Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Robust Stabilization of Nonlinear Systems by Quantized and Ternary Control

Claudio DE PERSIS *

* *Dipartimento di Informatica e Sistemistica A. Ruberti, Sapienza Università di Roma, Via Eudossiana 18, 00184 Roma, ITALY. (Tel: +39-06-44585959, e-mail: depersis@dis.uniroma1.it)*

Abstract: Results on the problem of stabilizing a nonlinear continuous-time system by a finite number of control or measurement values are presented. The basic tool is a discontinuous version of the so-called semi-global backstepping lemma. We derive robust practical stabilizability results by quantized and ternary controllers and apply them to a few control problems.

1. INTRODUCTION

The problem of controlling systems through a limited bandwidth channel has recently raised a great interest in the community, as thoroughly surveyed in Nair *et al.* (2007). In this paper, we consider *nonlinear continuous-time* systems. For these systems, the problem is twofold: (i) selecting a discrete set of control values, and (ii) scheduling the control values in the feedback loop. In Liberzon (2003), adopting a time-varying quantization, and assuming input-to-state stabilizability of the system, the author shows asymptotic convergence to the origin. In Liu and Elia (2004), Hayakawa *et al.* (2006) and Ceragioli and De Persis (2007), the role of static logarithmic quantization to prove practical semi-global stabilizability of nonlinear stabilizable systems has been investigated. The three papers mainly differ in the type of solution adopted. They also present results which rely on notions of robustness different from input-to-state stability.
In this paper we depart from the results of Ceragioli and De Persis (2007), where − knowing a Lyapunov function and the system model − a Lyapunov redesign was carried out to reject the perturbation due to quantization, to show how the quantization effect can be attenuated even when the Lyapunov function is hardly known and the model is affected by uncertainty. A discontinuous version of the *semi-global backstepping* lemma of Teel and Praly (1995), in which the measured state is logarithmically quantized, is applied to show that *minimum-phase* nonlinear systems, possibly with *uncertain parameters*, can be robustly semi-globally practically stabilized by a a quantized function of *partial-state* measurements. Other papers have dealt with uncertainty of the model. The work Phat *et al.* (2004) deals with the robust stabilization of *linear discrete-time* uncertain systems over a finite data-rate communication channel. A similar problem is considered in Zhang *et al.* (2006), but from the point of view of *adaptive* control. An adaptive scheme for nonlinear continuous time systems is finally proposed in Hayakawa *et al.* (2006).
We additionally show that semi-global practical stabilization is possible even using a simple *switched ternary* controller. Similar elementary controllers have been studied in Kaliora and Astolfi (2004) for a different class of nonlinear systems. A remarkable application of this result is the robust *output* feedback stabilization over a network of the class of nonlinear systems of Marino and Tomei (1993).Finally we present a result on semi-global practical regulation of the output when output measurements are finitely quantized and no other processing is carried out.
Preliminary facts are presented in Section 2. In Section 3, the semi-global backstepping tool in the presence of quantization is presented. The ternary controller is introduced in Section 4. Applications of these basic results to a number of control problems are illustrated in Section 5.

For lack of space proofs of the results could not be included. We refer the interested reader to the full version of the paper available at `www.dis.uniroma1.it/~depersis`

2. PRELIMINARIES

The system we focus our attention on is of the form

$$\begin{aligned}\dot{x} &= F(x,\mu) + G(x,\mu)\zeta \\ \dot{\zeta} &= q(x,\zeta,\mu) + b(x,\zeta,\mu)u\end{aligned} \tag{1}$$

with $x \in \mathbb{R}^{n-1}$, $\zeta \in \mathbb{R}$, μ an unknown parameter ranging over the compact set $\mathcal{P}$, $u \in \mathbb{R}$, $b(x,\zeta,\mu) \geq b_0 > 0$ for all (x,ζ,μ). Many systems of interest can be reduced to this form (cf. Section 5). We suppose that the upper subsystem satisfies the following property (Teel and Praly (1995), see also Isidori (1999)):

Definition. The system $\dot{x} = F(x,\mu)$, $x \in \mathbb{R}^{n-1}$, satisfies a Uniform Lyapunov Property if there exists an open set $\mathcal{A} \subset \mathbb{R}^{n-1}$, a real number $c \geq 1$, a continuously differentiable definite positive function $V : \mathcal{A} \to \mathbb{R}_+$ such that $\Gamma_{c+1} := \{x : V(x) \leq c+1\} \subset \mathcal{A}$ and

$$\frac{\partial V}{\partial x} F(x,\mu) < 0 \quad \forall x \in \Gamma_{c+1}\ ,\ x \neq 0\ .$$

Introduce the Lyapunov function (Teel and Praly (1995))

$$W(x,\zeta) = \frac{cV(x)}{c+1-V(x)} + \frac{d\zeta^2}{d+1-\zeta^2}$$

defined on the set $\{x : V(x) < c+1\} \times \{\zeta : \zeta^2 < d+1\}$, for some $d \geq 1$, and definite positive and proper therein. For an arbitrary $\sigma > 0$, consider the set

$$S = \{(x,\zeta) : \sigma \leq W(x,\zeta) \leq c^2 + d^2 + 1\}\ .$$

978-1-1234-7890-2/08/$20.00 © 2008 IFAC

10.3182/20080706-5-KR-1001.0700

The set is well-defined, because if $W(x,\zeta) \le c^2 + d^2 + 1$, then $V(x) < c+1$ and $\zeta^2 < d+1$. In Teel and Praly (1995) (see also Bacciotti (1989)) it is proven that a linear high-gain *partial-state* feedback $u = \bar{k}\zeta$ exists which makes $\dot{W}(x,\zeta)$ negative on S (thus allowing the authors to conclude that any trajectory starting in S is attracted by $\Omega_\sigma := \{(x,\zeta) : W(x,\zeta) \le \sigma\}$). In the next 2 sections, we carry out this investigation in 2 cases in which the feedback information ζ is available in a "limited" form.

