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Sampled-data Polynomial Modal Control of Linear Periodic Plants with Time-delay

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Abstract: The paper deals with the problem of sampled-data polynomial modal control for a linear continuous-time periodic plant with delays at the input and the output of the sampled controller. It is assumed that the plant and the digital controller have the same period. The characteristic matrix of the closed-loop system is constructed. An algorithm is given for constructing the set of causal discrete-time controllers which place the modes of the closed system at given points of the complex plane.

Keywords: Linear time-periodic systems, Time-delays, Sampled-data control systems, Modal control, Causality

1. INTRODUCTION

Due to the progress in technology, the control of finite dimensional linear continuous periodically time-varying (FDLCP) processes becomes realizable and obtains increasing interest in control theory and applications. Various aspects of theory and applications in this field are presented at the special conferences (PSY 2001, PSY 2004, PSY 2007) and the references therein.

In spite of the fact that due to complexity, only digital controllers are of practical relevance, these problems are mostly handled only in an approximate way as purely continuous or purely discrete-time systems. However, we have to consider a sampled-data system, and special methods must be applied, see (Chen and Francis 1995, Rosenwasser and Lampe 2006). Unfortunately, these problems are inadequately investigated and the associated literature is relatively unknown.

For sampled-data systems with continuous LTI processes, the influence of process delays have been considered earlier in a number of papers, e.g. in (Kwakernaak and Sivan 1972, Ackermann 1988, Lennartson 1989, Middleton and Freudenberg 1995, Middleton and Xie 1995, Åström and Wittenmark 1997, Rosenwasser and Lampe 2000, Rosenwasser and Lampe 2006, Polyakov 2006).

The paper (Lampe and Rosenwasser 2001) deals with the stabilization of FDLCP processes by digital LTI controllers. The paper (Lampe and Rosenwasser 2007b) considers the modal control problem for delayed FDLCP processes by digital controllers, where the delay acts on the output of the controller. The present contribution extends these results to the case, where the delays act on the input and the output of the digital controller. For this problem the results of (Lampe and Rosenwasser 2007b) cannot be easily extended, because due to the non-stationary character of the FDLCP process, its associated linear periodic operator is not commutative with the pure delay operator, even in the scalar case.

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2. SYSTEM DESCRIPTION

1. We consider the sampled-data control for a linear continuous-time periodic plant described by the following state equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t) + B(t)u(t-\tau_1) \tag{1}$$

and output equation

$$y(t) = C(t)x(t)\,. \tag{2}$$

In Eqs. (1) and (2), $x(t)$ is the state vector, $y(t)$ is the output vector and $u(t)$ is the control vector, their dimensions are $p \times 1$, $n \times 1$ and $m \times 1$, respectively. Moreover, $A(t) = A(t+T)$, $B(t) = B(t+T)$, and $C(t) = C(t+T)$ are continuous T-periodic matrices of the corresponding dimensions, and $\tau_1 \geq 0$ is a real constant denoting pure delay at the plant input.

2. It is assumed that the plant is controlled by a sampled-data controller described by the equations, see for instance (Rosenwasser and Lampe 2006):

$$\xi_k = y(kT - \tau_2) \tag{3}$$

$$\alpha_0\psi_k + \ldots + \alpha_q\psi_{k-q} = \beta_0\xi_k + \ldots + \beta_k\xi_{k-q}, \tag{4}$$

$$u(t) = h(t-kT)\psi_k, \qquad kT < t < (k+1)T\,. \tag{5}$$

It is important that the sampling period coincides with the period of the periodic plant (1), (2). Since the control needs a synchronization between process and controller, this assumption is the easiest case, and it could be extended to the case, where the sampling period is a multiple of T.

In Eqs. (4) and (5), the ψ_k are $\ell \times 1$ vectors which define the controlling sequence $\{\psi_k\}$, while α_i and β_i are constant matrices of dimensions $\ell\times\ell$ and $\ell\times n$, respectively. Moreover, in (5) $h(t)$ is an $m \times \ell$ matrix defining the form of the control pulses. This matrix is defined on the interval $0 < t < T$, and there it is piecewise-continuous.

The real constant $\tau_2 \geq 0$, appearing in (4) characterizes the pure delay at the input of the digital controller. In this paper all delays are considered to be given parameters.

