Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Arbitrarily Fast Tracking Feedback Systems for a Class of Nonlinear Plants

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Abstract: In the present article it is proved that stability can be ensured by quasi-linear compensators for plants with global Lipschitz nonlinearity. In the present article, it is shown that arbitrarily fast tracking can be obtained achieved using quasi-linear feedback are explored for various numbers of poles in excess of zeros in a given plant. This is a study of the intrinsic capability of feedback systems to achieve arbitrarily high performance because in industrial applications the raise of the gain is limited by hardware protection.

1. INTRODUCTION

Motivation. In many technological systems there is an increasing demand of fast and accurate responses to input signals and fast attenuation of disturbances. The responsibility for ensuring such a good performance falls on feedback control systems. The difficulty of the control design problem is amplified by the fact that many plant models are non-linear and with uncertain parameters, and subjected to unknown disturbances. In a recent article Bensoussan and Kelemen (2007), it was shown that global Lipsschitz non-linearities are Laplace transformable. We now show that arbitrarily fast tracking can be enforced by quasi linear compensators (Kelemen and Bensoussan (2004), Kelemen (2002)) for plants with global Lipschitz nonlinearity. It has been shown that quasi linear control allows the designer to benefit from all the advantages of high gains without any drawbacks, i.e. guarantied stability margins, non oscillatory and arbitrarily fast response, etc. There the plant was linear while now we consider global Lipschitz nonlinearity in the plant.

The arbitrarily fast tracking is also a stability result of the Bounded-input bounded output type and enriches the family of criterias of stability of non-linear feedback systems. Lurie and Postnikov (1944) addressed the problem of defining the restriction on a linear plant with a sector nonlinearity (0, k) in its feedback loop. Popov (1960) has found a very elegant solution to this problem. Zames (1966) proposed a mathematical framework which defined an algebra of sectors that could handle sector/conic nonlinearities, which translated into a graphical criterion, the circle criterion, independently developed by Sandberg (1964). In fact, the Popov criterion which applies to nonlinearities bounded by slopes (0, k) can be seen as a particular case of a sector/conic nonlinearities bounded by slopes (a, b). The Popov criterion can be translated in a graphical criterion in a Popov plot coordinates $\left(\omega \times \mathrm{Im}\{G(\omega)\},\ \mathrm{Re}\{G(\omega)\}\right)$ while the circle criteria can be translated in a graphical criterion is a regular Nyquist plot coordinates $\left(\mathrm{Im}\{G(\omega)\},\ \mathrm{Re}\{G(\omega)\}\right)$. Moreover, whenever the derivative $\frac{dN(y)}{dy}$ of the linearity is bounded by a sector (a, b), the output of a feedback system including in its loop such a nonlinearity cascaded with a linear plant is continuous. The same condition on the nonlinearity leads to the off-axis circle criteria Narendra and Goldwyn (1964) on a Nyquist plot, while a standard sector condition on a nonlinearity leads to a parabola criterion Narendra and Taylor (1973) on the Popov plot.

The algebra of relations and operators introduced by Zames allowed to combine sector nonlinearities $n(\cdot)$ with linear operators $G(\omega)$ in a unified framework. However the restriction $n(0)=0$ is needed to validate such a mathematical framework. Such an assumption excludes many nonlinearities and this limitation is not necessary in the present result which ties in with a previous one Bensoussan and Kelemen (2007) according to which it is justified to assume that Lipschitz nonlinearities have a Laplace transform and that they can therefore be analyzed in the frequency domain.