Following Teel and Praly (1995), consider the derivative $\dot{W}(x,\zeta) = (\partial W/\partial x)\dot{x} + (\partial W/\partial \zeta)\dot{\zeta}$. It is possible to obtain the following inequality to hold for all $(x,\zeta) \in \Omega_{c^2+d^2+1}$:

$$\begin{aligned}\dot{W}(x,\zeta) \le\ & \frac{c}{c+1}\frac{\partial V}{\partial x}F(x,\mu) + w(x,\zeta,\mu)\zeta + \\ & 2\frac{d(d+1)}{(d+1-\zeta^2)^2}\zeta b(x,\zeta,\mu)u\ , \\ w(x,\zeta,\mu) =\ & \frac{c(c+1)}{(c+1-V(x))^2}\frac{\partial V}{\partial x}G(x,\mu) + \\ & 2\frac{d(d+1)}{(d+1-\zeta^2)^2}q(x,\zeta,\mu)\ .\end{aligned}$$

Because of the ULP property, if the state belongs to $S_0 = \{(x,\zeta) \in S : \zeta = 0\}$, then $\dot{W}(x,\zeta) < 0$. By continuity, there exists a neighborhood U of S_0 where the sum of the first two terms on the right-hand side of the inequality above remains strictly negative. Without loss of generality, we can suppose that a constant $\eta > 0$ exists such that $U = \{(x,\zeta) \in S : |\zeta| < \eta\}$ (see Figure 1). Then, to show that $\dot{W}(x,\zeta)$ is negative on S, it is enough to investigate the sign of $\dot{W}(x,\zeta)$ on $\tilde{S} := S \setminus U$ only.

3. STABILIZATION BY QUANTIZED CONTROL

In what follows, we consider the case in which the measurement ζ is quantized by a logarithmic quantizer. Let $u_0 \in I\!R_+$, $j \in \mathbb{N}$ and $0 < \delta < 1$ be constants to design. Also let $u_i = \rho^i u_0$, with $\rho = \frac{1-\delta}{1+\delta}$ (Liu and Elia (2004)) . The following map is the quantizer

$$\Psi(r) = \begin{cases} u_0 & \dfrac{1}{1+\delta}u_0 < r \\ u_i & \dfrac{1}{1+\delta}u_i < r \le \dfrac{1}{1-\delta}u_i\ ,\ 1 \le i \le j \\ 0 & 0 \le r \le \dfrac{1}{1+\delta}u_j \\ -\Psi(-r) & r < 0\ , \end{cases} \tag{2}$$

and $u = -\Psi(\bar{k}\zeta)$ is the quantized input. Observe that it is *equivalent* to consider either the quantized control law $u = -\Psi(\bar{k}\zeta)$ or the control law $u = -\bar{k}\bar{\Psi}(\zeta)$, function of the *quantized partial-state* $\bar{\Psi}(\zeta)$, provided that $\bar{\Psi}$ is appropriately defined. As a matter of fact, define $\bar{\Psi}$ as Ψ in (2), but with a new set of quantization levels $\bar{u}_i$ (instead of u_i) defined as $\bar{u}_i = \rho^i \bar{u}_0$, with $\bar{u}_0 = \bar{k}^{-1}u_0$. Then, it is easy to show that $\bar{k}\bar{\Psi}(\zeta) = \Psi(\bar{k}\zeta)$, and all the results drawn with $u = -\Psi(\bar{k}\zeta)$ also hold for $u = -\bar{k}\bar{\Psi}(\zeta)$. In what follows, we only refer to the quantized input $u = -\Psi(\bar{k}\zeta)$.

Observe that the quantizer has $2j+3$ quantization levels, with u_0, j, $\bar{k}$ to determine. Of course, the size of the *deadzone* of the quantizer, i.e. the region around the zero where $\Psi = 0$, decreases as j increases. The parameter δ can be viewed as the quantization *density*, and we do not assume any constraint on its value (for open-loop unstable systems, $\delta \in (0,1)$ results in no loss of generality Ceragioli and De Persis (2007)). The closed-loop system is

$$\begin{aligned}\dot{x} &= F(x,\mu) + G(x,\mu)\zeta \\ \dot{\zeta} &= q(x,\zeta,\mu) - b(x,\zeta,\mu)\Psi(\bar{k}\zeta)\ .\end{aligned} \tag{3}$$

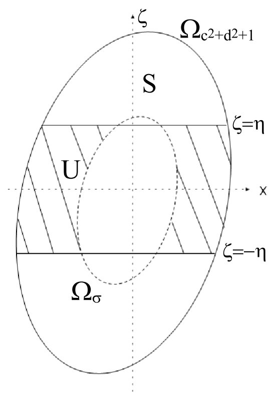

Fig. 1. The sets of interest in the paper. The regions at the top, center and bottom, delimited by the boundary of $\Omega_{c^2+d^2+1}$ and the 2 horizontal solid lines, are respectively Ω_-, Ω_0, Ω_+.

