatrix} 100^2 & 0 \\ 0 & 100^2 \end{bmatrix}$$

where $G = \left[\begin{smallmatrix} 0.5T_s^2 & T_s & 0 & 0 \\ 0 & 0 & 0.5T_s^2 & T_s \end{smallmatrix}\right]^T$, $w_\xi(k) = w_\eta(k) = 0.1$ for $40 < k < 50$, $w_\xi(k) = w_\eta(k) = 0.05$ otherwise.

Kalman filter models: The IMM algorithm consists of two Kalman filters, which are described by (1)-(2) with

$$A_i = \begin{bmatrix} 1 & T_s & 0 & -0.5\dot{\psi}_i T_s^2 \\ 0 & 1 & 0 & -\dot{\psi}_i T_s \\ 0 & 0.5\dot{\psi}_i T_s^2 & 1 & T_s \\ 0 & \dot{\psi}_i T_s & 0 & 1 \end{bmatrix} \qquad i = 1, 2$$

where $\dot{\psi}_1 = 0$, $\dot{\psi}_2 = 1.5$ *deg/s*; $B_1 = B_2 = 0$; $C_1 = C_2 = C_T$;

$$Q_1 = G \begin{bmatrix} 0.05^2 & 0 \\ 0 & 0.05^2 \end{bmatrix} G^T \qquad Q_2 = G \begin{bmatrix} 2^2 & 0 \\ 0 & 2^2 \end{bmatrix} G^T$$

and $R_1 = R_2 = R_T$. This example illustrates that the proposed algorithm applies to the case in which the Kalman filter models used do not match the true system.

Simulation results: Figure 1 compares the means of the Kalman filter residuals computed by the proposed algorithm with the corresponding means computed by averaging the residuals from 60 Monte Carlo simulation runs. Figures 2, 3 and 4 show the results for the means of the likelihood functions ($\bar{\Lambda}_j(k)$) and the root-mean-squared (RMS) estimation errors. The RMS error of $x_1(k)$ of Kalman filter 1 is the square-root of the first diagonal term of $F_1(k)$ in (41), and so on. The simulation results validate the accuracy of the proposed algorithm. On the other hand, the proposed algorithm requires much less computational time (0.27 *sec*) compared with that required by the Monte Carlo simulation (4.4 *sec*). This reduction in computational time is important when the performance analysis needs to be repeated for different algorithm designs and different operational scenarios. Furthermore, our analytical approach would provide more insight into the performance of the IMM algorithm.

5. CONCLUSIONS

We have presented an algorithm to compute the means and cross-covariances of the Kalman filter residuals, and the means of likelihood functions and mean-squared estimation errors as performance measures of the Interacting Multiple Model algorithm. The analysis carried out in this paper could be useful for understanding and evaluating the performances of general Kalman filter based hybrid estimation algorithms. As discussed in Section 2.2, the performances of HE algorithms are scenario dependent. However, some more general characteristics, such as stability, of the HE algorithm may be independent of the operating scenario. Thus, one possible extension of this analysis is to investigate such general characteristics of the HE algorithm. Note that it would not be possible to carry out such generalization with Monte Carlo simulations.

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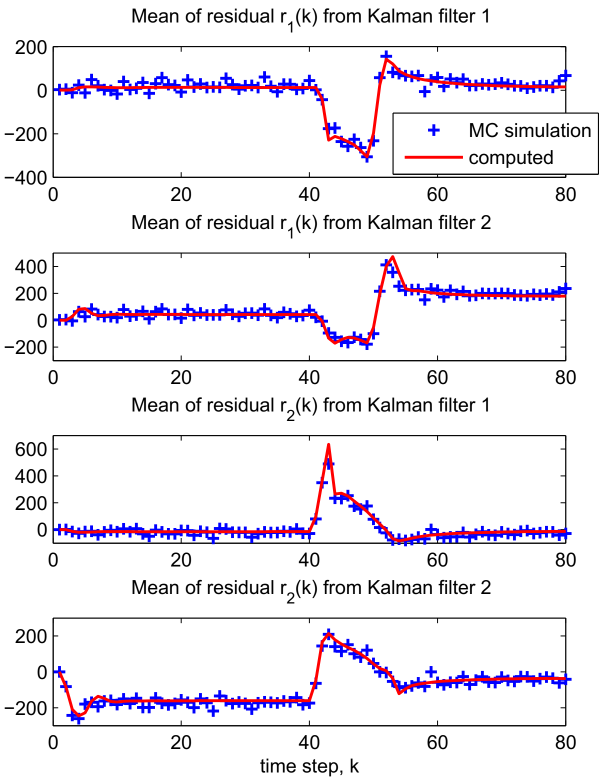

Fig. 1. Means of Kalman filter residuals computed by proposed algorithm and those from Monte Carlo simulations.