2. PRELIMINAIRIES ON QUASI-LINEAR FEEDBACK

A quasi-linear compensator is one with a finite number of poles and zeros but whose *poles* depend appropriately on the *compensator gain.*

Let be the following convolution system:

$$y(t)-w(t)=\int_0^t h_p(t-\tau)\left[c_e(\tau)+n\left(y(\tau)-w(\tau)\right)\right]d\tau,\qquad t\geq 0,\ h_p(t)=\mathcal{L}^{-1}\left(P(s)\right) \tag{1}$$

This can be seen as a feedback loop where $y(t)-w(t)$ *is the output, w(t) is a disturbance and* $c_e(\tau)$ *is an input to the plant. Here P(s) is the transfer function of the linear element in the forward path,* $\mathcal{L}^{-1}$ *is the inverse Laplace transform,*

$h_p(t)$ is the impulse response corresponding to *P(s)* and *n(z)* is the memoryless nonlinearity in the return path. Suppose $P(s)=P_2(s)/P_1(s)$, where $P_1(s)$ and $P_2(s)$ are polynomials in the complex variable *s* with $deg(P_1(s))>deg(P_2(s))$ and $deg(P_1(s))\geq 1$; by *deg* we denote the degree of a polynomial in the Laplace variable *s*. Also, *P(s)* has *d* poles more than zeros. Then the initial conditions which may affect the input respectively the output of the linear component alone can be written as:

$$i_0:\left\{c_e(0),\ldots,c_e(0)^{(deg\ P_2-1)}\right\},$$
$$o_0:\left\{(y(0)-w(0)),\ldots,(y(0)-w(0))^{(deg\ P_1-1)}\right\}$$

where $x(0)^{\ell}$ denotes the initial condition on the derivative ℓ of the variable *x*.

We apply now a feedback to the plant (1), see also figure 1. The feedback is represented in the Laplace domain because it is a *quasi-linear* one, i.e. in which the *poles of the compensator depend on its gain k:*

$$C_e(s)=G_k(s)\big(U(s)-Y(s)\big)=k\,N_2(s)/DG_k(s) \qquad (2)$$

here $C_e(s)$ *and* U(s) *are the Laplace transforms of* $c_e(t)$ *and of a reference input* u(t). *The compensator* $G_k(s)$ *is a rational transfer function in the variable* s, *with* k *a positive gain, and* N_2 (s) *and* $DG_k(s)$ *are two polynomials with* $deg(DG_k(s))\geq deg(N_2(s))$.

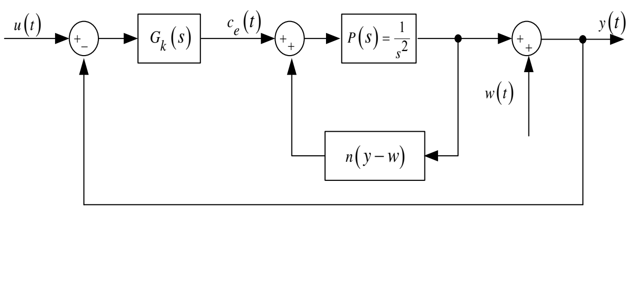

Fig. 1: Quasi-linear feedback control of a non-linear plant.

Following Kelemen and Bensoussan (2004) we can rewrite the closed loop transfer function $T_k(s)$ as:

$$T_k(s)=\frac{kN(s)}{D_k(s)+kN(s)},\ N(s)=P_2(s)N_2(s),\ D_k(s)=P_1(s)DG_k(s),$$
$$T_k(s)=\frac{kL_k(s)}{1+kL_k(s)},\ with\ L_k(s)=\frac{N(s)}{D_k(s)}$$
$$T_k(s)=T_{zk}(s)T_{dk}(s),\ \text{where}$$
$$T_{zk}(s)=\frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+\tilde{z}_1)(s+\tilde{z}_2)\ldots(s+\tilde{z}_m)},Re\ z_1\leq\ldots\leq Re\ z_m,$$
$$T_{dk}(s)=\frac{k}{(s+\tilde{p}_1)\ldots(s+\tilde{p}_d)},\ \ Re\ \tilde{p}_1\leq\ldots\leq Re\ \tilde{p}_d, \qquad (3)$$
$$\tilde{p}_\ell=a_\ell k^{n_{\ell R}}\left(1+o_{\ell_R}(1)\right)+ib_\ell k^{n_{\ell I}}\left(1+o_{\ell_I}(1)\right),$$
$$k>0\ \text{large}$$

are rational numbers and denote $n_\ell=\max\{n_{\ell_R},n_{\ell_I}\}$

Following (Kelemen and Bensoussan (2004)), we know that, provided $(d-1)f<1<df$ and $d\geq 2$, the asymptotic values in k of the poles o*f* $T_{dk}(s)$ are given by (see Kelemen and Bensoussan (2004)):.