3. Below, we will use the representations

$$\tau_i = \rho_i T + \theta_i, \quad 0 \leq \theta_i < T, \qquad i = 1, 2, \tag{6}$$

where ρ_i are non-negative integers. Moreover, we introduce

$$\gamma_i = T - \theta_i, \qquad 0 < \gamma_i \leq T\,. \tag{7}$$

4. Equation (4) will be called the equation of the discrete controller. Introducing the backward shift operator $\zeta = \mathrm{e}^{-sT}$, one can write discrete controller equation (4) in the form

$$\alpha(\zeta)\psi_k = \beta(\zeta)\xi_k\,,$$

where $\alpha(\zeta)$ and $\beta(\zeta)$ are polynomial matrices of the form

$$\alpha(\zeta) = \alpha_0 + \alpha_1\zeta + \ldots + \alpha_q\zeta^q\,,$$

$$\beta(\zeta) = \beta_0 + \beta_1\zeta + \ldots + \beta_q\zeta^q\,.$$

Below for brevity, we refer to a matrix pair $(\alpha(\zeta), \beta(\zeta))$ as a controller. If

$$\det\alpha(0) = \det\alpha_0 \neq 0\,, \tag{8}$$

the controller $(\alpha(\zeta), \beta(\zeta))$ will be called *causal*. It is known, e.g. (Åström and Wittenmark 1997, Rosenwasser and Lampe 2000), that only causal controllers can be implemented in practice. Therefore, below we assume that (8) holds.

5. Equations (1)-(5) taken in the aggregate define a system of linear differential-difference equations, which will be called the system $\mathcal{S}_\tau$. A solution of the system $\mathcal{S}_\tau$ is a set of continuous vector functions $x(t), y(t)$ and a numerical sequence $\{\psi_k\}$ such that (1)-(5) hold for all t and k.

3. STATEMENT OF THE SAMPLED-DATA POLYNOMIAL MODAL CONTROL PROBLEM

1. Let $H(t)$ be the fundamental matrix of Eq. (1) satisfying

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = A(t)H(t), \quad H(0) = I_p\,,$$

where I_p is the $p \times p$ identity matrix. Under the assumptions on $A(t)$, this matrix always exists. Let also

$$M = H(T)$$

be the corresponding monodromy matrix. As is known (Yakubovich and Starzhinskii 1975),

$$H(t+T) = H(t)M$$

and, therefore,

$$H^{-1}(t+T) = M^{-1}H^{-1}(t)\,.$$

2. Introduce the notations

$$\begin{aligned} x_k(\varepsilon) &= x(kT+\varepsilon)\,, \quad 0 \leq \varepsilon \leq T \\ x_k &= x(kT) = x_k(\varepsilon)|_{\varepsilon=0}\,, \end{aligned} \tag{9}$$

where k is any integer.

Theorem 1. The system S_τ can be configured to the modified (dependent on ε) discrete model defined by the relations

$$\begin{aligned} x_k(\varepsilon) &= H(\varepsilon)x_k + \Gamma_1(\varepsilon)\psi_{k-\rho_1} + \Gamma_2(\varepsilon)\psi_{k-\rho_1-1}\,, \\ \xi_k &= C(\gamma_2)H(\gamma_2)x_{k-\rho_2-1} + C(\gamma_2)\Gamma_1(\gamma_2)\psi_{k-\rho_1-\rho_2-1} \\ &\quad + C(\gamma_2)\Gamma_2(\gamma_2)\psi_{k-\rho_1-\rho_2-2}\,, \\ \alpha_0\psi_k &+ \ldots + \alpha_q\psi_{k-q} = \beta_0\psi_k + \ldots + \beta_q\xi_{k-q}\,. \end{aligned} \tag{10}$$

In (10) the following notations are used:

$$\Gamma_1(\varepsilon) = \begin{cases} O_{pl}, & \text{for } 0 \leq \varepsilon \leq \theta_1 \\ \displaystyle\int_{\theta_1}^{\varepsilon} H(\varepsilon)H^{-1}(\mu)B(\mu)h(\mu-\theta_1)\,\mathrm{d}\mu & \\ & \text{for } \theta_1 \leq \varepsilon \leq T\,, \end{cases} \tag{11}$$

where $O_{p\ell}$ means the $p \times \ell$ zero matrix. Moreover,

$$\Gamma_2(\varepsilon) = \begin{cases} \displaystyle\int_0^{\varepsilon} H(\varepsilon)H^{-1}(\mu)B(\mu)h(\mu+\gamma_1)\,\mathrm{d}\mu & \\ & \text{for } 0 \leq \varepsilon \leq \theta_1 \\ \displaystyle\int_0^{\theta_1} H(\varepsilon)H^{-1}(\mu)B(\mu)h(\mu+\gamma_1)\,\mathrm{d}\mu & \\ & \text{for } \theta_1 \leq \varepsilon \leq T\,. \end{cases} \tag{12}$$

The proof for Theorem 1 and the following statements are given in the Appendix.