Observe that the vector field on the right-hand side of (3) is discontinuous and solutions of the system must be intended in some generalized sense. Here we focus on Krasowskii solutions, but other types of solutions are possible (see e.g. Ceragioli and De Persis (2007) and references therein). The main reason to consider Krasowskii solutions lies in the fact that a rather complete Lyapunov theory for the study of the stability of these solutions is available.

Definition. A curve $\varphi : [0,+\infty) \to I\!R^n$ is a *Krasowskii solution* of a system of ordinary differential equations $\dot{x} = G(t,x)$, where $G : [0,+\infty) \times I\!R^n \to I\!R^n$, if it is absolutely continuous and for almost every $t \ge 0$ it satisfies the differential inclusion $\dot{x} \in K(G(t,x))$, where $K(G(t,x)) = \cap_{\delta>0}\overline{\mathrm{co}}\,G(t,B_\delta(x))$, with $\overline{\mathrm{co}}G$ the convex closure of the set G.

In the present case, Krasowskii solutions are absolutely continuous functions which satisfy the differential inclusion (see e.g. Ceragioli and De Persis (2007))

$$\begin{pmatrix}\dot{x}\\ \dot{\zeta}\end{pmatrix} \in \begin{pmatrix} F(x,\mu)+G(x,\mu)\zeta \\ q(x,\zeta,\mu)\end{pmatrix} + \left\{\begin{pmatrix}0\\ -b(x,\zeta,\mu)\end{pmatrix} v\ ,\ v \in K(\Psi(\bar{k}\zeta))\right\}$$

$$K(\Psi(\bar{k}\zeta)) \subseteq \begin{cases}\{(1+\lambda\delta)\bar{k}\zeta\ ,\ \lambda \in [-1,1]\} & \dfrac{u_j}{1+\delta} < |\bar{k}\zeta| \le \dfrac{u_0}{1-\delta} \\ \{\lambda(1+\delta)\bar{k}\zeta\ ,\ \lambda \in [0,1]\} & \dfrac{u_j}{1+\delta} \ge |\bar{k}\zeta|\ .\end{cases}$$

Then we claim the following version of the so-called "semi-global backstepping lemma" in Teel and Praly (1995) with quantized feedback:

Lemma 1. For any $\delta \in (0,1)$, there exist positive numbers k^*, j^*, and u_0 such that, for any gain $\bar{k} \ge k^*$ and any

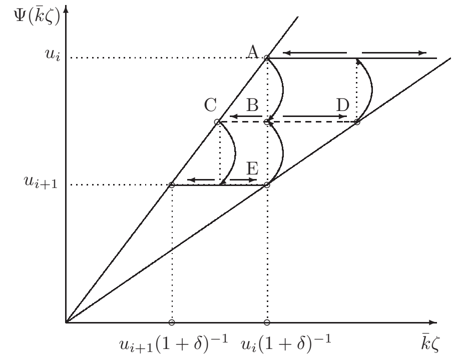

Fig. 2. Chattering-free implementation of the control $u = -\Psi(\bar{k}\zeta)$ in Hayakawa *et al.* (2006).

number of quantization levels $j \geq j^*$, any Krasowskii solution φ of the system (3) is such that, if $\varphi(0) \in \Omega_{c^2+d^2+1}$, then there exists $T > 0$ such that $\varphi(t) \in \Omega_\sigma$ for all $t \geq T$.

Proof. (Sketch) Consider $\dot{W}(x,\zeta)$ when $(x,\zeta) \in S$ and $v \in K(\Psi(\bar{k}\zeta))$, in the two cases in which the quantizer is inside and outside the deadzone. In both cases an appropriate design of $\bar{k}$, u_0 and j yields $\dot{W}(x,\zeta) < 0$. Pick:

$$\bar{k} \geq k^* = (d+1)M_0/(db_0m_0(1-\delta)) \,,$$

$$j \geq j^* = \left\lceil \log\left(\frac{d^2}{(c^2+d^2+d+1)^2}\frac{b_0m_0}{4M_1}\right)\log\left(\frac{1-\delta}{1+\delta}\right)^{-1}\right\rceil$$

and $u_0 = (1+\delta)\bar{k}\max_{(x,\zeta)\in\tilde{S}}|\zeta|$, with:

$$\begin{aligned} m_0 &= \min_{(x,\zeta)\in\tilde{S}} \zeta^2 \,,\; M_0 = \max_{(x,\zeta)\in\tilde{S}\,,\,\mu\in\mathcal{P}} |w(x,\zeta,\mu)\zeta| \,, \\ M_1 &= \max_{(x,\zeta)\in\tilde{S}\,,\,\mu\in\mathcal{P}} |b(x,\zeta,\mu)| \cdot (\max_{(x,\zeta)\in\tilde{S}} |\zeta|)^2 \,. \end{aligned} \tag{4}$$

Remark. The constant k^* differs from the one in Teel and Praly (1995), Isidori (1999) by the presence of the factor $(1-\delta_M)^{-1}$. That is, as expected, the error due to quantization is counteracted by raising the controller gain. Moreover, it is interesting to observe that the constant j^*, that is the number of quantization levels, only depends on the size of the domain of attraction and of the target set.