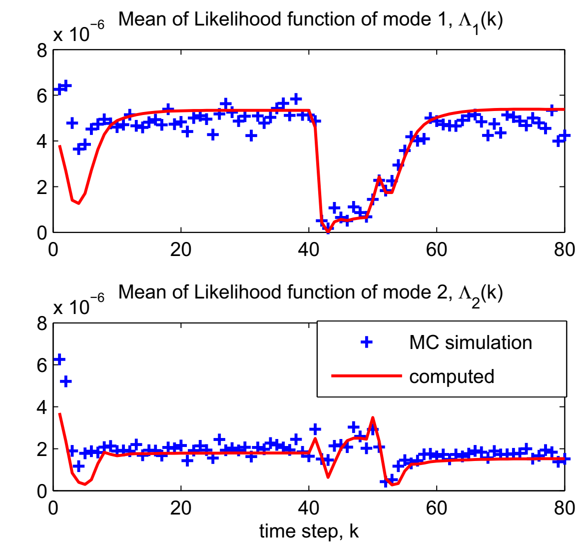

Fig. 2. Means of likelihood functions computed by proposed algorithm and those from Monte Carlo simulations.

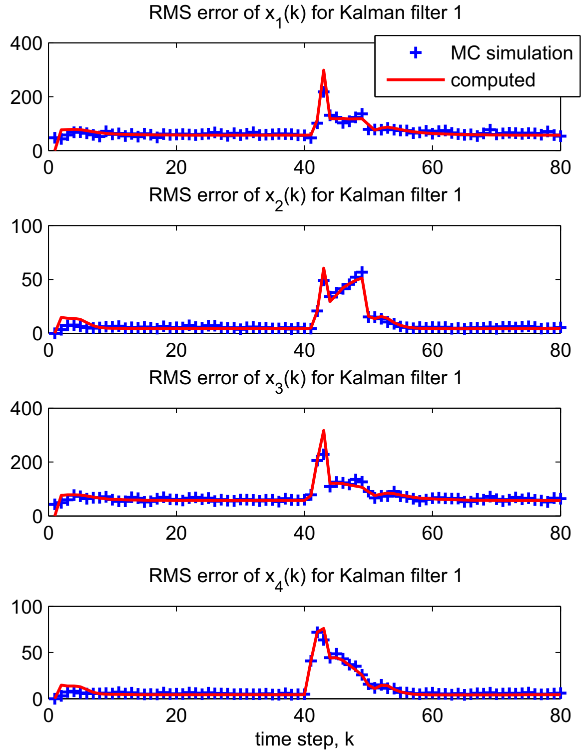

Fig. 3. Root-mean-squared estimation errors of Kalman filter 1

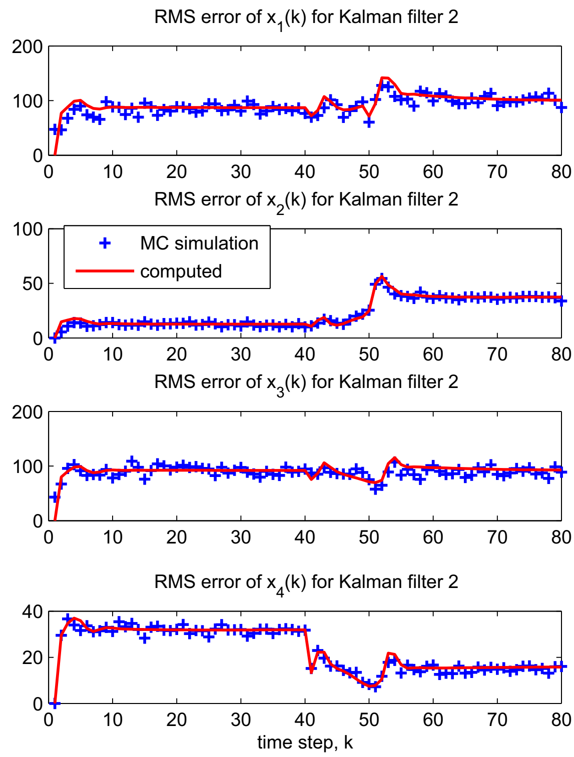

Fig. 4. Root-mean-squared estimation errors of Kalman filter 2