Corollary 2. Using the notation (6)-(7), the system $\mathcal{S}_\tau$, described by equations (1)-(5), can be represented by the discrete backward model $\mathcal{S}_d$

$$\begin{aligned} x_k &= Mx_{k-1} + \Gamma_1\psi_{k-\rho_1-1} + \Gamma_2\psi_{k-\rho_1-2}\,, \\ \xi_k &= C(\gamma_2)H(\gamma_2)x_{k-\rho_2-1} + C(\gamma_2)\Gamma_1(\gamma_2)\psi_{k-\rho_1-\rho_2-1} \\ &\quad + C(\gamma_2)\Gamma_2(\gamma_2)\psi_{k-\rho_1-\rho_2-2}, \\ \alpha_0\psi_k &+ \ldots + \alpha_q\psi_{k-q} = \beta_0\xi_k + \ldots + \beta_q\xi_{k-q}\,, \end{aligned} \tag{13}$$

where

$$\begin{aligned} \Gamma_1 &= \Gamma_1(\varepsilon)|_{\varepsilon=T} \\ &= M\int_0^{\gamma_1} H^{-1}(\lambda+\theta_1)B(\lambda+\theta_1)h(\lambda)\,\mathrm{d}\lambda\,, \\ \Gamma_2 &= \Gamma_2(\varepsilon)|_{\varepsilon=T} \\ &= M^2\int_{\gamma_1}^{T} H^{-1}(\lambda+\theta_1)B(\lambda+\theta_1)h(\lambda)\,\mathrm{d}\lambda\,. \end{aligned}$$

At the sampling instants, the signals in the system $\mathcal{S}_\tau$ coincide with those of the discrete model $\mathcal{S}_d$.

3. Introduce the polynomial matrices

$$a(\zeta) = I_p - \zeta M\,, \quad b(\zeta) = \Gamma_1 \zeta^{\rho_1+1} + \Gamma_2 \zeta^{\rho_1+2}\,,$$
$$c(\zeta) = C(\gamma_2) H(\gamma_2) \zeta^{\rho_2+1}\,,$$
$$d(\zeta) = C(\gamma_2)[\Gamma_1(\gamma_2)\zeta^{\rho_1+\rho_2+1} + \Gamma_2(\gamma_2)\zeta^{\rho_1+\rho_2+2}]\,,$$

where I_p is the $p \times p$ identity matrix.

Then we can write the characteristic matrix $Q(\zeta, \alpha, \beta)$ of the system $\mathcal{S}_d$ as

$$Q(\zeta, \alpha, \beta) = \begin{bmatrix} a(\zeta) & O_{pn} & -b(\zeta) \\ -c(\zeta) & I_n & -d(\zeta) \\ O_{\ell p} & -\beta(\zeta) & \alpha(\zeta) \end{bmatrix}. \tag{14}$$

4. The eigenvalues of the matrix $Q(\zeta, \alpha, \beta)$ will be called the modes of the system $\mathcal{S}_\tau$. In analogy with (Rosenwasser and Lampe 2006), it can be shown that the modes of the system $\mathcal{S}_\tau$ define elementary continuous-time processes in this system, like the roots of the characteristic equations for ordinary LTI systems. In this connection, one of the fundamental control problems for the system $\mathcal{S}_\tau$ is the sampled-data polynomial modal control problem, which is formulated as follows:

Sampled-data Polynomial Modal Control (SDPMC) Problem: Given a plant (1)-(2), and furthermore relations (3), (5)-(7). Find the set $\mathcal{R}_\Delta$ of all causal controllers $(\alpha(\zeta), \beta(\zeta))$ such that

$$\det \begin{bmatrix} a(\zeta) & O_{pn} & -b(\zeta) \\ -c(\zeta) & I_n & -d(\zeta) \\ O_{\ell p} & -\beta(\zeta) & \alpha(\zeta) \end{bmatrix} \approx \Delta(\zeta)\,, \tag{15}$$

where $\Delta(\zeta)$ is a given polynomial, and the symbol $\approx$ denotes the equivalence of polynomial matrices (and, as a special case, of scalar polynomials).