In Hayakawa *et al.* (2006), a switching rule to implement the quantized control (2) has been proposed and is illustrated in Figure 2. It is based on the introduction of the new quantization value $u_i(1+\delta)^{-1}$ between each pair (u_i, u_{i+1}). In this way (see Hayakawa *et al.* (2006), Section 3 for details), the existence of a unique solution is guaranteed, while avoiding the occurrence of chattering. It is important to stress that the analysis at the basis of the results in this paper remains valid also for this solution. As a matter of fact, we require $\dot{W}(x,\zeta) < 0$ to be negative for all $(x,\zeta) \in S$ and all $v \in K(\Psi(\bar{k}\zeta)) = \{(1+\lambda\delta)\bar{k}\zeta \,:\, \lambda \in [-1,1]\}$. This implies, for instance, that at the discontinuity point $u_i/(1+\delta)$ in the Figure, v is any point of the segment $\overline{\mathrm{AE}}$. Hence, $\dot{W}(x,\zeta)$ continues to be negative even at the transition, and we can conclude convergence in finite time to Ω_σ also for this solution.

Now suppose that the hysteresis-like mechanism discussed above has been introduced. When controlling a continuous-time system with a finite number of control values, two main parameters play a role, the number of quantization levels employed and the minimum time elapsed between 2 consecutive switchings. These parameters give a rough idea of the bandwidth needed to implement the controller through a network. The number of quantization levels in the present case is given by the estimate j^* derived in the proof of the result above. Bearing in mind (2), computing how often a transition occurs reduces to compute over the set of all the quantization levels $u_i \in \{u_i \,:\, u_i = \pm\rho^i u_0\,,\, i = 0,1,\ldots,j\}$ the smallest time needed for the solution of (1) with $u = u_i$ to cross the region of the state space $\{(x,\zeta) \,:\, u_i/(1+\delta) < \bar{k}\zeta \leq u_i/(1-\delta)\}$. Observe that the size of the set is independent of $\bar{k}$, for u_i depends on u_0 which is proportional to $\bar{k}$. In the next section we propose a different stabilization scheme which has the advantage of presenting only 3 quantization levels, although it does not necessarily yield a lower bandwidth.

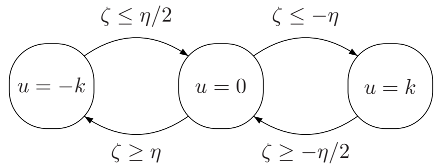

Fig. 3. The partial-state switched controller.

4. TERNARY CONTROLLER

Let η be the positive constant introduced at the end of Section 2 in the definition of the neighborhood U, and introduce the following sets, depicted in Figure 1:

$$\begin{aligned} \Omega_- &= \{(x,\zeta) \in \Omega_{c^2+d^2+1} \,:\, \zeta \geq \eta\} \,, \\ \Omega_0 &= \{(x,\zeta) \in \Omega_{c^2+d^2+1} \,:\, |\zeta| < \eta\} \,, \\ \Omega_+ &= \{(x,\zeta) \in \Omega_{c^2+d^2+1} \,:\, \zeta \leq -\eta\} \,. \end{aligned}$$

Assume without loss of generality that η is small enough that Ω_-, Ω_+ are not void. We propose the following controller. At the initial time $t = 0$, assume that $(x,\zeta) \in \Omega_{c^2+d^2+1}$, and set the control value as

$$u(0) = \begin{cases} -\bar{k} \;\, if \;\, \zeta(0) \geq \eta \\ \;\;0 \;\, if \;\, |\zeta(0)| < \eta \\ \;\;\bar{k} \;\, if \;\, \zeta(0) \leq -\eta \,. \end{cases} \tag{5}$$

For $t > 0$, the controller is chosen as

$$u(t) = \begin{cases} -\bar{k} \;\, if \;\, [u(t^-) = 0] \wedge [\zeta(t^-) \geq \eta] \\ \;\;0 \;\, if \;\, \begin{array}{c}\{[u(t^-) = -\bar{k}] \wedge [\zeta(t^-) \leq \eta/2]\}\vee \\ \{[u(t^-) = \bar{k}] \wedge [\zeta(t^-) \geq -\eta/2]\}\end{array} \\ \;\;\bar{k} \;\, if \;\, [u(t^-) = 0] \wedge [\zeta(t^-) \leq -\eta] \,, \end{cases} \tag{6}$$

with $\bar{k} > 0$ a parameter to design, and where the symbol $r(t^-)$ denotes the limit $\lim_{s\to t^-} r(s)$, that is the value of $r(t)$ immediately before the transition of the controller takes place. The transition graph in Figure 3 illustrates the behavior of the controller (6).

We state the analogous of Lemma 1 but with the ternary controller above.

Lemma 2. There exists a choice of $\bar{k}$ and η such that the Lyapunov function $W(x,\zeta)$, computed along any trajectory of the closed-loop system (1), (5), (6) which starts in S, satisfies $\dot{W}(x(t),\zeta(t)) < 0$ for all $(x(t),\zeta(t)) \in S$.