$$\left.\begin{array}{r}\left(-\tilde{p}_d(k)\right)=-a_{d-1}k^f,\ldots,\\ \left(-\tilde{p}_2(k)\right)=-a_1k^f,\\ \left(-\tilde{p}_1(k)\right)=\dfrac{-1}{a_1,\ldots,a_{d-1}}k^{1-(d-1)f}\end{array}\right\}$$

Throughout this article, $T_k(s)$ will be subjected to the following requirement as in (Kelemen and Bensoussan (2004)):

(r_1) k > 0 appears in $DG_k(s)$ only with rational exponents. Denote by f the maximal exponent of $D_k(s)$ in (3) and suppose f < 1;

(r_2) All the zeros of N(s) and the poles of $T_{dk}(s)$ from (3) are simple. Moreover, if two poles of $T_{dk}(s)$ have the same exponent n_ℓ in the expansion (3) then their difference behaves as $Ak^{n_\ell}\left(1+o(1)\right)$ too, for some positive constant A.

Simple poles and zeros are for computational convenience. Finally, we recall from (Kelemen and Bensoussan (2004)) the necessary and sufficient properties required for arbitrarily fast tracking by feedback AFTF, corresponding to $T_k(s)$. The 'physical' meaning of AFTF will be given in Corollary below. First we express the step response of $T_k(s)$ with the residue theorem as:

$$H_k(s)=\mathcal{L}^{-1}\left(T_k(s)/s\right)=\alpha_k-e^{-t\tilde{p}}\beta_k(t),$$
$$\alpha_k=T_k(0),\ \tilde{p}\ \text{real} \qquad (4)$$

Then these properties (5) are:

(pr$_1$) $\lim_{k\to\infty} \alpha_k = 1$;

(pr$_2$) for every $T > 0 \lim_{k\to\infty} |\beta_k(t)| = 0$ uniformly in $t \in [T, \infty]$; (5)

(pr$_3$) $\sup_{t\geq 0} |\beta_k(t)| \leq B_\beta < \infty$, $k > 0$ large, B_β independent of k;

(pr$_4$) there exists a number $\gamma > 0$ such that $-\tilde{p} \leq -\gamma < 0$.

It is useful to define for each $T > 0$ the sequence $\beta_{(k,T)} = \sup_{t\geq T} |\beta_k(t)|$. Then $(pr_2) \Leftrightarrow \lim_{k\to\infty} \beta_{(k,T)} = 0$ for every $T > 0$.

Fact: If the feedback is *linear,* i.e. $D_k(s)$ is independent of k, the conditions *(pr$_2$)* and *(pr$_4$)* are met if and only if $d = 1$ [1]. The properties (5) show that for any $T > 0$ the function $H_k(t)$ converges to 1 uniformly in $t \in [T, \infty]$ when k goes to infinity, and it is bounded independently of large k and $t > 0$. The Theorem below proves that these "good" properties for a step input lead to certain "good" properties for a large class of inputs and disturbances. Moreover, using the function $H_k(t)$ will allow to estimate the difference $y_k(t) - u(t)$ in *absolute value* not in integral norm, which is the figure of merit sought off in many applications.

We present now our main result. The theorem is an extension to plants having Lipschitz nonlinearity, with which it shares hypothesis (i_1) and the class of input and disturbance functions. This hypothesis is a sufficient condition to satisfy the AFTF properties of (5).