The sampled-data polynomial stabilization problem is a special case of the SDPMC problem. In this case $\Delta(\zeta)$ in (15) is an arbitrary polynomial that is free of roots inside the closed unit disk. For given matrices $a(\zeta)$, $b(\zeta)$, $c(\zeta)$, $d(\zeta)$ and a specified polynomial $\Delta(\zeta)$, Eq. (15) can be considered as a polynomial equation for the unknown matrices $\alpha(\zeta)$, $\beta(\zeta)$. Unlike Diophantine polynomial equations, which are traditionally considered in literature, e.g. (Kučera 1991, Kailath 1980), we will refer to equations of the form (15) as *determinant* polynomial equations.

4. MAIN RESULTS

1.

Theorem 3. For the solvability of the SDPMC problem, it is necessary that

$$\Delta(0) \neq 0\,. \tag{16}$$

If (16) holds, Eq. (15) can have only causal solutions. For $\Delta(0) = 0$, equation (15) does not possess causal solutions.

2. The system $\mathcal{S}_\tau$ will be called completely modal controllable if Eq. (15) is solvable for any polynomial $\Delta(\zeta)$. This means that an arbitrary set of modes can be assigned for the system $\mathcal{S}_\tau$ by a discrete-time controller.

The conditions for complete modal controllability can be represented in the following form.

Theorem 4. For complete modal controllability of the system $\mathcal{S}_\tau$, it is necessary and sufficient that the pair (M, G) be completely observable, and the pair (M, F) be completely controllable, where the matrices F, G are given by

$$\begin{aligned} F &= M^{-\rho_1} \int_0^T H^{-1}(\nu + \theta_1) B(\nu + \theta_1) h(\nu)\, \mathrm{d}\nu\,, \\ G &= C(\gamma_2) H(\gamma_2) M^{-(\rho_2+1)}\,. \end{aligned} \tag{17}$$

3.

Theorem 5. Let the system $\mathcal{S}_\tau$ be completely modal controllable and condition (16) be fulfilled. Then, for a fixed polynomial $\Delta(\zeta)$ there exists the set $\mathcal{R}_\Delta$ of solutions of Eq. (15), which contains only causal controllers and can be constructed using the following algorithm:

a) Construct the matrix fraction description (MFD) (Kailath 1980)
$$c(\zeta)(I - \zeta M)^{-1} = g^{-1}(\zeta) f(\zeta)\,,$$
where $g(\zeta)$ and $f(\zeta)$ are polynomial matrices such that for any ζ, we have
$$\operatorname{rank}[\, g(\zeta)\ \ f(\zeta)\,] = n\,.$$
In this case, $\det g(0) \neq 0$.

b) Find an arbitrary basic controller $(\alpha_0(\zeta), \beta_0(\zeta))$ satisfying
$$\det \begin{bmatrix} g(\zeta) & -f(\zeta)b(\zeta) - g(\zeta)d(\zeta) \\ -\beta_0(\zeta) & \alpha_0(\zeta) \end{bmatrix} = 1\,.$$
From (Rosenwasser and Lampe 2006) it follows that under the given assumptions the set of basic controllers is not empty.

c) The set $\mathcal{R}_\Delta$ of causal solutions to (15) is given by
$$\alpha(\zeta) = D(\zeta)\alpha_0(\zeta) - N(\zeta)[f(\zeta)b(\zeta) + g(\zeta)d(\zeta)]\,,$$
$$\beta(\zeta) = D(\zeta)\beta_0(\zeta) - N(\zeta)g(\zeta)\,,$$
where $N(\zeta)$ and $D(\zeta)$ are polynomial matrices, the former can be chosen arbitrarily, while the latter satisfies the single condition
$$\det D(\zeta) \approx \Delta(\zeta)\,.$$

4. If the system $\mathcal{S}_\tau$ is not completely modal controllable, Eq. (15) might have no solutions for some polynomials $\Delta(\zeta)$. The following theorem provides for conditions of solvability in this case.