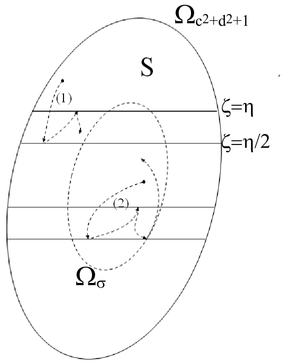

Fig. 4. Sample trajectories for the closed-loop system.

Two sample trajectories of the switched system are depicted in Figure 4. The first one starts in S and is such that $\zeta(0) \geq \eta$, and converges to Ω_σ hitting the 2 planes with $\zeta = \eta$ and $\zeta = \eta/2$. The second one starts in Ω_σ and never leaves the set. Observe that in the latter case, the Lyapunov function $W(x, \zeta)$ may increase along the trajectory, but never exceed σ.

The previous result shows polynomial decay of the Lyapunov function. Nevertheless, a slight variation of its proof shows that, if the Lyapunov function for the x subsystem decays exponentially, so does the entire closed-loop system as far as $u \neq 0$.

5. APPLICATIONS

Systems with uniform relative degree
It is well-known that a nonlinear input-affine system is said to have a *uniform* relative degree r if it has a relative degree r at x_0 for each $x_0 \in \mathbb{R}^n$. It is also well-known that there exists a globally defined diffeomorphism which changes the system into one of the following form (see e.g. Proposition 9.1.1. in Isidori (1995)):

$$\begin{aligned} \dot{z} &= f(z, \xi_1) \\ \dot{\xi}_i &= \xi_{i+1} \ , \ 1 \leq i \leq r-1 \\ \dot{\xi}_r &= \bar{q}(z, \xi) + \bar{b}(z, \xi) u \end{aligned} \tag{7}$$

with $z \in \mathbb{R}^{n-r}$, and $\bar{b}(z, \xi) \geq b_0 > 0$ for all (z, ξ). Systems like the one above restricted to the components $z, \xi_1, \ldots, \xi_{r-1}$, with ξ_r viewed as an input, can be always stabilized by means of a linear high-gain partial-state feedback (Isidori (1995), Theorem 9.3.1), provided that the origin $z = 0$ is a globally asymptotically stable equilibrium point for $\dot{z} = f(z, 0)$, i.e. system (7) is minimum-phase. As a matter of fact, let $\lambda^{r-1} + a_{r-2}\lambda^{r-2} + \ldots + a_1\lambda + a_0$ be a polynomial with all its roots with strictly negative real parts, and $\xi_r = -(k^{r-1}a_0\xi_1 + k^{r-2}a_1\xi_2 + \ldots + ka_{r-2}\xi_{r-1})$ the candidate "control law". Then, for any $R > 0$ there exists $k^* > 0$ such that, if $k \geq k^*$, every solution of

$$\begin{aligned} \dot{z} &= f(z, \xi_1) \\ \dot{\xi}_i &= \xi_{i+1} \ , \ 1 \leq i \leq r-2 \\ \dot{\xi}_{r-1} &= -(k^{r-1}a_0\xi_1 + k^{r-2}a_1\xi_2 + \ldots + ka_{r-2}\xi_r) \end{aligned} \tag{8}$$

starting from the cube in $\mathbb{R}^{n-1}$ whose edges are $2R$ long, asymptotically converges to the origin. Perform the change of coordinates

$$\xi_r = -(k^{r-1}a_0\xi_1 + k^{r-2}a_1\xi_2 + \ldots + ka_{r-2}\xi_r) + \zeta \ , \tag{9}$$

let $x = (z, \xi_1, \ldots, \xi_{r-1})$, and rewrite (7) as

$$\begin{aligned} \dot{x} &= F(x) + G\zeta \\ \dot{\zeta} &= q(x, \zeta) + b(x, \zeta)u \ , \end{aligned}$$

where $F(x)$ is the vector field on the right-hand side of (8), and G, q, b are understood from the context. The system $\dot{x} = F(x)$ satisfies the ULP property. We conclude that both Lemma 1 and 2 can be applied to system (7) to obtain

Proposition 1. Consider a minimum-phase nonlinear system of the form (7). For any $R > 0$ and any $\varepsilon > 0$, there exist a quantized feedback law $u = -\Psi(\bar{k}\zeta)$, or a ternary feedback law (5), (6), with ζ given by (9), and a time $T > 0$, such that any trajectory φ of the closed-loop system which starts in the cube centered at the origin of side $2R$ lies in the cube centered at the origin of side 2ε for all $t \geq T$.

In particular, the proposition shows that it is as simple as in the non-quantized case to stabilize nonlinear minimum-phase systems with *quantized measurements* provided that the *relative degree* of the system is *one*. In fact, if this is the case, then ζ coincides with the output of the system.

In the remaining, systems for which results similar to Proposition 1 apply are referred to as *semi-globally practically stabilizable* systems.