3. THEOREM

We assume that:

(i_1) in addition to (r_1) and (r_2) the poles and zeros *of* $T_k(s)$ from (3) satisfy:

$a_\ell > 0$ and $n_{\ell_R} > 0$ for all the poles $-\tilde{p}_\ell$ of $T_{dk}(s)$, and $Re\, z_\ell > 0$ for all the zeros $-z_\ell$ of $T_{zk}(s)$;

(i_2) the reference input *u(t)* and the disturbance input *w(t)* are bounded functions with support $t > 0$. Moreover, for $t > 0$ they are continuous and continuously differentiable with bounded derivatives. The bounds on the functions are denoted by $B_u = \sup_{t\geq 0} |u| < \infty$, $B_w = \sup_{t\geq 0} |w| < \infty$, $B_{\dot{u}} = \sup_{t>0} |\dot{u}(t)| < \infty$, and $B_{\dot{w}} = \sup_{t>0} |\dot{w}(t)| < \infty$.

(i_3) the nonlinearity *n(z)* satisfies a global Lipschitz property on the real axis with Lipschitz constant $L_{ip} > 0$, i.e. the nonlinearity $n: R \to R$ is piecewise continuous and satisfies:

$$|n(x_1) - n(x_2)| \leq L_{ip} |x_1 - x_2|$$

Then:

(c$_1$) for every set $R_N = \{u,\, w,\, o_0,\, i_0,\, L_{ip},\, n(0)\}$ and numbers $T > 0$, $\varepsilon > 0$ there is a number $K(R_N, T, \varepsilon) > 0$ so that if $k \geq K(R_N, T, \varepsilon)$.

$$|y_k(t) - u(t)| \leq \left[(B_{\dot{u}} + B_{\dot{w}}) B_\beta T + \varepsilon\right](1+\varepsilon),\ t \in [T, \infty]$$

$$|y_k(t) - u(t)| \leq \left[B_\beta(|u(0)| + |w(0)|) + (B_{\dot{u}} + B_{\dot{w}}) B_\beta T + B_1 + \varepsilon\right](1+\varepsilon),\ t \in [0, T]$$

here B_1 is a positive constant; there are also the coefficients B_2, B_3 and B_4 which depend on L_{ip} (see proof below).

Corollary: It follows from theorem 1 that the tracking precision $|y_k(t) - u(t)|$ can be made *arbitrarily small,* uniformly in $t \in [T, \infty]$, by taking $T > 0$ and $\varepsilon > 0$ small enough and $k > 0$ sufficiently large; at the same time the peak of the tracking error is *bounded* independently of $t > 0$ and large *k.*

Proof : Note that if *n(z) = 0,* (c_1) follows directly from Kelemen and Bensoussan (2004). Our approach is to work with the plant (1) in implicit form and thus considering the non-linear term as an input adding up to $c_e(t)$ assumed bounded and continuous. This makes easier the computations including the fact that the initial conditions could be treated as in open loop. Only toward the end of the proof, we convert an implicit inequality involving $y(t)$ into an explicit one by using a generalization of the Gronwall inequality.

We apply now the Laplace transform as it is easier to visualize the application of (a dynamic) feedback and the actual feedback design can be performed in the frequency domain. Thus we obtain the open loop transfer function:

$$Y(s) = \frac{P_2(s)}{P_1(s)} C_e(s) + \frac{P_2(s)}{P_1(s)} \overline{n(y-w)}(s) + W(s) - \frac{C(i_0, n_o, s)}{P_1(s)} + \frac{C(o_o, s)}{P_1(s)}, \quad (6)$$

where $W(s)$ is the Laplace transform of $w(t)$. The functions $C(i_0, n_o, s)$ and $C(o_o, s)$ are the polynomials in s of degrees $\leq deg(P_2(s)) - 1$ and $\leq deg(P_1(s)) - 1$ with coefficients depending on the initial conditions of the input respectively the output of the open loop [10]. Also $\overline{n(y-w)}(s)$ is the Laplace transform of $n(y(t) - w(t))$. Concerning $C(i_0, n_o, s)$ the notations means that to $c_e(0)$ is added $n(y(0) - w(0))$ because the function $n(z)$ "reads" only $\left(y(t)-w(t)\right)$ not its time derivatives. And also since the effective input to the linear plant is $c_e(t)+n\left(y(t)-w(t)\right)$, see (1).