Theorem 6. Let the system $\mathcal{S}_\tau$ be not completely modal controllable. Let also $\varphi_1(\zeta)$ be a greatest common left divisor of the matrices $a(\zeta)$ and $b(\zeta)$ such that for all ζ we have

$$[\, a(\zeta)\ \ b(\zeta)\,] = \varphi_1(\zeta)\,[\, a_1(\zeta)\ \ b_1(\zeta)\,]\,,$$

and for all ζ

$$\operatorname{rank}[\, a_1(\zeta)\ \ b_1(\zeta)\,] = p\,. \tag{18}$$

Let also $\varphi_2(\zeta)$ be a greatest common right divisor of the matrices $a_1(\zeta)$ and $c(\zeta)$ such that

$$\begin{bmatrix} a_1(\zeta) \\ c(\zeta) \end{bmatrix} = \begin{bmatrix} a_2(\zeta) \\ c_1(\zeta) \end{bmatrix} \varphi_2(\zeta)\,,$$

and for all ζ

$$\operatorname{rank}\begin{bmatrix} a_2(\zeta) \\ c_1(\zeta) \end{bmatrix} = p\,.$$

Denote

$$\det \varphi_1(\zeta) = \delta_1(\zeta)\,, \qquad \det \varphi_2(\zeta) = \delta_2(\zeta)\,.$$

Then the following propositions hold:

a) For all ζ, the following equality is true:
$$\operatorname{rank}\,[\, a_2(\zeta)\ b_1(\zeta)\,] = p\,. \tag{19}$$
b) Equation (15) is solvable if and only if
$$\Delta_1(\zeta) = \frac{\Delta(\zeta)}{\delta_1(\zeta)\delta_2(\zeta)}$$
turns out to be a polynomial.
c) If b) holds and $\Delta_1(0) \neq 0$, the set of causal solutions of Eq. (15) coincides with the set of solutions of
$$\det\begin{bmatrix} a_2(\zeta) & O_{pn} & -b_1(\zeta) \\ -c_1(\zeta) & I_n & -d(\zeta) \\ O_{\ell p} & -\beta(\zeta) & \alpha(\zeta) \end{bmatrix} \approx \Delta_1(\zeta)\,,$$
which, with account for (18) and (19), can be constructed using the algorithm of Theorem 5.

5. EXAMPLE

1. Consider the modular control problem for the FDLCP process with

$$A(t) = -\frac{\sin t}{2-\cos t}\,, \quad B(t) = 1, \quad C(t) = 1\,, \quad T = 2\pi\,.$$

In this case, we get

$$H(t) = \frac{1}{2-\cos t}\,, \quad H^{-1}(t) = 2-\cos t\,,$$

and the monodromy matrix becomes

$$M = H(2\pi) = 1\,.$$

Furthermore, for concreteness we assume

$$h(t) = 1, \qquad 0 < t < T\,,$$

and

$$0 < \tau_1 < 2\pi, \qquad 0 < \tau_2 < 2\pi\,, \tag{20}$$

which implies $\rho_1 = \rho_2 = 0$, $\tau_1 = \theta_1$, $\tau_2 = \theta_2$.

2. Denote

$$r(\lambda) = 2\lambda - \sin\lambda\,.$$

Then, we obtain from (11)

$$\Gamma_1(\varepsilon) = \begin{cases} 0\,, & 0 \le \varepsilon \le \tau_1\,, \\ H(\varepsilon)[r(\varepsilon) - r(\tau_1)]\,, & \tau_1 \le \varepsilon \le 2\pi, \end{cases} \tag{21}$$

and as follows from (12)

$$\Gamma_2(\varepsilon) = \begin{cases} H(\varepsilon)r(\varepsilon)\,, & 0 \le \varepsilon \le \tau\,, \\ H(\varepsilon)r(\tau_1)\,, & 0 \le \varepsilon \le 2\pi\,. \end{cases} \tag{22}$$

From (21) and (22), we find

$$\Gamma_1 = \Gamma_1(\varepsilon)|_{\varepsilon=2\pi} = r(2\pi) - r(\tau_1)$$

$$\Gamma_2 = \Gamma_2(\varepsilon)|_{\varepsilon=2\pi} = r(\tau_1)\,.$$

3. For calculating the coefficients $\Gamma_1(\gamma_2)$ and $\Gamma_2(\gamma_2)$, we have to decide between the two cases

$$\tau_1 + \tau_2 \le T \tag{23}$$

and

$$\tau_1 + \tau_2 \ge T\,. \tag{24}$$

When (23) is fulfilled, we obtain $\gamma_2 \ge \tau_1$, and therefore

$$\Gamma_1(\gamma_2) = H(\gamma_2)[r(\gamma_2) - r(\tau_1)]\,,$$
$$\Gamma_2(\gamma_2) = H(\gamma_2)r(\tau_1)\,.$$