Robust switched stabilization of nonlinear systems
In this section we propose a simple switched controller to stabilize nonlinear systems of the form

$$\begin{aligned} &\dot{z} = F(\mu)z + G(\xi_1, \mu)\xi_1 \\ &\dot{\xi}_1 = q_{10}(\xi_1, \mu)z + q_{11}(\xi_1, \mu)\xi_1 + b_1(\xi_1, \mu)\xi_2 \\ &\quad \vdots \\ &\dot{\xi}_{r-1} = q_{r-1,0}(\xi_1, \ldots, \xi_{r-1}, \mu)z + \\ &\quad \sum_{i=1}^{r-1} q_{r-1,i}(\xi_1, \ldots, \xi_{r-1}, \mu)\xi_i + b_{r-1}(\xi_1, \ldots, \xi_{r-1}, \mu)\xi_r \\ &\dot{\xi}_r = q_{r0}(\xi_1, \ldots, \xi_r, \mu)z + \\ &\quad \sum_{i=1}^{r} q_{ri}(\xi_1, \ldots, \xi_r, \mu)\xi_i + b_r(\xi_1, \ldots, \xi_r, \mu)u \ , \end{aligned} \tag{10}$$

where $z \in \mathbb{R}^{n-r}$, and $b_i(z, \xi_1, \ldots, \xi_i, \mu) \geq b_{i0} > 0$ for all $(z, \xi_1, \ldots, \xi_i, \mu) \in \mathbb{R}^{n-r+i}$. We also assume that, for all $\mu \in \mathcal{P}$, there exists $P(\mu) = P^T(\mu) > 0$ such that

$$F^T(\mu)P(\mu) + P(\mu)F(\mu) \leq -I \ .$$

The first fact we recall is the following:

Lemma 3. Set $\xi = \mathrm{col}(\xi_1, \ldots, \xi_{r-1})$. There exists an $(r-1) \times (r-1)$ matrix $M_0(\xi)$ and a $1 \times (r-1)$ vector $\delta(\xi)$ of smooth functions such that $\xi^T M(\xi)\xi$ is a definite positive and proper function, and the function $V(z, \xi) = z^T P(\mu)z + \xi^T M_0(\xi)\xi$ satisfies $\dot{V}(z, \xi) \leq -\varepsilon V(z, \xi)$, where $\dot{V}(z, \xi)$ is the derivative of $V(z, \xi)$ along the trajectories of (10) with $u = \delta(\xi)\xi$.

By the change of coordinates $\zeta = \xi_r - \delta(\xi)\xi$, letting as before $x = (z, \xi)$, it is not hard to see that we are in the setting of Section 4, and systems of the form (10) can be semi-globally practically stabilized by the ternary controller (5), (6), where now the guards depend on $\zeta =$

$\xi_r - \delta(\xi)\xi$. This result can be also employed to give a simple robust output-feedback switched stabilization scheme for a class of nonlinear systems, as shown in the next subsection.

A simple output-feedback switched stabilization scheme Consider the nonlinear system

$$\begin{aligned} \dot{x} &= F(\mu)x + G(y,\mu)y + \bar{g}(\mu)\gamma(y)u \\ \dot{y} &= H(\mu)x + K(y,\mu)y \ , \end{aligned} \tag{11}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ the measured output, and $\gamma(y)$ a smooth function bounded away from zero. Under appropriate conditions, namely (Marino and Tomei (1993), and also Isidori (1999), Section 11.3) (i) the system has a well-defined uniform relative degree $r \geq 2$ and (ii) its zero dynamics is globally asymptotically stable, one can prove that, for the system above, to which it is appended the additional dynamics

$$\begin{aligned} \dot{\xi}_i &= -\lambda_i \xi_i + \xi_{i+1} \ , \ 2 \leq i \leq r-1 \\ \dot{\xi}_r &= -\lambda_{r-1}\xi_r + \gamma(y)u \ , \end{aligned} \tag{12}$$

there exists a change of coordinates $z = T(x, y, \xi, \mu)$, linear in (x, y, ξ, μ), which transforms the extended system into

$$\begin{aligned} \dot{z} &= \tilde{F}(\mu)z + \tilde{G}(y,\mu)y \\ \dot{y} &= \tilde{H}(\mu)z + \tilde{K}(y,\mu)y + b(\mu)\xi_2 \\ \dot{\xi}_i &= -\lambda_i\xi_i + \xi_{i+1} \ , \ 2 \leq i \leq r-1 \\ \dot{\xi}_r &= -\lambda_{r-1}\xi_r + \gamma(y)u \ , \end{aligned}$$

with $b(\mu)$ bounded away from zero. This system is a special case of (10), and therefore there exists a ternary switched controller depending on $y, \xi_2, \ldots, \xi_r$ for it. The appended dynamics (12) with u given by (5), (6), and $\zeta = \xi_r - \delta(\xi)\xi$, $\xi = (y, \xi_2, \ldots, \xi_{r-1})$, is a switched dynamic output feedback controller which semi-globally practically stabilizes the system (11). The implementation of the closed-loop system through a network is illustrated in Figure 5, with the encoder depicted in Figure 6. The decoder, on the other hand, is simply a device which converts the packets received by the channel into one of the 3 values $\{-1, 0, 1\}$, which are then multiplied by the gain $\bar{k}$.

The approach to output feedback stabilization through a network outlined above is different from the one in De Persis (2006) where uniformly completely observable systems were considered. Although the former is less general, it applies to uncertain systems, and employs a *linear* "encoder" on the sensor side. Similar considerations hold for the results of Cheng and Savkin (2007) in which the approach of De Persis (2006) is applied to a different class of nonlinear systems. A similar class as (11) was considered in Liberzon (2007), where the output is quantized with no pre-processing. However, in that paper, the control law u must be designed so as to guarantee input-to-state stability with respect to state measurements errors, a task which may be considerably harder than designing the control law as in Lemma 3. Observe that we do *not* employ a dense quantization, that is we do not require a small quantization error (the quantization density can be any number in $(0, 1)$) to compensate for the lack of input-to-state stability.