We prove now that the Laplace transform was applied in a legitimate way. For this we have to show, according to Schwartz (1966), that $y(t)$ is locally L^1 integrable and that $|y(t)|$ does not grow faster than an exponential, i.e. there are real numbers $A>0$, c, such that $|y(t)|\le Ae^{ct}; t\le 0$. Now the plant (1) with the initial conditions included is:

$$\begin{aligned}
&y(\theta)-w(\theta)=\int_0^{\theta} h_p(\theta-\tau)c_e(\tau)\,d\tau+\\
&\int_0^{\theta} h_p(\theta-\tau)n\left(y(\tau)-w(\tau)\right)d\tau-\\
&\mathcal{L}^{-1}\left(\frac{C(i_0,n_0,s)}{P_1(s)}\right)+\mathcal{L}^{-1}\left(\frac{C(o_0,s)}{P_1(s)}\right),\quad \theta\in[0,t]
\end{aligned}\tag{7}$$

What we obtained is a non-linear Volterra integral convolution equation. Due to the Lipschitz property of $n(z)$ and the continuity and boundedness of $c_e(0)$ this equation has a unique and continuous solution on the interval $[0;t]$, $t>0$ arbitrary but given; the solution is continuously differentiable for $\theta>0$. These statements are proved at conclusion (c_1) of lemma in the Appendix (Kelemen (2002)). Therefore the solution is locally L^1. In particular this shows that there can not be a finite escape time. As for the growth condition it was proved in the same lemma at (c_2). Thus the Laplace transform was applied correctly.

Note that also $n\left(y(t)-w(t)\right)$ is Laplace transformable and the corresponding abscissa of convergence is the maximum between that of $\left(y(t)-w(t)\right)$ and 0 (which appears if $n(0)\ne 0$). This can be checked directly by using the Lipschitz property of $n(z)$ in the Laplace integral (Schwartz (1966)).

Moreover, both $Y(s)$ and $\overline{n(y-w)}(s)$ are inverse Laplace transformable. For this, we have to show by [12] that both $\left(y(t)-w(t)\right)$ and $n\left(y(t)-w(t)\right)$ are locally of bounded variation around every $t>0$. This is true for $\left(y(t)-w(t)\right)$ because we showed it is of C^1 class for $\theta>0$. Thus we apply the mean value theorem for derivatives and get the required result (Doetsch, (1974)). For $n\left(y(t)-w(t)\right)$, the property follows as before but using first the Lipschitz condition.

Now with (2), (5), and the notations from (3) the closed loop response in Laplace domain becomes for every given $k>0$.

$$\begin{aligned}
Y_k(s)&=T_k(s)U(s)+T_{Mk}(s)\overline{n(y_k-w)}(s)\\
&+W(s)-T_k(s)W(s)\\
&-\frac{DG_k(s)C(i_0,n_0,s)}{D_k(s)+kN(s)}+\frac{DG_k(s)C(o_0,s)}{D_k(s)+kN(s)},
\end{aligned}\tag{8}$$

where (the index *M* stands for "modified")

$$T_{Mk}(s)=\frac{P_2(s)DG_k(s)}{D_k(s)+kN(s)}.$$

By definition, see also (6), (3), (2) and (1), we have: $deg\left(DG_k(s)C(o_0,s)\right)\le deg\left(D_k(s)\right)-1$, $deg\left(DG_k(s)C(i_0,n_0,s)\right)< deg\left(D_k(s)\right)-1$ and $deg\left(P_2(s)DG_k(s)\right)\le deg\left(D_k(s)\right)-1$.