In case of (24), we have $\gamma_2 \le \tau_1$, and therefore

$$\Gamma_1(\gamma_2) = 0\,, \quad \Gamma_2(\gamma_2) = H(\gamma_2)r(\gamma_2)\,.$$

Using the above expressions, we find for the present example

$$a(\zeta) = 1-\zeta\,, \quad b(\zeta) = \Gamma_1\zeta + \Gamma_2\zeta^2$$
$$c(\zeta) = H(\gamma_2)\zeta$$
$$d(\zeta) = \Gamma_1(\gamma_2)\zeta^{\rho_1+\rho_2+1} + \Gamma_2(\gamma_2)\zeta^{\rho_1+\rho_2+2}\,.$$

4. Determinant polynomial equation (15) takes the form

$$\det\begin{bmatrix} 1-\zeta & 0 & -b(\zeta) \\ -c(\zeta) & 1 & -d(\zeta) \\ 0 & -\beta(\zeta) & \alpha(\zeta) \end{bmatrix} \approx \Delta(\zeta)\,,$$

which is equivalent to the set of Diophantine equations

$$\alpha(\zeta)a(\zeta) - \beta(\zeta)v(\zeta) \approx \Delta(\zeta)\,, \tag{25}$$

where

$$v(\zeta) = d(\zeta)(1-\zeta) + b(\zeta)c(\zeta)\,. \tag{26}$$

In the given case, we have

$$v(1) = b(1) = \Gamma_1 + \Gamma_2 = 4\pi \neq 0\,. \tag{27}$$

Therefore, the polynomials $a(\zeta)$ and $b(\zeta)$ are coprime, and equation (25) is solvable for any polynomial $\Delta(\zeta)$. Hence, the investigated system is completely modal controllable for all τ_1 and τ_2 satisfying condition (20).

5. It follows from (27), that there exists a controller $\alpha_0(\zeta)$, $\beta_0(\zeta)$ with

$$\alpha_0(\zeta)a(\zeta) - \beta_0(\zeta)v(\zeta) = 1\,.$$

Thus for a fixed polynomial $\Delta(\zeta)$ the solution of the modal control problem takes the form

$$\alpha(\zeta) = \ell\Delta(\zeta)\alpha_0(\zeta) - m(\zeta)v(\zeta)\,,$$
$$\beta(\zeta) = \ell\Delta(\zeta)\beta_0(\zeta) - m(\zeta)a(\zeta)\,,$$

where $m(\zeta)$ is an arbitrary polynomial and ℓ is an arbitrary constant.

CONCLUSIONS

The report deals with closed-loop systems that consist of a linear continuous-time periodic plant and a sampled linear controller where the sampling period coincides with the period of the process. It is assumed that pure delays act on the input and the output of the sampled controller.

The characteristic matrix of the closed-loop system is constructed and the set of causal discrete controllers is found, such that the set of eigenvalues of the characteristic matrix has a given form. The solution opens new possibilities for the practical design of digital controllers for this complicated class of processes.

Appendix A. PROOFS FOR THE BASIC STATEMENTS

A.1 Proof of Theorem 1

Integrating equation (1), we obtain

$$x(t) = H(t)M^{-k}x(kT) + \int_{kT}^{t} H(t)H^{-1}(\nu)B(\nu)u(\nu-\tau_1)\,\mathrm{d}\nu\,.$$

Let us take here

$$t = kT + \varepsilon\,, \quad \nu = kT + \mu\,.$$

Using the notation (9), we find

$$x_k(\varepsilon) = H(\varepsilon)x_k + \int_0^{\varepsilon} H(\varepsilon)H^{-1}(\mu)u(kT+\mu-\tau_1)\,\mathrm{d}\mu\,. \tag{A.1}$$

From (5), (6) and (7), we obtain

$$u(kT+\mu-\tau_1) = \begin{cases} h(\mu+\gamma_1)\psi_{k-\rho_1-1} \\ \quad \text{for } -T<\mu-\theta_1<0\,, \\ h(\mu-\theta_1)\psi_{k-\rho_1} \\ \quad \text{for } 0<\mu-\theta_1<T\,. \end{cases} \tag{A.2}$$