Remarks on quantized output feedback stabilization In the preceding subsection we focused on an output feedback stabilization scheme based on "quantizing" the state of a linear filter which preliminarily processes the output.

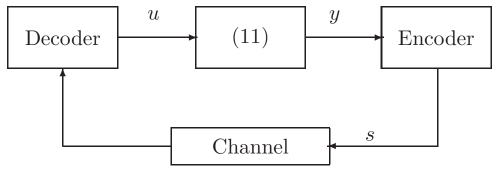

Fig. 5. The switched output feedback controller for system (11) implemented through a network.

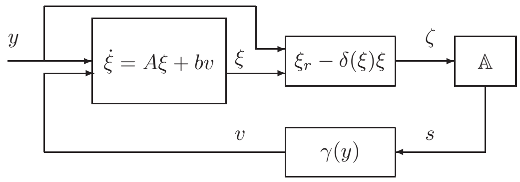

Fig. 6. The encoder. The block labeled with $\mathbb{A}$ is the automaton depicted in Figure 3. This block outputs 3 possible values. The device which converts these values into packets of 2 bits which can be transmitted through the network is not depicted for the sake of simplicity.

Except for the case of minimum-phase relative-degree-one systems (see Remark following Proposition 1), if no pre-processing of the output is allowed, then the problem becomes more difficult. Here we discuss some of these difficulties for a special case of system (10), the one with $\dot{z} = f(z, \xi_1)$, $b_i(\xi_1, \ldots, \xi_i, \mu) = 1$, measured output $y = \xi_1$, and such that, for all i, the sums involving the functions q_{ij} are simply replaced by a function $q_i(z, \xi_1)$. These systems are sometimes said to be in *observer canonical form* (Teel and Praly (1995)). Available for feedback is the quantity $\Psi_y(y)$, where the map Ψ_y is defined as in (2), with u_i replaced by $y_i = \rho^i y_0$, $0 \leq i \leq j$, and y_0 and j parameters to determine. Consider the dynamic controller

$$\begin{aligned} \dot{\eta} &= A\eta + bu \\ u &= -(k^r a_0 \Psi_y(y) + k^{r-1} a_1 \eta_2 + \ldots + a_{r-1}\eta_r) \ , \end{aligned}$$

with $b \in \mathbb{R}^r$ a vector whose entries are all equal to zero except for the last one which is equal to 1,

$$A = \begin{bmatrix} -c_{r-1} & 1 & 0 & \ldots & 0 & 0 \\ -c_{r-2} & 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -c_1 & 0 & 0 & \ldots & 0 & 1 \\ -c_0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} ,$$

and $\lambda^r + c_{r-1}\lambda^{r-1} + \ldots + c_1\lambda + c_0$ is a polynomial whose roots have all strictly negative real parts. Perform the change of coordinates $e = \xi - \eta$, $\zeta_1 = \xi_1$, $\zeta_i = k^{-(i-1)}\eta_i$, $i = 2, \ldots, r$, to obtain, with u specified above, the closed-loop system

$$\begin{aligned} \dot{z} &= f(z, \zeta_1) \\ \dot{e} &= Ae + Q(z, \zeta_1) \\ \dot{\zeta} &= kA_c\zeta + Q_k(z, e, \zeta_1) + G_k(\Psi(\zeta_1) - \zeta_1) \\ y &= \zeta_1 \ , \end{aligned} \tag{13}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{r-1} \end{bmatrix}, Q(z,\xi_1) = c\xi_1 +$$

$$\begin{bmatrix} q_1(z,\xi_1) \\ q_2(z,\xi_1) \\ \dots \\ q_r(z,\xi_1) \end{bmatrix}, Q_k(z,e,\zeta_1) = \begin{bmatrix} q_1(z,\zeta_1) \\ \frac{1}{k}(c_2 e_1 - c_2\zeta_1) \\ \dots \\ \frac{1}{k^{r-1}}(c_r e_1 - c_r x_1) \end{bmatrix},$$

and $G_k = -ka_0\,[\,0\ 0\ \dots\ 1\,]^T$. We set

$$\begin{aligned} \dot{x} &\triangleq \begin{bmatrix} \dot{z} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} f(z,0) \\ Ae \end{bmatrix} + \begin{bmatrix} f(z,\zeta_1) - f(z,0) \\ Q(z,\zeta_1) \end{bmatrix} \\ &\triangleq F(x) + G(x,\zeta_1)\zeta_1 \ , \end{aligned}$$

and obtain

$$\begin{aligned} \dot{x} &= F(x) + G(x,\zeta_1)\zeta_1 \\ \dot{\zeta} &= kA_c\zeta + Q_k(z,e,\zeta_1) + G_k(\Psi(\zeta_1) - \zeta_1) \ . \end{aligned} \tag{14}$$