We apply now the inverse Laplace transform $\mathcal{L}^{-1}$ to (8). To prove this is possible, we apply it first formally and obtain the non-linear Volterra integral equation from below.

$$\begin{aligned}
&y_k(t)=\int_0^t h_k(t-\tau)\,u(\tau)d\tau+\\
&\int_0^t h_{Mk}(t-\tau)n\left(y_k(\tau)-w(\tau)\right)d\tau+\\
&w(t)-\int_0^t h_k(t-\tau)\,w(\tau)d\tau-C_i(t,k)+C_o(t,k),\ t\ge 0
\end{aligned}\tag{9}$$

Here:

$$\begin{aligned}
&y_k(t)=\mathcal{L}^{-1}\left(T_k(s)\right),\ h_{Mk}(t)=\mathcal{L}^{-1}\left(T_{Mk}(s)\right),\\
&C_i(t,k)=\mathcal{L}^{-1}\left(\frac{DG_k(s)C(i_0,n_0,s)}{D_k(s)+kN(s)}\right),\\
&C_o(t,k)=\mathcal{L}^{-1}\left(\frac{DG_k(s)C(o_0,s)}{D_k(s)+kN(s)}\right).
\end{aligned}\tag{10}$$

Then observe that (9) has exactly the form of (7). Indeed, for every fixed $k>0$ the feedback is linear. Thus in Laplace domain the closed loop (8) has exactly the form of the open loop (6), including the fact that all the transfer functions are strictly proper. Hence, $y_k(t)$ and $h_{Mk}(t)$ are the impulse responses corresponding to strictly proper transfer functions as was $h_p(t)$ from (1) before. What has changed are the singularities of $Y_k(s)$ and $\overline{n(y_k-w)}(s)$, which might be shifted but with a finite amount for every *k*. Thus we can use the same arguments for (9) as for (7) including lemma 2 from appendix Kelemen (2002). Therefore we are led to the conclusion that both the direct and inverse Laplace transforms are applicable to the closed loop and that $y_k(t)-w(t)$ is continuous for $\theta\ge 0$ and class C^1 for $\theta\ge 0$.

To get some useful estimates, we proceed by integrating by parts in (9) the integrals containing $u(t)$ and $w(t)$ (first and fourth term). Then replacing $H_k(t)$ thus obtained with (4), we get:

$$\begin{aligned} y_k(t)-u(t) &= (\alpha_k u(t)-u(t)) - e^{-t\tilde{p}}\beta_k(t)u(0) \\ &-\int_0^t \beta_k(\tau)e^{-t\tilde{p}}\dot{u}(t-\tau)d\tau \\ &+(w(t)-\alpha_k w(t)) + e^{-t\tilde{p}}\beta_k(t)w(0) \\ &+\int_0^t \beta_k(\tau)e^{-t\tilde{p}}\dot{w}(t-\tau)d\tau \\ &+\int_0^t h_{Mk}(t-\tau)\left(n\left(y_k(\tau)-w(\tau)\right)\right)d\tau \\ &-C_i(t,k)+C_o(t,k),\ t\geq 0. \end{aligned}$$

The absolute value of the terms from the right hand side, except the last 3, can be estimated as in (Kelemen 2002) (c_1). This result is applicable in view of hypotheses (i_1) and (i_2) of the present theorem, see also (Kelemen 2002), (c_{11}), (c_{12}). Thus with (5), we obtain for every $T>0$ and large $k>0$.

$$\begin{aligned} |y_k(t)-u(t)| &\leq |\alpha_k-1|(B_u+B_w)+\beta_{(k,T)}C_{(u,w,\gamma)} \\ &+(B_{\dot{u}}+B_{\dot{w}})B_\beta T \\ &+\int_0^t |h_{Mk}(t-\tau)|\,\left|\left(n\left(y_k(\tau)-w(\tau)\right)\right)\right|d\tau \\ &+|C_i(t,k)|+|C_o(t,k)|,\ t\in[T,\infty], \end{aligned}$$

$$\infty > C_{(u,w,\gamma)} = \begin{cases} |u(0)|+|w(0)|, \text{ } u \text{ and } w \text{ are constant,} \\ |u(0)|+|w(0)|+(B_{\dot{u}}+B_{\dot{w}})\int_0^\infty e^{-\gamma\tau}d\tau, \\ \text{otherwise,} \end{cases} \quad (11)$$

$$\begin{aligned} |y_k(t)-u(t)| &\leq |\alpha_k-1|(B_u+B_w)+ \\ \beta_\beta(|u(0)|+|w(0)|)&+(B_{\dot{u}}+B_{\dot{w}})B_\beta T+ \\ \int_0^t |h_{Mk}(t-\tau)|&\,\left|\left(n\left(y_k(\tau)-w(\tau)\right)\right)\right|d\tau+ \\ |C_i(t,k)|&+|C_o(t,k)|,\ t\in[0,T]. \end{aligned}$$

We have to estimate now the last three terms from the two inequalities from above, which were defined at (8). We begin with $|C_o(t,k)|$.