Inserting (A.2) into (A.1), we find

$$\begin{aligned} x_k(\varepsilon) &= H(\varepsilon)x_k + \int_0^{\varepsilon} H(\varepsilon)H^{-1}(\mu)B(\mu)h(\mu+\gamma_1)\,\mathrm{d}\mu\,\psi_{k-\rho_1-1} \\ &\qquad \text{for } 0\le\varepsilon\le\theta_1 \\ x_k(\varepsilon) &= H(\varepsilon)x_k + \int_0^{\theta_1} H(\varepsilon)H^{-1}(\mu)B(\mu)h(\mu+\gamma_1)\,\mathrm{d}\mu\,\psi_{k-\rho_1-1} + \\ &\quad \int_{\theta_1}^{\varepsilon} H(\varepsilon)H^{-1}(\mu)B(\mu)h(\mu-\theta_1)\,\mathrm{d}\mu\,\psi_{k-\rho_1} \\ &\qquad \text{for } \theta_1\le\varepsilon\le T\,, \end{aligned} \tag{A.3}$$

which is equivalent to the first equation in (10). For the proof of the second equation in (10), we assume

$$\xi(t) = y(t-\tau_2) = C(t-\tau_2)x(t-\tau_2)\,.$$

Hence

$$\xi_k = \xi(kT) = C(kT-\tau_2)x(kT-\tau_2)\,.$$

But, obviously

$$C(kT-\tau_2) = C(\gamma_2)\,.$$

Moreover,

$$\begin{aligned} x(kT-\tau_2) &= x(kT-\rho_2T-\theta_2) \\ &= x(kT-\rho_2T-T+\gamma_2)\,. \end{aligned}$$

which together with the first equation in (10) leads to the second equation in (10). ■

Hereinafter, the symbol ■ indicates the end of a proof.

A.2 Proof of Corollary 2

Inserting in equation (A.3) $\varepsilon = T$ and substituting k by $k-1$, we obtain the first equation in (13). ■

A.3 Proof of Theorem 3

From (14), we obtain

$$\begin{aligned} \det Q(\zeta,\alpha,\beta)|_{\zeta=0} &= \det a(0)\det\alpha(0) \\ &= \det\alpha(0)\,. \end{aligned}$$

If in (15) $\Delta(0) = 0$, then $\det\alpha(0) = 0$, i.e. the controller $(\alpha(\zeta),\beta(\zeta))$ is non-causal. If however, $\Delta(0)\neq 0$, then $\det\alpha(0)\neq 0$, so that the controller $(\alpha(\zeta),\beta(\zeta))$ becomes causal. ■

A.4 Proof of Theorem 4

It was shown in (Lampe and Rosenwasser 2007a), that the determinant polynomial equation (15) is solvable for any polynomial $\Delta(\zeta)$, if and only if for all ζ

$$\operatorname{rank}\,[\,I_p-\zeta M\;\; b(\zeta)\,] = p\,, \tag{A.4}$$

$$\operatorname{rank}\begin{bmatrix} I_p-\zeta M \\ c(\zeta) \end{bmatrix} = p\,. \tag{A.5}$$

The generalized theorem of Bezout (Gantmacher 1959) yields the existence of polynomial matrices $b_1(\zeta)$ satisfying

$$b(\zeta) = (I_p-\zeta M)b_1(\zeta) + B_1\,,$$

where

$$B_1 = M^{-(\rho_1+1)}[\Gamma_1 + M^{-1}\Gamma_2] = F\,,$$

and F is the matrix determined by (17). Hence, from the equality

$$[\,I_p-\zeta M\;\; b(\zeta)\,]\begin{bmatrix} I_p & -b_1(\zeta) \\ O_{lp} & I_l \end{bmatrix} = [\,I_p-\zeta M\;\; F\,]\,,$$

we conclude that the matrices $[\,I_p-\zeta M\;\; b(\zeta)\,]$ and $[\,I_p-\zeta M\;\; F\,]$ are equivalent. Thus, taking advantage from the fact that the monodromy matrix M is non-singular, we reason that condition (A.4) is fulfilled in the case and only in the case, when the pair (M, F) is completely controllable. Analogously, we can show, that condition (A.5) is fulfilled if and only if the pair (M, G) is completely observable. ■

A.5 Proofs for Theorems 5 and 6

The proofs directly emerge from the general properties of MFD, proven in (Lampe and Rosenwasser 2007a).

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