If we assume the origin to be a globally asymptotically stable equilibrium for the system $\dot{z} = f(z,0)$, we can guarantee the existence of a positive definite and proper Lyapunov function $V(x)$ such that

$$\frac{\partial V}{\partial x} F(x) < 0 \quad \text{for all } x \neq 0 \ .$$

Furthermore, let $P_c = P_c^T > 0$ be the matrix such that $A_c^T P_c + P_c A_c = -I$, and set $U(\zeta) = \zeta^T P_c \zeta$. Finally, define the Lyapunov function $W(x,\zeta)$ as before, except for ζ^2 replaced by $U(\zeta)$. It is shown in Teel and Praly (1995) that, no matter how small σ in the set S is chosen, there always exists $\bar{k}^*$ such that for all $k \geq \bar{k}^*$, the Lyapunov function $W(x,\zeta)$ computed along the solutions of the system (14), is strictly decreasing as far as the state evolves in the set S, provided that no quantization is present, i.e. $G_k(\Psi(\zeta_1) - \zeta_1) = 0$. It is possible to check that the system retains the same property even when $G_k(\Psi(\zeta_1) - \zeta_1) \neq 0$. In particular, we have the analogous of Lemma 1:

Lemma 4. For any $\delta \in (0,1)$, there exist positive numbers k^*, j^* such that, for any gain $\bar{k} \geq k^*$ and any number of quantization levels $j \geq j^*$, each Krasowskii solution φ of system (14) is such that, if $\varphi(0) \in \Omega_{c^2+d^2+1}$, then there exists $T > 0$ such that $\varphi(t) \in \Omega_\sigma$ for all $t \geq T$.

The proof is not dissimilar from that of Lemma 1. In the original coordinates, the result only guarantees practical regulation of the output $y = \xi_1$ and not of all the coordinates, for $\eta_i = k^{i-1}\zeta_i$. The adoption of the "zooming-in" technique of Liberzon (2003) does not overcome this limitation, because there is no guarantee for the gain k to remain bounded as the state converges to the origin. One could turn to Teel and Praly (1995) where, to surmount the obstacle, it is additionally supposed that $z = 0$ is a locally exponentially stable equilibrium for $\dot{z} = f(z,0)$. This shows that the state of (14) with $G_k(\Psi(\zeta_1) - \zeta_1) = 0$ asymptotically converges to zero. In the case of quantized measurements considered above, however, the same hypothesis does not suffice, and one has to additionally suppose a quantizer Ψ_y with an *infinite* number of quantization levels, i.e. in (2) $1 \leq i < \infty$ and $\Psi(r) = 0$ if and only if $r = 0$. Then one can prove that all the Krasowskii solutions of (14) converge to zero, and so do the trajectories of the original system.

6. CONCLUSION

We have discussed a few results on the problem of stabilizing nonlinear systems using a finite number of control or measurement values and in the presence of parametric uncertainty.

REFERENCES

A. Bacciotti. Further remarks on potentially global stabilizability. *IEEE Trans. Auto. Contr.*, 34 (6), 637–639, 1989.

F. Ceragioli and C. De Persis. Discontinuous stabilization of nonlinear systems: Quantized and switching controls. *Systems & Control Letters*, Vol. 56, 7-8, 461-473, 2007.

T.M. Cheng and A.V. Savkin. Output feedback stabilization of nonlinear networked control systems with non-decreasing nonlinearities: A matrix inequalities approach. *Int. J. of Robust and Nonlinear Control*, 17 (5-6), 387–404, 2007.

C. De Persis. On stabilization of nonlinear systems under data rate constraints using output measurements. *Int. J. of Robust and Nonlinear Control*, 16, 315-332, 2006.

R.A. Freeman and P. Kokotovic. Design of 'softer' robust nonlinear control laws. *Automatica*, 32, 733-746, 1993.

T. Hayakawa, H. Ishii, and K. Tsumura. Adaptive quantized control for nonlinear uncertain systems. In *Proc. 2006 American Control Conf.*, Minneapolis, MN, 2006.

A. Isidori. *Nonlinear Control Systems*, 3rd edition. Springer-Verlag, New York, 1995.

A. Isidori. *Nonlinear Control Systems*, Vol. 2. Springer, London, 1999.

G. Kaliora and A. Astolfi. Nonlinear control of feedforward systems with bounded signals. *IEEE Trans. Auto. Contr.*, 49 (11), 1975–1990, 2004.

D. Liberzon. Hybrid feedback stabilization of systems with quantized signals. *Automatica*, 39: 1543–1554, 2003.

D. Liberzon. Observer-based quantized output feedback control of nonlinear systems. In *Proc. MED'07*, Athens, Greece, 2007.

J. Liu and N. Elia. Quantized feedback stabilization of non-linear affine systems. *Int. J. Control*, **77**, 3, 239-249, 2004.

R. Marino and P. Tomei. Global adaptive output-feedback control of nonlinear systems, Part I & II. *IEEE Trans. Auto. Contr.*, 38, 17–32 & 33–48, 1993.

G. Nair, F. Fagnani, S. Zampieri, and R.J. Evans. Feedback control under data rate constraints: An overview. In *Proc. 2007 of the IEEE*, 95: 108–137, 2007.

V. Phat, J. Jiang, A.V. Savkin, and I.R. Petersen. Robust stabilization of linear uncertain discret-time systems via a limited capacity communication channel *Systems & Control Letters*, 53, 5, 347–360, 2004.

A.R. Teel and L. Praly. Tools for semi-global stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, 33, 1443–1488, 1995.

G.Z. Zhang, G.N. Nair, R.J. Evans, and B. Wittenmark. A data-rate limited view of adaptive control. In *Proc. 14th IFAC Symp. Sys. Ident.*, Newcastle, Australia, 2006.