We recall that $deg\left(DG_k(s)C(o_0,s)\right)\leq deg\left(D_k(s)\right)-1$ and the maximal exponent of k of $DG_k(s)$ is $f<1$ by (r_1) from hypothesis (i_1). So, with the full power of hypothesis (i_1) – which includes requirement (r_2) – we can apply the method of proof from Kelemen (2002), theorem 2, (c_{12}) based on formula (27) (from there). Thus, there are some numbers $B_{op}\geq 0$, $B_{oz}\geq 0$, $0<a=\min a_\ell$; so that for any T > 0 and large $k>0$, $0<n_R=\min n_{\ell_R}$ $0<\tilde{z}<\min Re z_i$.

$$\begin{aligned} |C_o(t,k)| &\leq B_{op}e^{-Tak^{(n_R)}}+B_{oz}e^{-T\tilde{z}}/k^{1-f}, \\ t&\in[T,\infty],\ (f<1) \\ |C_o(t,k)| &\leq B_{op}+B_{oz}/k^{1-f}, \\ t&\in[0,T] \end{aligned} \quad (12)$$

Similarly, since $deg\left(DG_k(s)C(o_0,s)\right)\leq deg\left(D_k(s)\right)-1$ and $deg\left(P_2(s)DG_k(s)\right)\leq deg\left(D_k(s)\right)-1$, we get for some numbers $B_{ip}\geq 0$, $B_{iz}\geq 0$, $B_{Mp}\geq 0$, $B_{Mz}\geq 0$.

$$\begin{aligned} |C_i(t,k)| &\leq B_{ip}e^{-Tak^{(n_R)}}+B_{iz}e^{-T\tilde{z}}/k^{1-f},\ t\in[T,\infty], \\ |C_i(t,k)| &\leq B_{ip}+B_{iz}/k^{1-f},\ t\in[0,T]; \\ |h_{Mk}(t)| &\leq B_{Mp}e^{-Tak^{(n_R)}}+B_{Mz}e^{-T\tilde{z}}/k^{1-f},\ t\in[T,\infty], \\ |h_{Mk}(t)| &\leq B_{Mp}+B_{Mz}/k^{1-f},\ t\in[0,T]. \end{aligned} \quad (13)$$

However, owing to the integrals appearing in (11) a more appropriate estimate is:

$$\int_0^t |h_{Mk}(t-\tau)|d\tau \leq \frac{B_{Mp}}{ak^{(n_R)}}+\frac{B_{Mz}}{\tilde{z}k^{(1-f)}}, \quad (14)$$

which follows from the previous two inequalities by replacing *T* with *t* (we are no longer concerned with the uniform in *t* bound from there).

Due to the global Lipschitz property of $n(z)$, there is a Lipschitz constant $L_{ip}>0$, so that:

$$\begin{aligned} |n(y_k(t)-w(t))| &\leq |n(y_k(t)-w(t))-n(0)|+|n(0)| \\ &\leq L_{ip}|y_k(t)-w(t)-0|+|n(0)| \\ \leq L_{ip}|y_k(t)-u(t)|&+L_{ip}|u(t)|+L_{ip}|w(t)|+|n(0)|. \end{aligned} \quad (15)$$

We note that all the numbers found in the estimates beginning with (10) are independent of *t, T* and large *k* > 0. Now employing (12), (13), (14) and (15) in (11) we obtain the new estimate:

$$\begin{aligned} |y_k(t)-u(t)| &\leq a(T,k)+ \\ \int_0^t |h_{Mk}(t-\tau)|&L_{ip}|y_k(\tau)-u(\tau)|d\tau,\ t\in[T,\infty], \\ |y_k(t)-u(t)| &\leq a_1(T,k)+ \\ \int_0^T |h_{Mk}(t-\tau)|&L_{ip}|y_k(\tau)-u(\tau)|d\tau,\ t\in[0,T], \end{aligned} \quad (16)$$

where:

$$a(T,k) = |\alpha_k - 1|(B_u + B_w) + \beta_{(k,T)} C_{(u,w,\gamma)} +$$
$$(B_{\dot{u}} + B_{\dot{w}}) B_\beta T +$$
$$B_1 e^{-Tak^{(n_R)}} + B_2 / ak^{(n_R)} + (B_3 e^{-T\tilde{z}} + B_4) / k^{(1-f)},$$
$$a_1(T,k) = |\alpha_k - 1|(B_u + B_w) + B_\beta (|u(0)| + |w(0)|) +$$
$$(B_{\dot{u}} + B_{\dot{w}}) B_\beta T +$$
$$B_1 + B_2 / ak^{(n_R)} + (B_3 + B_4) / k^{(1-f)},$$

and

$$B_1 = B_{op} + B_{ip},\ B_2 = B_{Mp}\left(L_{ip} B_u + L_{ip} B_w + |n(0)|\right),$$

$$B_3 = B_{oz} + B_{iz}, B_4 = (B_{Mz}/\tilde{z})\left(L_{ip} B_u + L_{ip} B_w + |n(0)|\right).$$

Next we use a generalization of the Gronwall (Hirsch and Smale (1974)). Since $u(t)$ and $w(t)$ are continuous by ((i_2) and $y_k(t)$) was proved to be so after (10), we get:

$$|y_k(t) - u(t)| \le a(T,k) \times \int_0^t L_{ip} |h_{Mk}(t-\tau)| d\tau,\ t \in [T, \infty],$$
$$|y_k(t) - u(t)| \le a_1(T,k) \times \int_0^t L_{ip} |h_{Mk}(t-\tau)| d\tau,\ t \in [0, T].$$

Carrying out the computations by using again (14), we obtain for every $T > 0$ and large $k > 0$.

$$|y_k(t) - u(t)| \le a(T,k) \times e^{L_{ip} X},\ t \in [T, \infty]$$
$$|y_k(t) - u(t)| \le a_1(T,k) \times e^{L_{ip} X},\ t \in [0, T]$$

with $X = B_{Mp} / ak^{(n_R)} + B_{Mz} / \tilde{z} k^{(1-f)}$.

Finally, these estimates lead to the following inequalities: for every set $R_N = \{u, w, o_0, i_0, L_{ip}, n(0)\}$ and numbers $T > 0$ and $\varepsilon > 0$, there is a number $K(R_N, T, \varepsilon) > 0$ so that if $k \ge K(R_N, T, \varepsilon)$, then:

$$|y_k(t) - u(t)| \le \left[(B_{\dot{u}} + B_{\dot{w}}) B_\beta T + \varepsilon\right](1+\varepsilon),\ t \in [T, \infty]$$
$$|y_k(t) - u(t)| \le \left[B_\beta (|u(0)| + |w(0)|) + (B_{\dot{u}} + B_{\dot{w}}) B_\beta T + B_1 + \varepsilon\right]$$
$$(1+\varepsilon),\ t \in [0, T]$$

Indeed this follows from the coefficients $a(T,k)$, $a_1(T,k)$ for large $k > 0$, and noting that the expression X from above approaches 0 as k goes to infinity.

Which proves theorem 1.

4. CONCLUSIONS

It has been shown that global Lipschitz nonlinearities can be controlled in a feedback system in order to get arbitrarily fast tracking of the input. The time response to an echelon input can be squeezed in a tube which can be made narrower by increasing the gain of the quasi-linear controller. How this bounded input-bounded output stability combined with faster time response preserves the robustness properties of quasi-linear control is a topic which deserves to be studied in the next future.

ACKNOWLEGDMENT

The present work was started with the late Dr Matei Kelemen of the École de technologie supérieure in Montreal. The authors wants to thank him for the many discussions that led to this result.

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